The classification of Zamolodchikov periodic quivers

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Joint work with Pavlo Pylyavskyy

Part 1: Statement

The result

Theorem

Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1.
2.
3.
4.
5.

The result

Theorem

Let Q be a bipartite recurrent **quiver**. *Then the following are equivalent.*

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4.
5.

The result

Theorem

Let Q be a bipartite **recurrent** *quiver. Then the following are equivalent.*

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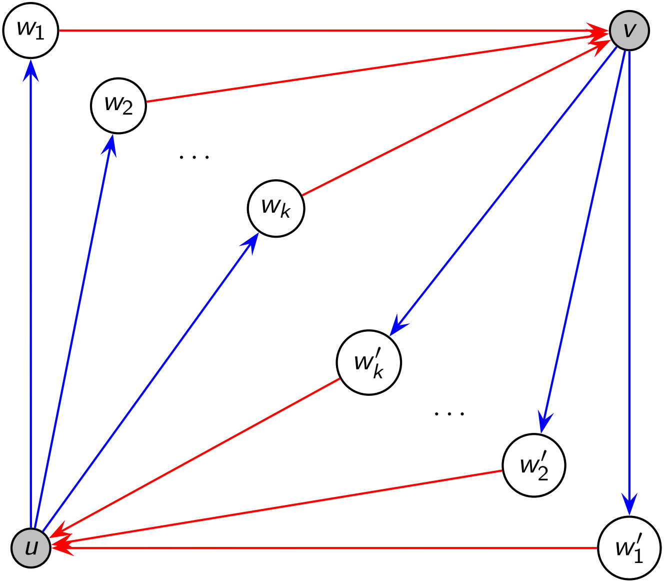
Recurrent quivers



The result

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Let Q be a bipartite recurrent quiver. Then the following are equivalent.

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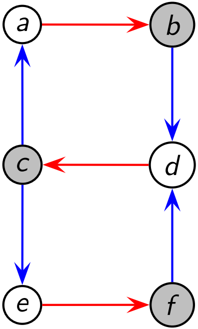
Theorem

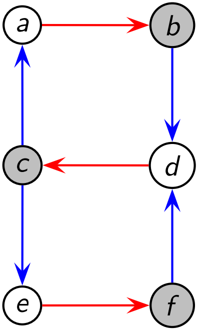
Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1.
2.
3.
4.
5. **The T-system associated with Q is periodic.**

