# MATH 206A: SYMMETRIC FUNCTIONS
# HOMEWORK #1

• The homework is due on Gradescope on *Monday, October 10th at 4pm*. Late homework is generally not accepted (unless you have a good reason).
• Each problem is worth the same number of points.
• Collaboration is encouraged, but you have to write up the solutions by yourself. For each problem, all sources and collaborators must be clearly listed.
• LaTeX is much preferred (hand-drawn pictures may be scanned). Alternatively, please submit good quality scans of your work.
• Justify your answers by rigorous proofs.

**Problem 1.** Give a combinatorial proof that Schur polynomials

$$s_\lambda(x_1, x_2, \ldots, x_n) := \sum_{T \in \mathrm{SSYT}_n(\lambda)} x^T$$

are symmetric.

*Hint:* For each $i \leq n-1$, find a bijection on $\mathrm{SSYT}_n(\lambda)$ swapping the number of $i$'s and $(i+1)$'s in $T$. Then show that this is enough to deduce the result. For a solution, see the proof of [Sta99, Thm. 7.10.2].

**Problem 2.** Let

$$e_\lambda = \sum_\mu M_{\lambda\mu} m_\mu \quad \text{and} \quad h_\lambda = \sum_\mu N_{\lambda\mu} m_\mu.$$

Added on 10/02/22: Given a matrix $A = (a_{ij})_{i,j\geq 1}$ with finitely many nonzero entries, let $\mathrm{row}(A)$ and $\mathrm{col}(A)$ be its sequences of row and column sums:

$$\mathrm{row}(A)_i := \sum_{j=1}^{\infty} a_{ij} \quad \text{and} \quad \mathrm{col}(A)_j := \sum_{i=1}^{\infty} a_{ij}.$$

(1) Show that $M_{\lambda\mu}$ is the number of $0,1$-matrices $A = (a_{ij})$ satisfying $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \mu$.
(2) Show that $N_{\lambda\mu}$ is the number of $\mathbb{Z}_{\geq 0}$-matrices $A = (a_{ij})$ satisfying $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \mu$.

*Hint:* For a solution, see [Sta99, Prop. 7.4.1, 7.5.1].

**In addition, solve the following exercises from** [Sta99]**:**

- Exercise 7.2(a,b,c). The *dominance ordering* $\leq$ (which we will study in detail later) is defined as follows: write $\lambda \leq \mu$ if $|\lambda| = |\mu|$ and for each $i \geq 1$, we have

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i.$$

The relevant definitions can be found on wiki. Links: lattice, dual, graded.
- Exercise 7.3.

*Hint:* You can find the solutions at the end of the book.

*Date*: October 2, 2022.

## References

[Sta99] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.