MATH 206A: GEOMETRIC COMBINATORICS
HOMEWORK #2

• The homework is due on Gradescope on *Friday, December 3rd at 12pm*. Late homework is generally not accepted (unless you have a good reason).
• Each problem is worth the same number of points.
• Collaboration is encouraged, but you have to write up the solutions by yourself. For each problem, all sources and collaborators must be clearly listed.
• LaTeX is preferred (hand-drawn pictures may be scanned). Alternatively, please submit good quality scans of your work! (e.g. google "phone scan app")
• Justify your answers by rigorous proofs.

• Turn in *at least one* problem from each of the two sections below. Turn in *at most five* problems total. Each problem is worth 10 points, thus the total is 50 points. If you do some of the extra credit parts, you can earn up to 150 additional points.

1. Flips, triangulations, matroids

Problem 1.1. Show that Stanley's triangulation of the hypersimplex (as discussed in Lecture 13) is actually a triangulation.

Problem 1.2. Show that the triangulation in Fig. 1 is not regular.
Extra credit (+10pts): Let $\mathcal{A}$ be the point configuration in Fig. 1. Draw the graph whose vertices are triangulations of $\mathcal{A}$ and whose edges correspond to flips of triangulations. Mark the vertices corresponding to non-regular triangulations. Determine how close is your graph to the 1-skeleton of a 3-dimensional polytope.

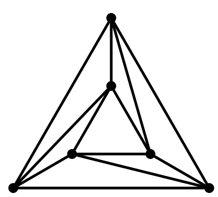


Figure 1. A triangulation of a configuration $\mathcal{A}$ of six points in $\mathbb{R}^2$.

Problem 1.3. Let $\mathcal{V} = (v_0, \dots, v_n) \in \mathbb{R}^{d\cdot(n+1)}$ be a vector configuration which spans $\mathbb{R}^d$, and let $v_0 \neq 0$ be such that the vectors in $\mathcal{V} - v_0 := (v_1, \cdots, v_n)$ still span $\mathbb{R}^d$. Let $\mathcal{C}$ be a fine zonotopal tiling of the zonotope $\mathcal{Z}_{\mathcal{V}}$. Construct (cf. Lecture 21) fine zonotopal tilings $\mathcal{C} - v_0$ and $\mathcal{C}/v_0$ of the zonotopes $\mathcal{Z}_{\mathcal{V}-v_0}$ and $\mathcal{Z}_{\mathcal{V}/v_0}$, respectively, and show that the number of top-dimensional tiles in $\mathcal{C}$ equals that of $\mathcal{C} - v_0$ plus that of $\mathcal{C}/v_0$. (Here $\mathcal{V}/v_0$ is obtained by projecting the vectors $v_1, \dots, v_n$ onto the orthogonal complement of v_0.)

Problem 1.4. Let $\mathcal{B} \subset \binom{[n]}{k}$ be a nonempty collection of k-element subsets of $[n]$. Consider the following three statements:

Date: November 22, 2021.

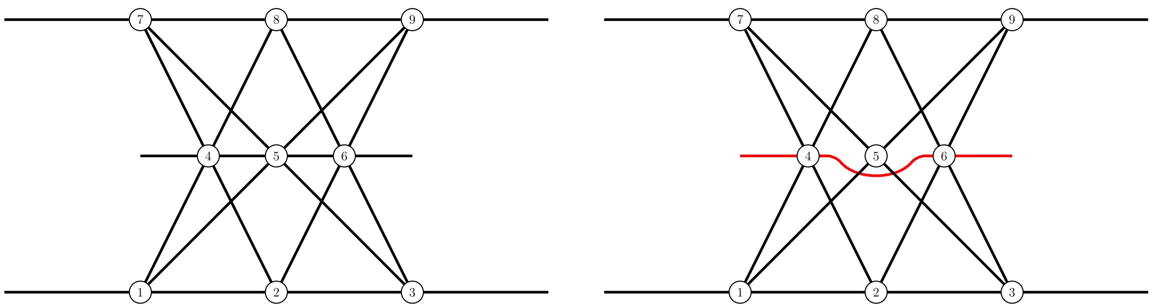
(1) For any $I, J \in \mathcal{B}$ and any $i \in I$, there exists $j \in J$ such that $(I \setminus \{i\}) \cup \{j\}$ belongs to $\mathcal{B}$.
(2) For any $I, J \in \mathcal{B}$ and any $i \in I$, there exists $j \in J$ such that both $(I \setminus \{i\}) \cup \{j\}$ and $(J \setminus \{j\}) \cup \{i\}$ belong to $\mathcal{B}$.
(3) For any $I, J \in \mathcal{B}$, any $r \geq 1$, and any $i_1, \dots, i_r \in I$, there exist $j_1, \dots, j_r \in J$ such that both $(I \setminus \{i_1, \dots, i_r\}) \cup \{j_1, \dots, j_r\}$ and $(J \setminus \{j_1, \dots, j_r\}) \cup \{i_1, \dots, i_r\}$ belong to $\mathcal{B}$.