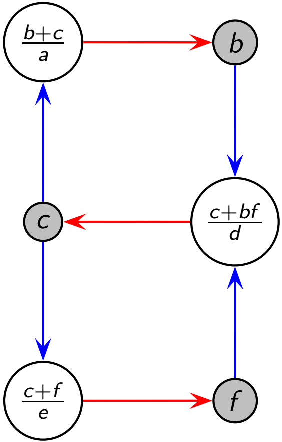
T-system



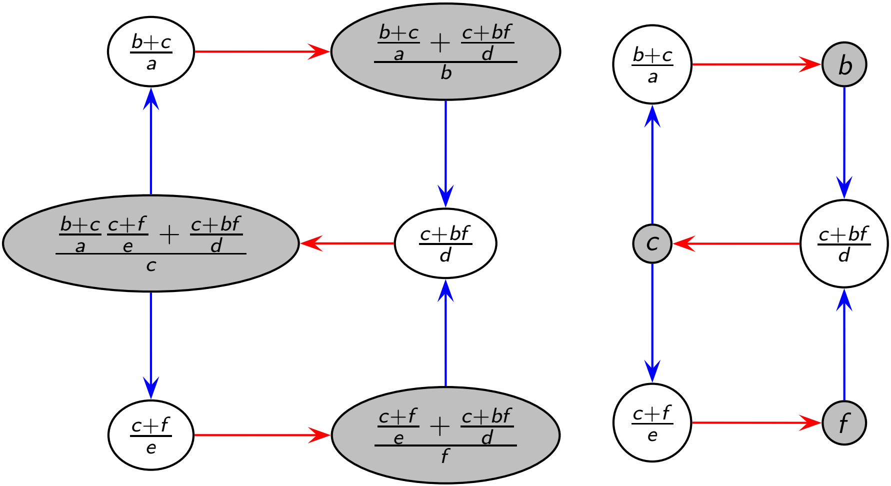
T-system



$\longrightarrow$

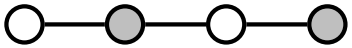
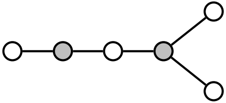
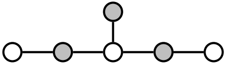
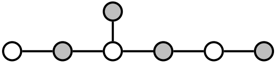
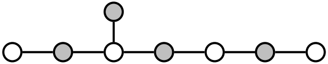


T-system

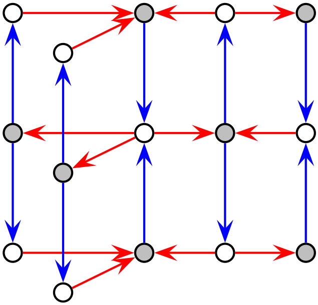


$\frac{b+c}{a}$
$\frac{\frac{b+c}{a}+\frac{c+bf}{d}}{b}$
$\frac{\frac{b+c}{a}\frac{c+f}{e}+\frac{c+bf}{d}}{c}$
$\frac{c+bf}{d}$
$\frac{c+f}{e}$
$\frac{\frac{c+f}{e}+\frac{c+bf}{d}}{f}$
$\frac{b+c}{a}$
b
c
$\frac{c+bf}{d}$
$\frac{c+f}{e}$
f

ADE Dynkin diagrams: bad definitions

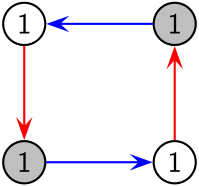
Name	**Picture**	**h**
A_n		$n+1$
D_n		$2n-2$
E_6		12
E_7		18
E_8		30

Tensor product



$D_5 \otimes A_3$

Example



The result

Theorem

Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1.
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3.
4.
5. *The T-system associated with Q is periodic.*

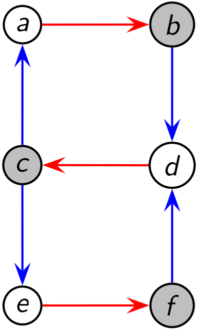
The result

Theorem

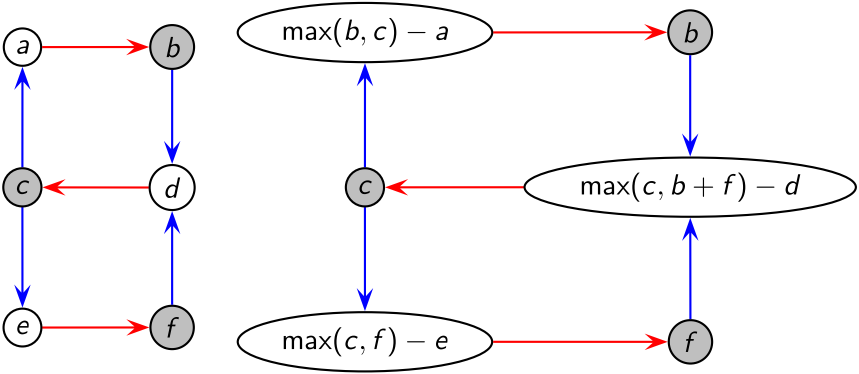
Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1.
2.
3.
4. **The tropical T-system is periodic for any initial value.**
5. *The T-system associated with Q is periodic.*

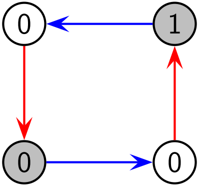
Tropical T-system



Tropical T-system



Example



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Let Q be a bipartite recurrent quiver. Then the following are equivalent.

