MATH 206A: GEOMETRIC COMBINATORICS
HOMEWORK #1

• The homework is due on Gradescope on *Wednesday, October 27th at 12pm*. Late homework is generally not accepted (unless you have a good reason).
• Each problem is worth the same number of points.
• Collaboration is encouraged, but you have to write up the solutions by yourself. For each problem, all sources and collaborators must be clearly listed.
• LaTeX is preferred (hand-drawn pictures may be scanned). Alternatively, please submit good quality scans of your work! (e.g. google "phone scan app")
• Justify your answers by rigorous proofs.
• The total number of problems you turn in should be *five*.

1. Mandatory problems

Turn in *all three* of the below problems.

Problem 1. Let $\mathbf{x} := (x_1, x_2, \dots, x_n)$ be a collection of formal variables. For $\mathbf{a} \in \mathbb{Z}^n$, denote $\mathbf{x}^{\mathbf{a}} := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. Given a multivariate polynomial

$$P(\mathbf{x}) = \sum_{\mathbf{a}} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \quad \in \mathbb{C}[\mathbf{x}],$$

define the *Newton polytope* $N(P) \subset \mathbb{R}^n$ as the convex hull of the exponent vectors of the monomials appearing in P:

$$N(P) := \mathrm{Conv}\{\mathbf{a} \mid c_{\mathbf{a}} \neq 0\}.$$

(i) Show that for any two polynomials $P(\mathbf{x}), Q(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$, $N(PQ)$ is the Minkowski sum

$$N(PQ) = N(P) + N(Q).$$

(ii) Find the Newton polytope of

$$P(\mathbf{x}) := \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

(iii) Use the Vandermonde determinant identity to deduce that the permutohedron Π_n is the Minkowski sum of line segments.

Problem 2. Define the *moment curve* $\mathbf{y} : \mathbb{R} \to \mathbb{R}^d$ by

$$\mathbf{y}(t) := (t, t^2, \dots, t^d).$$

Let $2 \leq d < n$ and choose $t_1 < t_2 < \cdots < t_n$. Consider the following *cyclic polytope*

$$C_d(n) := \mathrm{Conv}\{\mathbf{y}(t_1), \mathbf{y}(t_2), \dots, \mathbf{y}(t_n)\}.$$

(i) Describe the facets of $C_d(n)$.
(ii) Show that $C_d(n)$ is *neighborly*: the convex hull of any $\lfloor d/2 \rfloor$ vertices of $C_d(n)$ is a face of $C_d(n)$.

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Problem 3. Let

$$\Phi := \{\pm e_i \mid i \in [n]\} \sqcup \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}$$

be the root system of type B_n. Compute the f-vector of the type B_n permutohedron $\Pi_\Phi(\rho)$. Your formulas may involve some summation signs.

2. Optional problems

Solve all of the below problems, but turn in exactly *two* of them which you found the most interesting. All exercises that were mentioned (or could have been mentioned) in class are also considered members of this list.

Problem 4. Compute the h-vector of the dual of the type B_n permutohedron.

Problem 5. Let P_n be the set of all $n \times n$ real matrices with nonnegative entries such that each row and column sum is equal to 1. (It is known as the *Birkhoff polytope*).

(i) Show that P_n has $n!$ vertices.
(ii) Show that there exists an affine surjective map $P_n \to \Pi_n$.
(iii) Decide whether P_n is combinatorially equivalent to the permutohedron Π_n.

Problem 6. Recall that a Dyck path is a northeast lattice path from $(0,0)$ to (n,n) staying weakly above the diagonal, and a peak of a Dyck path is a north step followed by an east step. For $1 \leq k \leq n$, find a bijection between Dyck paths with k peaks and Dyck paths with $n+1-k$ peaks. (That is, find a combinatorial proof of the h-vector symmetry relation for the associahedron.)

Problem 7. Show that for each $n \geq 3$, the weak Bruhat order on S_n is a lattice and the strong Bruhat order on S_n is not a lattice.

Problem 8. Prove that $\mathbb{R}^{n-1}$ can be tiled by shifts of the permutohedron Π_n.

Problem 9. Recall the *GKZ realization* of the associahedron. Choose a convex $(n+2)$-gon R. To each triangulation T of R, associate a vector $v_T := (v_1, v_2, \ldots, v_{n+2}) \in \mathbb{R}^{n+2}$ such that v_i is the sum of areas of the triangles in T which contain the vertex i. Show that

$$\operatorname{Conv}\{v_T \mid T \text{ is a triangulation of } R\}$$

belongs to an $(n-1)$-dimensional affine subspace of $\mathbb{R}^{n+2}$.