Thus (1) states that $\mathcal{B}$ is the set of bases of a matroid. Decide whether (1) and (2) are equivalent.

Extra credit (+10pts): decide whether (3) is equivalent to either (1) or (2).

Problem 1.5. Consider a configuration $\mathcal{A} \subset \mathbb{R}^2$ of nine points shown on the left in Fig. 2. Thus $\{4, 5, 6\} \notin \mathcal{B}(\mathcal{M}_{\mathcal{A}})$ is not a basis of $\mathcal{M}_{\mathcal{A}}$. Let the *non-Pappus* matroid $\mathcal{M}$ be defined by $\mathcal{B}(\mathcal{M}) := \mathcal{B}(\mathcal{M}_{\mathcal{A}}) \sqcup \{\{4, 5, 6\}\}$. Check that $\mathcal{M}$ is a matroid and that it is not realizable over $\mathbb{R}$.

Extra credit (+10pts): check that it is in fact not realizable over any field.



Point configuration $\mathcal{A}$ non-Pappus matroid $\mathcal{M}$

FIGURE 2. The non-Pappus matroid

Problem 1.6. For a k-dimensional subspace $U \in \mathrm{Gr}(k, n; \mathbb{R})$ of $\mathbb{R}^n$, let $U^\perp \in \mathrm{Gr}(n - k, n; \mathbb{R})$ denote the orthogonal complement of U. Show that $\mathcal{M}_U^* = \mathcal{M}_{U^\perp}$.

Extra credit (+10pts): show that the same holds over an arbitrary field $\mathbb{F}$. More precisely, consider a dot product on $\mathbb{F}^n$ given by $\langle v, w \rangle := \sum_{i=1}^n v_i w_i$. We say that $v, w \in \mathbb{F}^n$ are *orthogonal* if $\langle v, w \rangle = 0$. Show that for $U \in \mathrm{Gr}(k, n; \mathbb{F})$ a k-dimensional linear subspace of $\mathbb{F}^n$, the orthogonal complement $U^\perp \in \mathrm{Gr}(n - k, n; \mathbb{F})$ is $(n - k)$-dimensional, and that we still have $\mathcal{M}_U^* = \mathcal{M}_{U^\perp}$.

2. Hyperplane arrangements

Problem 2.1. Recall that given a root system Φ, one can consider the hyperplane arrangement

$$\mathcal{A}_\Phi := \{\alpha^\perp \mid \alpha \in \Phi^+\} = \{\alpha^\perp \mid \alpha \in \Phi\}.$$

Find the characteristic polynomial of $\mathcal{A}_\Phi$ for each of the following two root systems:

$$\Phi_{B_n} := \{\pm e_i \mid i \in [n]\} \sqcup \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\};$$
$$\Phi_{D_n} := \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}.$$

All references below are to the web version of Stanley's book [Sta07], which you can find at `https://www.cis.upenn.edu/~cis610/sp06stanley.pdf`. I'm adding a short summary for each problem so that it would be easier to match it to Stanley's notes.

Problem 2.2. [Sta07, Chapter 1, p. 12, Ex. (3)]: characteristic polynomial of $x_1 = x_2, x_2 = x_3, \dots, x_n = x_1$.

Problem 2.3. [Sta07, Chapter 1, p. 12 Ex. (7)(b,c)]: the union of bounded faces of $\mathcal{A}$ need not be homeomorphic to a ball or even starshaped.

Problem 2.4. [Sta07, Chapter 2, p. 30, Ex. (5)]: adding coordinate hyperplanes to a graphical arrangement.

Problem 2.5. [Sta07, Chapter 2, p. 30, Ex. (8)]: counting acyclic orientations of G with a given source.

Problem 2.6. [Sta07, Chapter 5, p. 81, Ex. (3)(a,b)]: bijective proof for the number of regions of the Shi arrangement.
Extra credit (+10pts): Do part (c) of this problem.

Problem 2.7. [Sta07, Chapter 5, p. 82, Ex. (11)]: poset of regions of the Catalan arrangement.

Problem 2.8. [Sta07, Chapter 5, p. 83, Ex. (12)]: characteristic polynomial of the Catalan arrangement.

Problem 2.9. [Sta07, Chapter 5, p. 84, Ex. (19)(a,b)]: characteristic polynomial of the Linial arrangement.

Problem 2.10. [Sta07, Chapter 5, p. 85, Ex. (20)(a)]: a combinatorial interpretation for the number of regions of the Linial arrangement in terms of alternating trees.
Extra credit (+100pts): solve part (b) of this problem and get an automatic A+ for this class.

References

[Sta07] Richard P. Stanley. An introduction to hyperplane arrangements. `https://www.cis.upenn.edu/~cis610/sp06stanley.pdf`. 2007.