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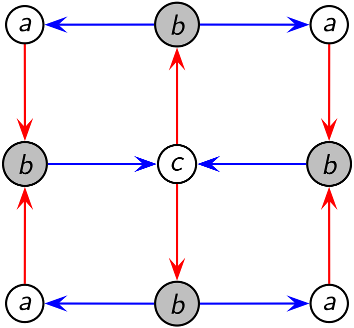
The result

Theorem

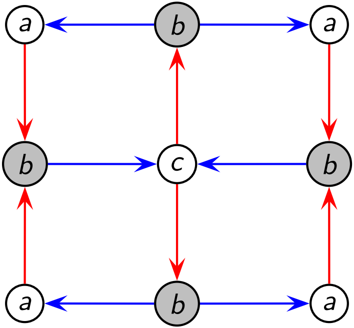
Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1.
2.
3. **Q has a fixed point.**
4. *The tropical T-system is periodic for any initial value.*
5. *The T-system associated with Q is periodic.*

Fixed point

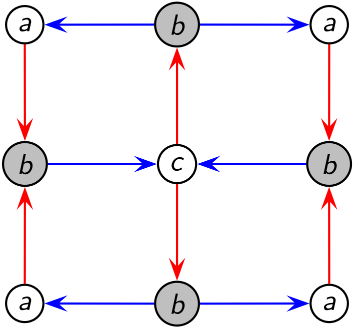


Fixed point



$$a^2 = b + b; \quad b^2 = a^2 + c; \quad c^2 = b^2 + b^2.$$

Fixed point



$$a^2 = b + b; \quad b^2 = a^2 + c; \quad c^2 = b^2 + b^2.$$

$$a = \sqrt{4 + 2\sqrt{2}}; \quad b = 2 + \sqrt{2}; \quad c = 2 + 2\sqrt{2}.$$

The result

Theorem

Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1.
2.
3. *Q has a fixed point.*
4. *The tropical T-system is periodic for any initial value.*
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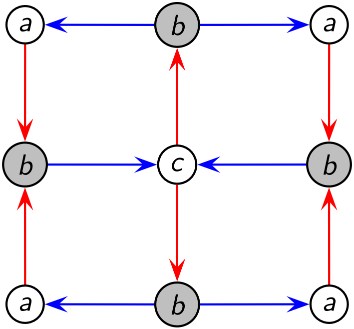
The result

Theorem

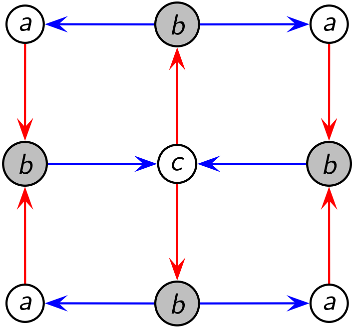
Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1.
2. **Q has a strictly subadditive labeling.**
3. *Q has a fixed point.*
4. *The tropical T-system is periodic for any initial value.*
5. *The T-system associated with Q is periodic.*

Strictly subadditive labeling

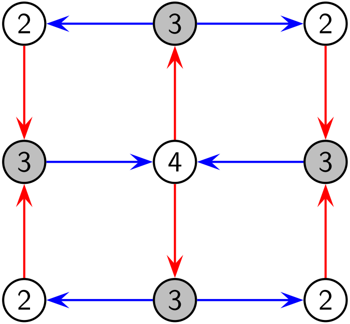


Strictly subadditive labeling



$$2a > \max(b, b); \quad 2b > \max(a + a, c); \quad 2c > \max(b + b, b + b).$$

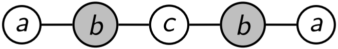
Strictly subadditive labeling



$$2a > \max(b, b); \quad 2b > \max(a + a, c); \quad 2c > \max(b + b, b + b).$$

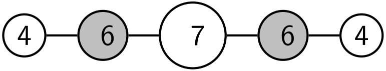
$$a = 2; \quad b = 3; \quad c = 4.$$

ADE Dynkin diagrams: good definitions



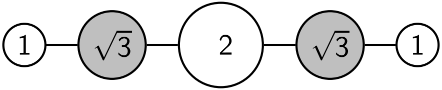
$$2a > b; \quad 2b > a + c; \quad 2c > b + b.$$

