MATH 206 CLUSTER ALGEBRAS: HOMEWORK #2

• The homework is due on Gradescope on *Wednesday, February 24th at 1pm*. Late homework is generally not accepted (unless you have a good reason).
• Each problem is worth the same number of points.
• Collaboration is encouraged, but you have to write up the solutions by yourself. For each problem, all sources and collaborators must be clearly listed.
• LaTeX is preferred (hand-drawn pictures may be scanned). Alternatively, please submit good quality scans of your work! (e.g. google "phone scan app")
• Justify your answers by rigorous proofs.

Solve all of the below problems, but turn in exactly *three* of them which you found the most interesting. All exercises that were mentioned (or could have been mentioned) in class are also considered to be members of this list.

Problem 1. In class, we showed that the quiver $1 \rightrightarrows 2$ is finite mutation type but not finite type. Find some other example of such a quiver.

Problem 2. Consider a sequence $x_0, x_1, \dots$ defined by a recurrence relation

$$x_{n+1} = \frac{x_n^2 + 1}{x_{n-1}}, \quad n \geq 1,$$

with initial values $x_0 = a$ and $x_1 = b$. Show that this sequence satisfies a linear recurrence with constant coefficients, and find these coefficients.

Problem 3. Prove Vinberg's characterization:

Theorem. *Let G be a finite simple graph.*
• G admits a subadditive labeling if and only if it is an ADE Dynkin diagram.
• G admits an additive labeling if and only if it is an affine ADE Dynkin diagram.

Problem 4. Let G be a finite simple graph. Let $c : V(G) \to \mathbb{Z}$ be a labeling of its vertices by integers. We say that $v \in V(G)$ is *unhappy* if

$$c(v) < \frac{1}{2} \sum_{\{u,v\} \in E(G)} c(u).$$

Consider the following game:

- Choose some vertex w and let the initial labeling be defined by $c(w) = 1$ and $c(v) = 0$ for $v \neq w$.
- Pick any vertex v which is unhappy and increase $c(v)$ by 1.
- Stop if there are no unhappy vertices.

Classify all graphs G for which this game stops after finitely many steps. Show that for all such graphs, the total number of steps and the final labeling do not depend on the choices made during the game.

Problem 5. Find all Coxeter elements in S_n.

Problem 6. Let Φ be a crystallographic root system. Define the *root poset* $(\Phi^+, \leq)$ on its set of positive roots as follows: for $\alpha, \beta \in \Phi^+$, write $\alpha \leq \beta$ if $\beta - \alpha$ is a nonnegative linear combination of simple roots. Show that $(\Phi^+, \leq)$ has a unique maximal element.

Problem 7. Let Φ be a crystallographic root system and W the corresponding Weyl group. A subset $I \subset \Phi^+$ is called *biconvex* if for any triple $\alpha, \beta, \gamma \in \Phi^+$ such that $\alpha + \gamma = \beta$, we have

(1) if $\alpha, \gamma \in I$ then $\beta \in I$;
(2) if $\alpha, \gamma \notin I$ then $\beta \notin I$.

For an element $w \in W$, define its *inversion set* $\mathrm{Inv}(w) := \{\alpha \in \Phi^+ \mid w(\alpha) \notin \Phi^+\}$. Show that a subset $I \subset \Phi^+$ is biconvex if and only if it is equal to $\mathrm{Inv}(w)$ for some $w \in W$.

Problem 8. A permutation $w \in S_n$ is called *fully commutative* if all reduced expressions for w are related by commutation relations $s_i s_j = s_j s_i$ (for $|i - j| > 1$). Show that S_n contains Catalan-many fully commutative elements.

Problem 9. [FWZ16, Exercise 2.7.7] (properties of matrix mutations)

Problem 10. [FWZ16, Exercise 3.2.8] (cluster variables of B_2)

Problem 11. [FWZ17, Exercise 4.4.10] (globally foldable quiver)

References

[FWZ16] Sergey Fomin, Lauren Williams, and Andrei Zelevinsky. Introduction to Cluster Algebras. Chapters 1-3. `arXiv:1608.05735v3`, 2016.

[FWZ17] Sergey Fomin, Lauren Williams, and Andrei Zelevinsky. Introduction to Cluster Algebras. Chapters 4-5. `arXiv:1707.07190v2`, 2017.