206B (TOTAL POSITIVITY): HOMEWORK #1

• The homework is due in class on *Monday, February 10th.* Late homework will be penalized 20% per each late day.
• Submit **at most 3 problems** out of the below list. Additionally, you can prove any statement that I left as an exercise in class but did not include in the list.
• Each non-open problem is worth 10 points, so 30 is the maximal possible score. Solving (or making substantial progress on) an open problem is worth 100 points and results in an automatic A+ for the class. If a problem has multiple parts, all parts are required unless stated otherwise.
• Feel free to collaborate and use any sources. For each problem, all sources and collaborators must be clearly listed.
• Please use LaTeX. Pictures can be drawn by hand and scanned.

Problem 1. Let B_n denote the polytope of *doubly stochastic $n \times n$ matrices*: these are matrices with nonnegative entries such that all row and column sums are equal to 1:

$$B_n := \{(a_{i,j})_{i,j=1}^n \mid a_{i,1}+a_{i,2}+\cdots+a_{i,n} = a_{1,j}+a_{2,j}+\cdots+a_{n,j} = 1 \text{ and } a_{i,j} \geq 0 \text{ for all } i,j \in [n]\}.$$

(a) Find the dimension of B_n.
(b) Find the number of vertices of B_n.
(c) Describe the edges of B_n.

Problem 2. Prove the Cauchy–Binet formula: for $n \times n$ matrices A and B and all $I, J \subset [n]$ with $|I| = |J| = k$, we have

$$\Delta_{I,J}(AB) = \sum_{K \in \binom{[n]}{k}} \Delta_{I,K}(A)\Delta_{K,J}(B).$$

Problem 3 (Hard)**.** Recall that $\leq$ denotes the Bruhat order on S_n.[1] For $a, b \in [n]$ and $w \in S_n$, let $r_{a,b}(w) := |\{i \leq a \mid w(i) \geq b\}|$. Show that for $v, w \in S_n$, we have $v \leq w$ if and only if $r_{a,b}(v) \leq r_{a,b}(w)$ for all $a, b \in [n]$.

Problem 4 (Very hard)**.** Fix $n \geq 2$ and let $w_0 \in S_n$ be the longest element. Denote

$$M_n := \sum_{w_0 = s_{i_1}\cdots s_{i_r}} i_1 \cdots i_r, \quad M'_n := \sum_{\mathrm{id} \xrightarrow{t_{i_1,j_1}} v_1 \xrightarrow{t_{i_2,j_2}} \cdots \xrightarrow{t_{i_r,j_r}} w_0} (i_1 - j_1)(i_2 - j_2)\cdots(i_r - j_r),$$

where $r = \binom{n}{2}$ and $u \xrightarrow{t_{i,j}} v$ means $v = ut_{i,j}$ and $\ell(v) = \ell(u) + 1$. Prove any of the following (a bijective proof of either (b) or (c) is worth 100 points):

(a) $M_n = \binom{n}{2}!$;
(b) $M'_n = \binom{n}{2}!$;
(c) $M_n = M'_n$.

[1] You are allowed to use any of the four characterizations of the Bruhat order whose equivalence we proved in class.

Problem 5. Define the *Demazure product* $*$ on S_n: for $w \in S_n$ and $i \in [n-1]$, set

$$w * s_i := \begin{cases} ws_i, & \text{if } \ell(ws_i) > \ell(w) \\ w, & \text{otherwise.} \end{cases}$$

Show that for any sequence $i_1, \dots, i_r$ and any positive real numbers $t_1, \dots, t_r \in \mathbb{R}_{>0}$, we have

$$x_{i_1}(t_1) \cdots x_{i_r}(t_r) \in U^w_{>0}, \quad \text{where} \quad w = s_{i_1} * s_{i_2} * \cdots * s_{i_r}.$$

Problem 6. Let $w = s_{i_1} \cdots s_{i_r}$ be a reduced word, and let $A := x_{i_1}(t_1) \cdots x_{i_r}(t_r)$ for some $t_1, \dots, t_r \in \mathbb{R}_{>0}$.