ADE Dynkin diagrams: good definitions



$$2a > b; \quad 2b > a + c; \quad 2c > b + b.$$

ADE Dynkin diagrams: good definitions



$$2a > b; \quad 2b > a + c; \quad 2c > b + b.$$

$$2a = \frac{b}{\cos(\pi/h)}; \quad 2b = \frac{a + c}{\cos(\pi/h)}; \quad 2c = \frac{b + b}{\cos(\pi/h)}.$$

The result

Theorem

Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1.
2. *Q has a strictly subadditive labeling.*
3. *Q has a fixed point.*
4. *The tropical T-system is periodic for any initial value.*
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The result

Theorem

Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1. **Q is an admissible ADE bigraph.**
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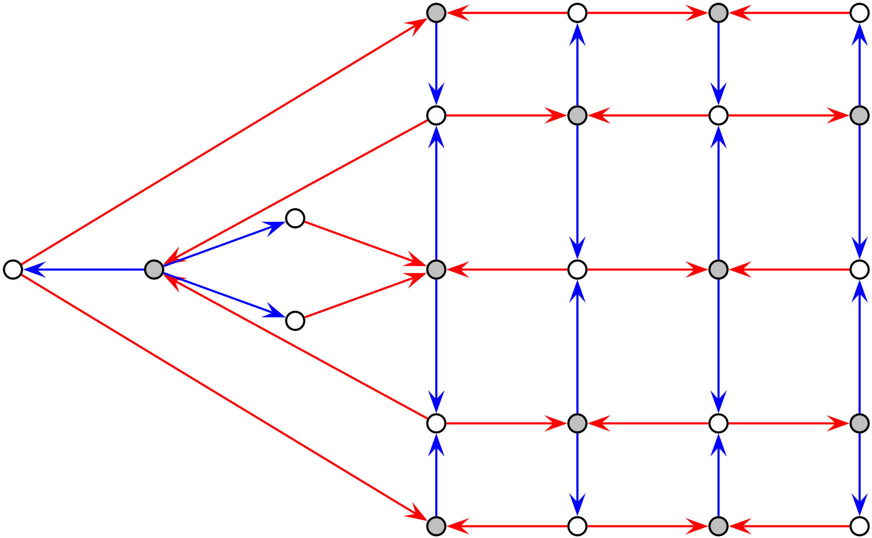
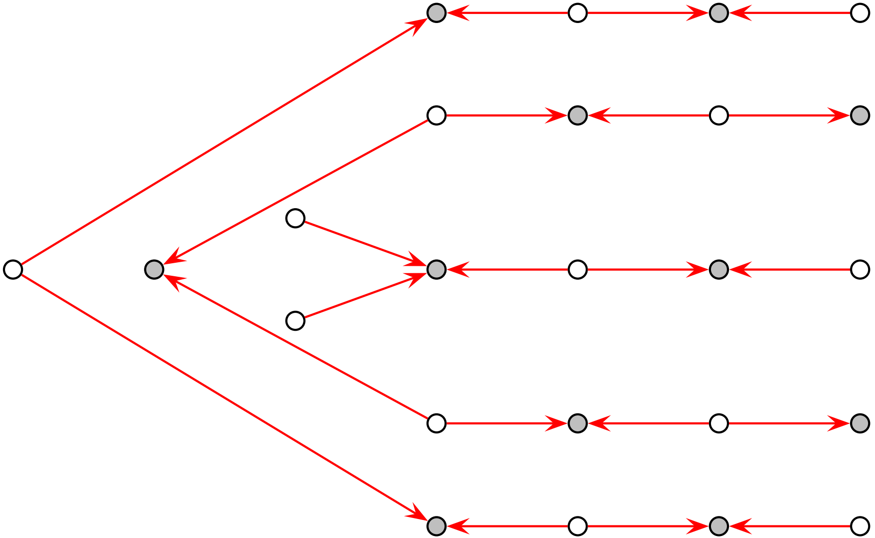
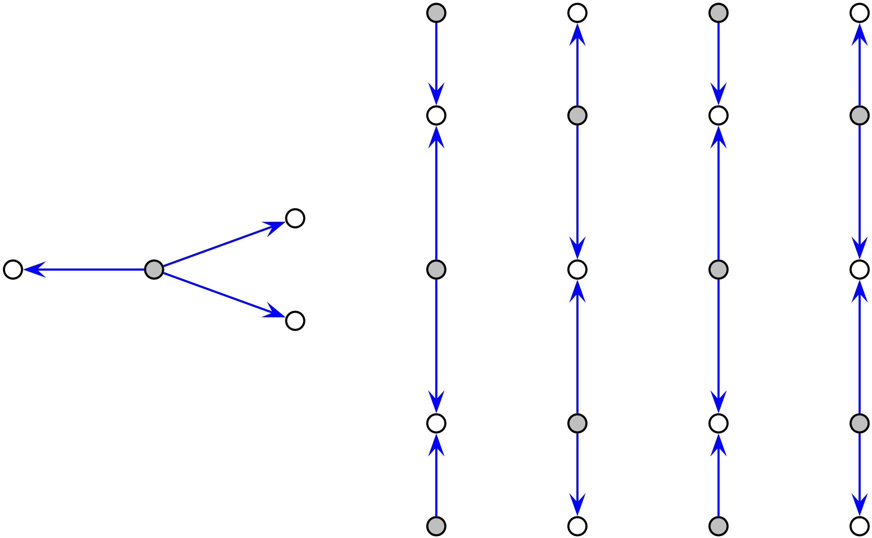
Admissible *ADE* bigraphs



