Boundary stratification

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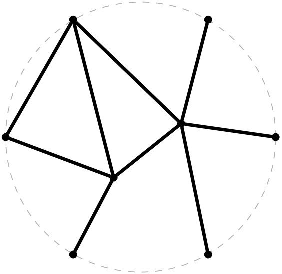

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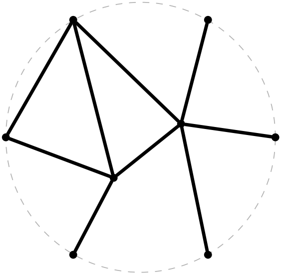

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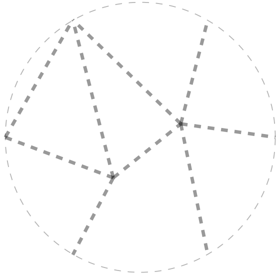

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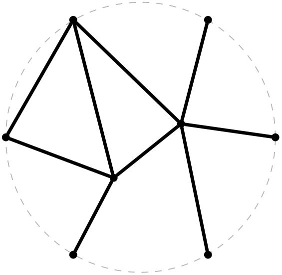

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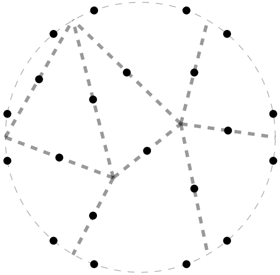

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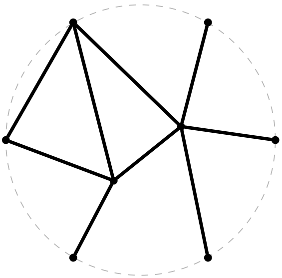

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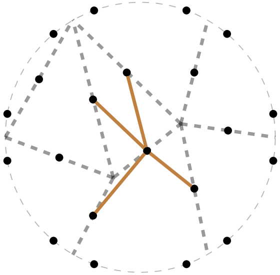

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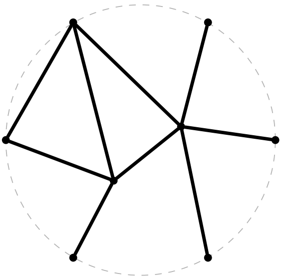

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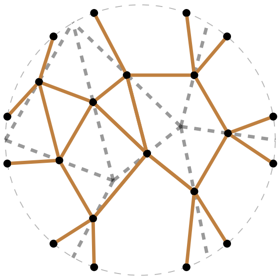

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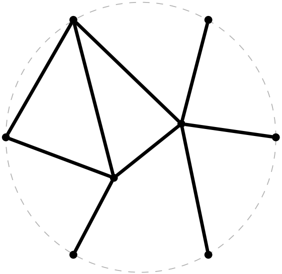

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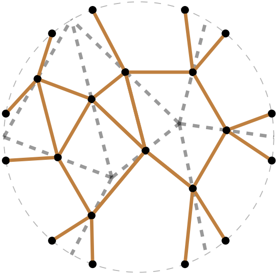

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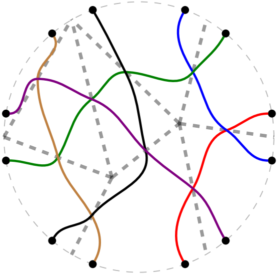

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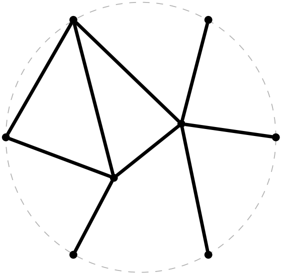

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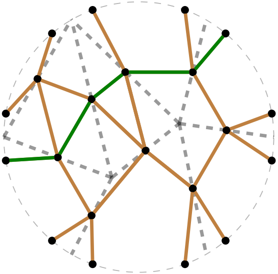

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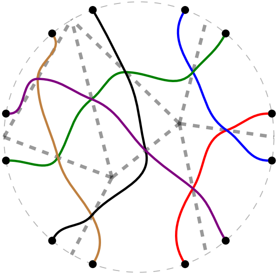

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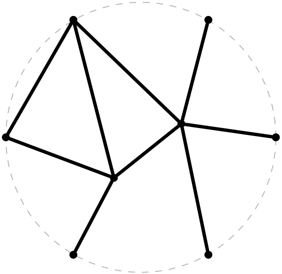

