

206B (TOTAL POSITIVITY): HOMEWORK #2

- The homework is due BY EMAIL on *Wednesday, March 11th* (this time we won't have any presentations). Late homework will be penalized 20% per each late day.
- Submit **at most 3 problems** out of the below list. Additionally, you can prove any statement that I left as an exercise in class but did not include in the list.
- Each non-open problem is worth 10 points, so 30 is the maximal possible score. Solving (or making substantial progress on) an open problem is worth 100 points and results in an automatic A+ for the class. If a problem has multiple parts, all parts are required unless stated otherwise.
- Feel free to collaborate and use any sources. For each problem, all sources and collaborators must be clearly listed.
- Please use L^AT_EX. Pictures can be drawn by hand and scanned.

Given $\mathcal{M} \subset \binom{[n]}{k}$, consider the following properties:

- (1) For all $I, J \in \mathcal{M}$ and $i \in I$, there exists $j \in J$ such that $(I \setminus \{i\}) \cup \{j\} \in \mathcal{M}$.
- (2) For all $I, J \in \mathcal{M}$ and $i \in I$, there exists $j \in J$ such that both $(I \setminus \{i\}) \cup \{j\}$ and $(J \setminus \{j\}) \cup \{i\}$ belong to \mathcal{M} .
- (3) For all $I, J \in \mathcal{M}$, $r \geq 1$, and $i_1, \dots, i_r \in I$, there exist $j_1, \dots, j_r \in J$ such that both $(I \setminus \{i_1, \dots, i_r\}) \cup \{j_1, \dots, j_r\}$ and $(J \setminus \{j_1, \dots, j_r\}) \cup \{i_1, \dots, i_r\}$ belong to \mathcal{M} .

Problem 1. Prove or disprove the equivalence of properties (1) and (2) above.

Problem 2 (Hard). Prove or disprove the equivalence of properties (1) and (3) or (2) and (3) above.

Problem 3. Check that the Fano matroid satisfies the exchange axiom but is not realizable over \mathbb{R} .

Problem 4. Check that the non-Pappus matroid satisfies the exchange axiom and but is not realizable over \mathbb{R} .

Problem 5. Let $\lambda \subseteq k \times (n - k)$ be a Young diagram inside a $k \times (n - k)$ rectangle. Show that the closure of the Schubert cell $\Omega_\lambda \subset \text{Gr}(k, n)$ is given by

$$\overline{\Omega_\lambda} = \bigsqcup_{\mu \subseteq \lambda} \Omega_\mu.$$

For the next few problems, we introduce the following embedding ϕ of $n \times n$ matrices inside $\text{Gr}(n, 2n)$. For $A = (a_{i,j})_{i,j \in [n]}$, let $B = (b_{i,j})_{i,j \in [n]}$ be defined by $b_{i,j} = (-1)^{n-i} a_{n+1-i,j}$, and let $\phi(A)$ be (the row span of) the $n \times 2n$ matrix $[\text{Id}_n \mid B]$, where Id_n is the $n \times n$ identity matrix. For example, for $n = 2, 3$, ϕ sends

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & -a_{21} & -a_{22} \\ 0 & 1 & a_{11} & a_{12} \end{pmatrix}, \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & a_{31} & a_{32} & a_{33} \\ 0 & 1 & 0 & -a_{21} & -a_{22} & -a_{23} \\ 0 & 0 & 1 & a_{11} & a_{12} & a_{13} \end{pmatrix}.$$

Recall that $\text{Gr}_{>0}(k, n) = \{V \in \text{Gr}(k, n) \mid \Delta_I(V) > 0 \text{ for all } I\}$ and let $(\text{GL}_n(R))_{\geq 0} = \{A \in \text{GL}_n(\mathbb{R}) \mid A \text{ is TNN}\}$.

Problem 6.

- (i) Show that A is TNN if and only if $\phi(A) \in \text{Gr}_{\geq 0}(n, 2n)$.
- (ii) Show that A is TP if and only if $\phi(A) \in \text{Gr}_{> 0}(n, 2n)$.

Problem 7. Describe the image of $(\text{GL}_n(R))_{\geq 0}$ under ϕ as a union of positroid cells inside $\text{Gr}_{\geq 0}(n, 2n)$. (Hint: see next problem.)

Problem 8. Recall that the $(n!)^2$ cells of $(\text{GL}_n(R))_{\geq 0}$ correspond to pairs (v, w) of permutations. Recall also that a double reduced word for (v, w) is just a shuffle of a reduced word for v and a reduced word for w . Find a simple procedure for converting the corresponding double wiring diagram into a plabic graph G so that the strands of v and w in the double wiring diagram would correspond to the strands inside G .

Because $\mathbb{R} \subset \mathbb{C}$, we naturally have $\text{Gr}_{\mathbb{R}}(k, n) \subset \text{Gr}_{\mathbb{C}}(k, n)$ consisting of those subspaces $V \subset \mathbb{C}^n$ contained inside $\mathbb{R}^n \subset \mathbb{C}^n$. Also, recall that the cyclic shift map $\sigma : \text{Gr}(k, n) \rightarrow \text{Gr}(k, n)$ takes a matrix with columns (u_1, \dots, u_n) to a matrix with columns $((-1)^{k-1}u_n, u_1, \dots, u_{n-1})$.