(a) Express t_r in terms of minors of A.
(b) Deduce that the map $(\mathbb{R}_{>0})^r \to U^w_{>0}$ sending $(t_1, \dots t_r) \mapsto x_{i_1}(t_1) \cdots x_{i_r}(t_r)$ is injective.

Problem 7. Let $A = (a_{i,j})_{i,j=1}^n$ be given by $A = x_{i_1}(t_1) \cdots x_{i_r}(t_r)$ for some $i_1, \dots, i_r \in [n-1]$ and $t_1, \dots, t_r \in \mathbb{R}_{>0}$. Choose $t > 0$ and define another matrix $B = (b_{i,j})_{i,j=1}^n$ by $b_{i,j} := a_{i,j} t^{j-i}$ for all i, j. Show that $B = x_{i_1}(t'_1) \cdots x_{i_r}(t'_r)$ for some $t'_1, \dots, t'_r \in \mathbb{R}_{>0}$ and express $t'_1, \dots, t'_r$ in terms of $i_1, \dots, i_r$ and $t_1, \dots, t_r$.

Problem 8. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ be a partition with $\lambda_1 + \cdots + \lambda_n = n$.

(a) Show that the number $\#\,\mathrm{SSYT}_n(\lambda)$ of semistandard Young tableaux of shape λ with entries in $[n]$ equals

$$\#\,\mathrm{SSYT}_n(\lambda) = \det(a_{i,j})_{i,j=1}^n, \quad \text{where} \quad a_{i,j} = \left(\!\!\binom{\lambda_i + j - i}{n}\!\!\right).$$

Here $\left(\!\binom{m}{k}\!\right) := \binom{m+k-1}{k-1}$ means "m multichoose k".

(b) More generally, let $h_k(x_1, \dots, x_n) := \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}$ be the *complete homogeneous symmetric polynomial*, and recall that $s_\lambda(x_1, \dots, x_n) := \sum_{T \in \mathrm{SSYT}_n(\lambda)} x^T$ is the Schur polynomial. Show that

$$s_\lambda(x_1, \dots, x_n) = \det(h_{\lambda_i + j - i}(x_1, \dots, x_n))_{i,j=1}^n,$$

where we set $h_k := 0$ for $k < 0$ and $h_0 = 1$.

Problem 9. Using the definition $s_\lambda(x_1, \dots, x_n) := \sum_{T \in \mathrm{SSYT}_n(\lambda)} x^T$, show that the Schur polynomial $s_\lambda(x_1, \dots, x_n)$ is symmetric in $x_1, \dots, x_n$, that is, for all $w \in S_n$, we have

$$s_\lambda(x_1, \dots, x_n) = s_\lambda(x_{w(1)}, \dots, x_{w(n)}).$$

For $T \in \mathrm{SSYT}_n(\lambda)$, recall that $x^T := \prod_{i=1}^n x_i^{\text{number of entries of } T \text{ equal to } i}$. The symmetric group acts on such monomials by permuting $x_1, \dots, x_n$, so for $u \in S_n$, let $u \cdot x_1^{a_1} \cdots x_n^{a_n} := x_{u(1)}^{a_1} \cdots x_{u(n)}^{a_n}$. The next problem asks to lift this action to an action on the set $\mathrm{SSYT}_n(\lambda)$.