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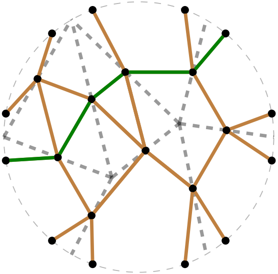

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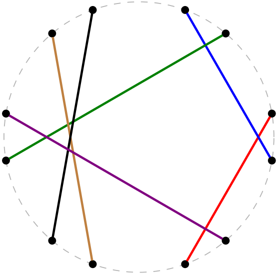

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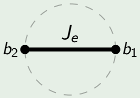

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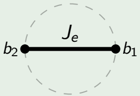

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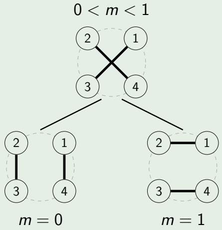

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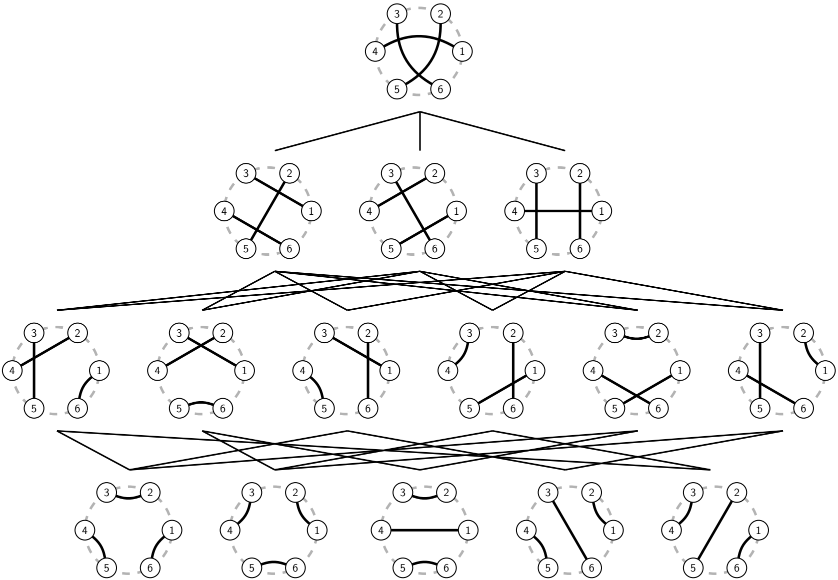

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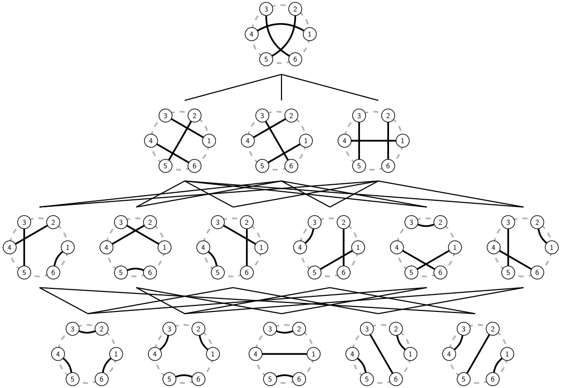

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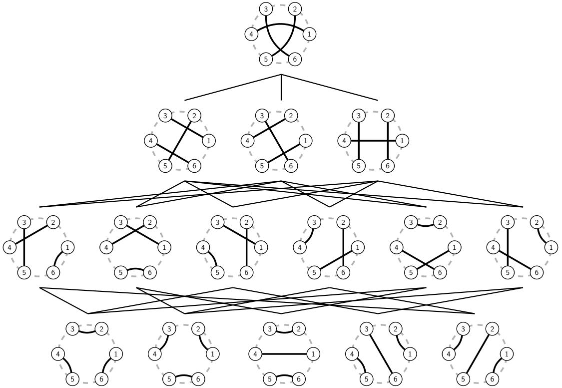

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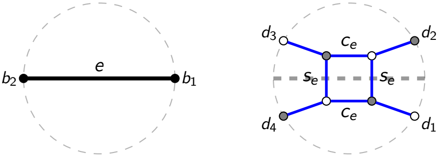
Conjecture (G.–Pylyavskyy (2018))