Problem 9.

- (i) Suppose that $V \in \text{Gr}_{\mathbb{C}}(k, n)$ is the row span of a complex $k \times n$ matrix M . Find a necessary and sufficient condition for V to belong to $\text{Gr}_{\mathbb{R}}(k, n)$ in terms of the maximal minors of M .
- (ii) Find all *cyclically symmetric points* inside $\text{Gr}_{\mathbb{C}}(k, n)$, i.e., points $V \in \text{Gr}_{\mathbb{C}}(k, n)$ such that $\sigma(V) = V$.
- (iii) Which of these points belong to $\text{Gr}_{\mathbb{R}}(k, n)$?

Problem 10. Find the number of d -dimensional cells in $\text{Gr}_{\geq 0}(2, n)$ for all n .

Recall that $\mathcal{B}(k, n)$ denotes the set of bounded affine permutations and $\ell : \mathcal{B}(k, n) \rightarrow \mathbb{Z}$ denotes the length of an affine permutation:

$$\ell(f) = \#\{i \in [n], j \in \mathbb{Z} \mid i < j \text{ and } f(i) > f(j)\}.$$

Problem 11. Show that $\mathcal{B}(k, n)$ is a finite set and show that it has $\binom{n}{k}$ elements of maximal length.

For $i \in \mathbb{Z}$, let $s_i : \mathbb{Z} \rightarrow \mathbb{Z}$ be the affine permutation given by

$$s_i(j) = \begin{cases} j + 1, & \text{if } j \equiv i \pmod{n} \\ j - 1, & \text{if } j \equiv i + 1 \pmod{n} \\ j, & \text{otherwise.} \end{cases}$$

Recall that \tilde{S}_n^0 is the group of affine permutations satisfying $\sum_{i=1}^n (f(i) - i) = 0$.

Problem 12. (i) Show that the group \tilde{S}_n^0 is generated by s_0, s_1, \dots, s_{n-1} .

- (ii) Given an affine permutation $f \in \tilde{S}_n^0$ written as a product $f = s_{i_1} \cdots s_{i_r}$ (for $i_1, \dots, i_r \in \{0, 1, \dots, n-1\}$), show that the minimal number r of terms in the right hand side is given by $\ell(f)$.

For an element $V \in \text{Gr}(k, n)$ represented by a matrix with columns u_1, \dots, u_n , we have defined u_i for $i \in \mathbb{Z}$ by the condition that $u_{i+n} = (-1)^{k-1}u_i$. Recall that $f_V : \mathbb{Z} \rightarrow \mathbb{Z}$ is given by

$$f_V(i) = \min\{j \geq i \mid u_i \in \text{Span}(u_{i+1}, \dots, u_j)\}.$$

Problem 13. Show that $\sum_{i=1}^n (f_V(i) - i) = kn$.

For a linear subspace $V \subset \mathbb{R}^n$, we denote by V^\perp its orthogonal complement with respect to the standard scalar product on \mathbb{R}^n .

Problem 14.

- (i) Express the Plücker coordinates of $V^\perp \in \text{Gr}(n-k, n)$ in terms of that of $V \in \text{Gr}(k, n)$.
- (ii) Express f_{V^\perp} in terms of f_V .

Problem 15. For a perfectly oriented plabic network (N, O) , we have defined the point $X(N, O) \in \text{Gr}_{\geq 0}(k, n)$ given in terms of flows in (N, O) . We have also defined the point $X(N) \in \text{Gr}_{\geq 0}(k, n)$ given in terms of almost perfect matchings in N . Explain the relationship between $X(N, O)$ and $X(N)$. (Hint: they are *not* equal as elements of $\text{Gr}_{\geq 0}(k, n)$.)

Let $w \in S_n$ be a permutation. A *bridge decomposition* of w is an expression $w = t_{i_1, j_1} t_{i_2, j_2} \cdots t_{i_r, j_r}$ of w as a product of transpositions such that for any $l \in [r]$ and any $p \in [n]$ such that $i_l < p < j_l$, we have $p \notin \{i_{l+1}, j_{l+1}, \dots, i_r, j_r\}$. (In other words, once an element p falls strictly between i_l and j_l , you are not allowed to use it as an index in the terms that come after t_{i_l, j_l} .) The *bridge length* of w is the minimal length of a bridge decomposition of w . In particular, it is bounded from above by $\ell(w)$.

Problem 16. Show that the bridge length of w equals that of w^{-1} .

Problem 17 (Open, 100 points). Solve the previous problem without using plabic graphs or anything we have learned about the TNN Grassmannian.

Problem 18 (Open, 100 points). Let $n = 2k$ and consider a reduced plabic graph G for the top cell $\text{Gr}_{>0}(k, n)$. Let G' be the plabic graph obtained from G by changing the colors of all vertices (white \leftrightarrow black) and rotating the whole graph by 180 degrees. Show that the minimal number of square moves connecting G to G' equals $\frac{1}{2}k(k-1)^2$. (Hint: you can try to play with it using [this link](#). There is a button that says “complement” which corresponds to the operation described above.)