MATH 206A: SYMMETRIC FUNCTIONS
HOMEWORK #4

• The homework is due on Gradescope on *Monday, October 31st at 11:59pm*. Late homework is generally not accepted (unless you have a good reason).
• The lowest homework score will be dropped.
• Each problem is worth the same number of points.
• Collaboration is encouraged, but you have to write up the solutions by yourself. For each problem, all sources and collaborators must be clearly listed.
• LaTeX is much preferred (hand-drawn pictures may be scanned). Alternatively, please submit good quality scans of your work.
• Justify your answers by rigorous proofs.

Problem 1. Choose a random permutation $w \in S_n$ for $n \geq 7$. Compute Fomin's growth diagram for w. Compute the image (P, Q) of w under the RSK algorithm and compare it to the growth diagram you got.

Hint: [Sta99, Theorem 7.13.5].

Problem 2. A symmetric function $f \in \Lambda$ is called *s-positive* if it is a nonnegative linear combination of Schur functions: $f = \sum_\lambda c_\lambda s_\lambda$ with $c_\lambda \geq 0$ for all λ. We similarly define m-positive, e-positive, etc. For each of the below statements, using your computer, decide whether it is "False" (in which case, give a counterexample) or "Probably True" (in which case, provide some convincing evidence) or "Definitely True" (in which case, provide a proof).

(1) $h_\lambda \cdot e_\mu$ is m-positive for all λ, μ.
(2) $e_\lambda \cdot s_\mu$ is s-positive for all λ, μ.
(3) $s_\lambda \cdot h_\mu$ is h-positive for all λ, μ.
(4) $s_\lambda \cdot s_\mu$ is s-positive for all λ, μ.
(5) $e_\lambda \cdot e_\mu$ is e-positive for all λ, μ.
(6) $m_\lambda \cdot m_\mu$ is p-positive for all λ, μ.

Hint: For some problems on e-positivity which are actually open, see this link.

Problem 3. Show that

$$\frac{1}{\prod_i (1 - x_i) \cdot \prod_{i<j}(1 - x_i x_j)} = \sum_\lambda s_\lambda(x).$$

Hint: [Sta99, Corollary 7.13.8]. This identity is actually due to Schur.

Problem 4. For $w = (w_1, w_2, \dots, w_n) \in \mathfrak{S}_n$, define the *descent set*

$$\mathrm{Des}(w) := \{1 \leq i \leq n - 1 : w_i > w_{i+1}\}.$$

Let (P, Q) be the image of w under RSK.

(1) Can $\mathrm{Des}(w)$ be uniquely recovered from P? (If yes, how? If not, why not?)
(2) Can $\mathrm{Des}(w)$ be uniquely recovered from Q? (If yes, how? If not, why not?)

Date: October 23, 2022.

References

[Sta99] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.