$\overline{\mathcal{X}}_n \cong \mathrm{OG}_{\geqslant 0}(n, 2n)$ *is a regular CW complex.*

Plabic graphs



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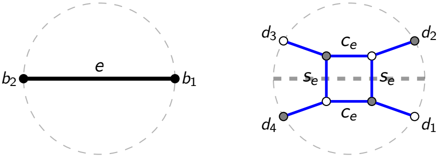

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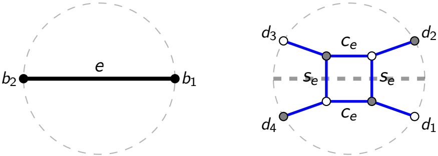

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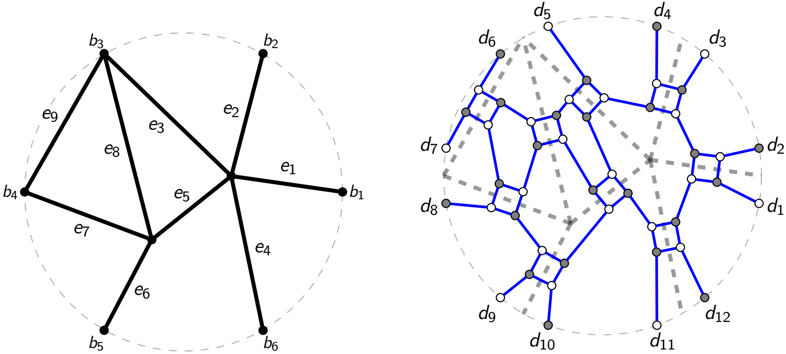
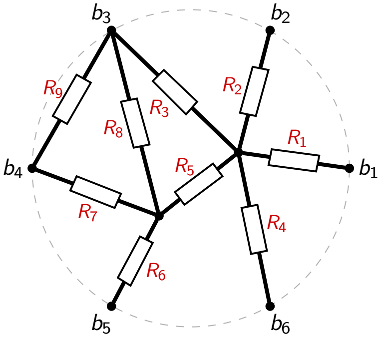


Electrical networks

Let $R : E \to \mathbb{R}_{>0}$ be an assignment of resistances to the edges of G.



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Definition

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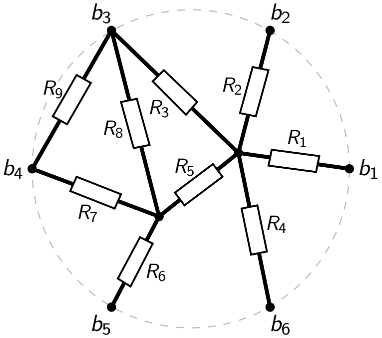

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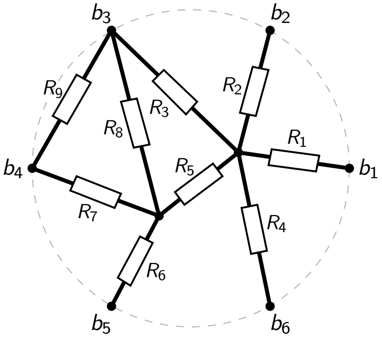

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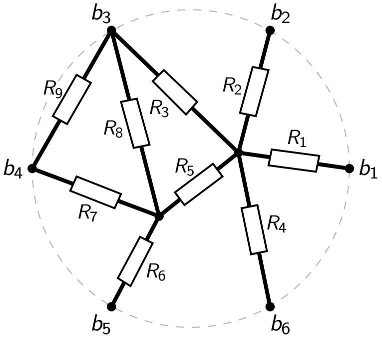

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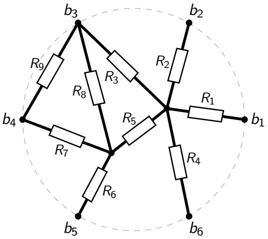

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Theorem (G.–Karp–Lam (2017))

$\overline{E}_n$ is homeomorphic to an $\binom{n}{2}$-dimensional closed ball

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Problem

Construct a stratification-preserving homeomorphism between $\overline{\mathcal{X}}_n$ and $\overline{E}_n$.

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