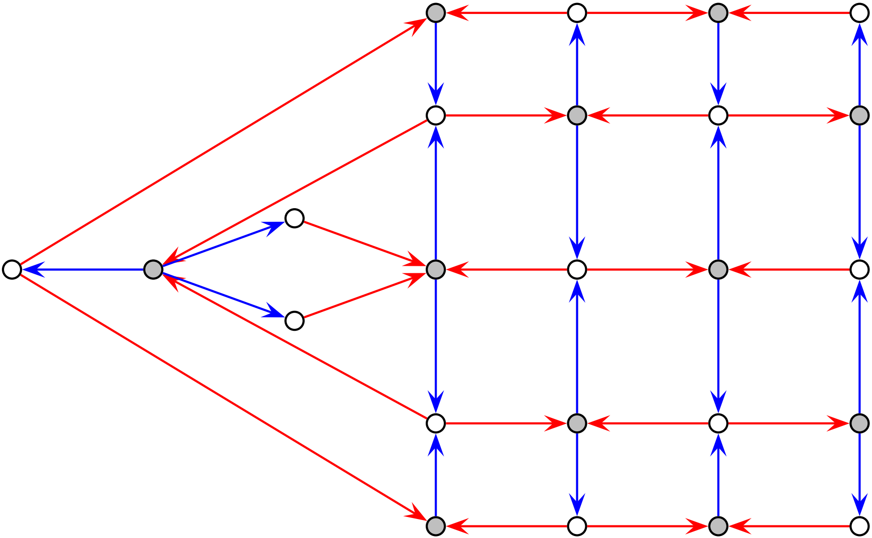
Admissible *ADE* bigraphs



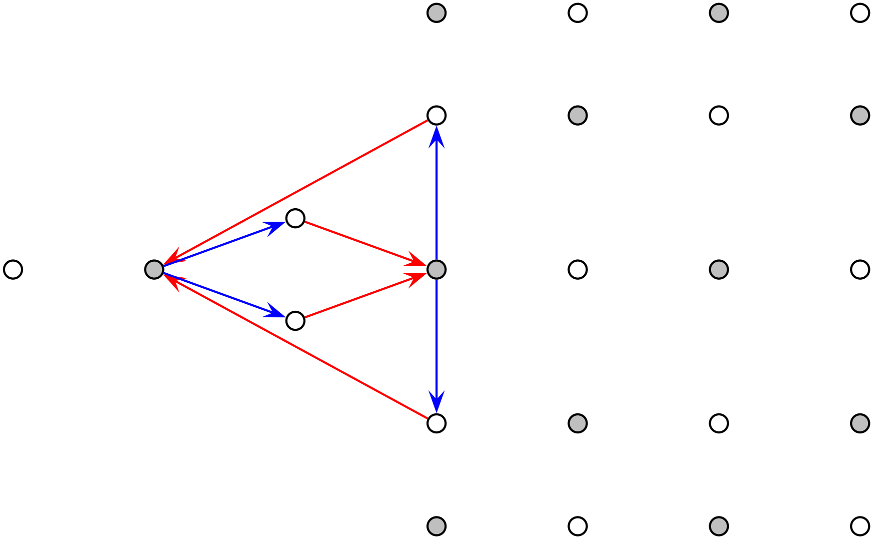
Admissible *ADE* bigraphs



Admissible ADE bigraphs



Admissible *ADE* bigraphs



The result

Theorem (G.-Pylyavskyy, 2016)

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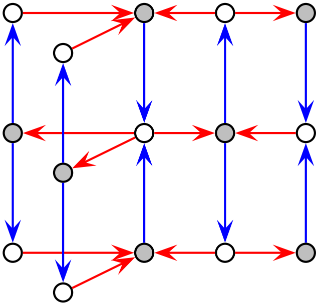