Problem 10 (Hard)**.** For each $i \in [n-1]$, find a bijection $\mathcal{S}_i : \mathrm{SSYT}_n(\lambda) \to \mathrm{SSYT}_n(\lambda)$ satisfying the following requirements:

- for each $T \in \mathrm{SSYT}_n(\lambda)$, we have $s_i \cdot x^T = x^{\mathcal{S}_i(T)}$;
- the $\mathcal{S}_i$-s give rise to an action of S_n, i.e., satisfy

$$\mathcal{S}_i^2 = \mathrm{id} \quad (i \in [n-1]), \quad \mathcal{S}_i\mathcal{S}_{i+1}\mathcal{S}_i = \mathcal{S}_{i+1}\mathcal{S}_i\mathcal{S}_{i+1} \quad (i \in [n-2]), \quad \mathcal{S}_i\mathcal{S}_j = \mathcal{S}_j\mathcal{S}_i \quad (|i-j| > 1).$$

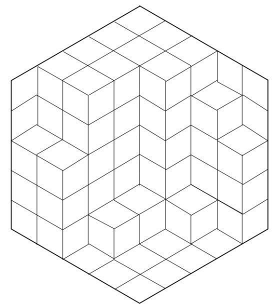


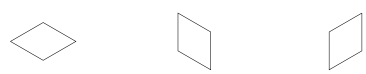
FIGURE 1. A lozenge tiling of $H_{5,5,5}$.

Problem 11. Let $a, b, c \geq 1$ be integers. An (a, b, c)-*plane partition* is an $a \times b$ matrix $P = (p_{i,j})$ with entries $p_{i,j} \in \{0, 1, \dots, c\}$ weakly increasing along rows and columns. For example,

$$\begin{pmatrix} 0 & 0 & 1 & 3 \\ 0 & 2 & 3 & 3 \\ 1 & 3 & 3 & 4 \end{pmatrix}$$

is an (a, b, c)-plane partition for $a = 3$, $b = 4$, $c = 4$.

Consider a hexagon $H_{a,b,c}$ with all angles $120°$ and edge lengths a, b, c, a, b, c in clockwise order. A *lozenge tiling* of $H_{a,b,c}$ is a tiling of $H_{a,b,c}$ using rhombi whose sides have unit length and are parallel to the sides of $H_{a,b,c}$: there are three types of rhombi available:



and an example of a lozenge tiling of $H_{5,5,5}$ is given in Figure 1.

(a) Give a bijection between (a, b, c)-plane partitions and lozenge tilings of $H_{a,b,c}$.

(b) Show that the number of (a, b, c)-plane partitions equals

$$\det (m_{i,j})_{i,j=1}^{c}, \quad \text{where} \quad m_{i,j} = \binom{a+b}{a+i-j}.$$

For example, the number of $(2, 2, 2)$-plane partitions equals $20 = \det \begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}$.

Problem 12. For a set $J \subset [n]$ of size r, denote $F_J := \Delta_{[r],J}$. Prove the following relation: for all $i < j < k$ and a set $S \subset [n]$ not containing i, j, k, we have

$$F_{Sik} F_{Sj} = F_{Sij} F_{Sk} + F_{Si} F_{Sjk},$$

where $Sik := S \sqcup \{i, k\}$, etc.

Problem 13. Recall that U and U^- denote the sets of upper (resp., lower) unitriangular $n \times n$ matrices, and T is the set of diagonal invertible matrices. Show that an $n \times n$ matrix A belongs to $U^- \cdot T \cdot U$ if and only if $\Delta_{[k],[k]}(A) \neq 0$ for $k = 1, 2, \dots, n$.

Problem 14. Let $v, w \in S_n$ and choose reduced words $v = s_{j_1} \cdots s_{j_p}$, $w = s_{i_1} \cdots s_{i_r}$. Recall that we have defined

$$G_{>0}^{v,w} := \{y_{j_1}(t'_1) \cdots y_{j_p}(t'_p) \cdot a \cdot x_{i_1}(t_1) \cdots x_{i_r}(t_r) \mid (t'_1, \dots, t'_p) \in (\mathbb{R}_{>0})^p,\ a \in T_{>0},\ (t_1, \dots, t_r) \in (\mathbb{R}_{>0})^r\}.$$

Describe $G_{>0}^{v,w}$ in terms of which minors are zero and which are positive.