In all cases, both the T-system and its tropicalization have period dividing

$$h + h'.$$

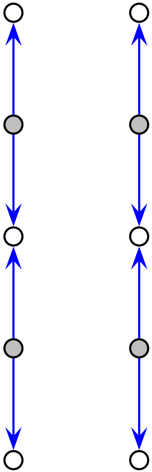
Part 2: Stembridge's Classification

Tensor products

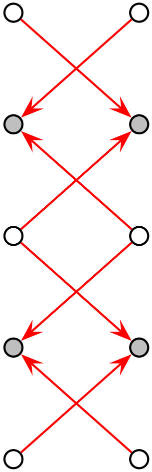


$D_5 \otimes A_3$

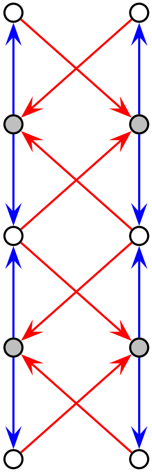
Twists



Twists

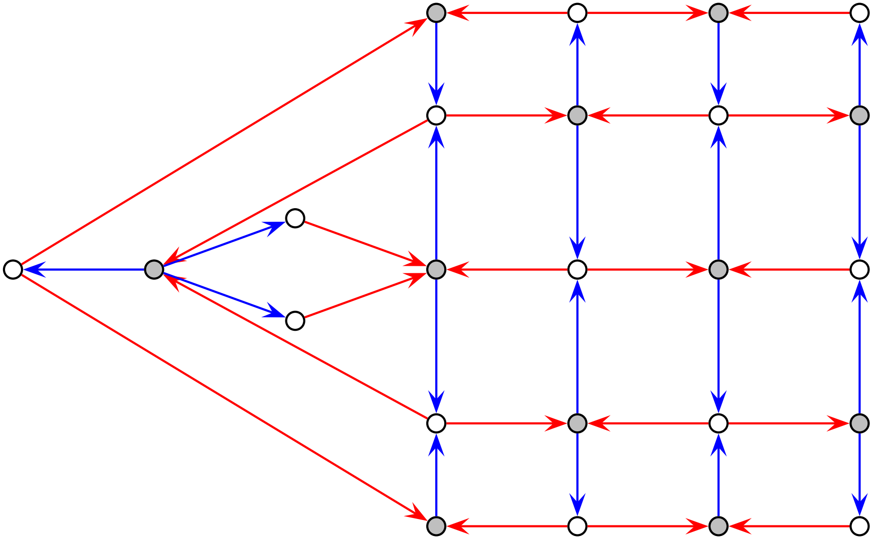


Twists

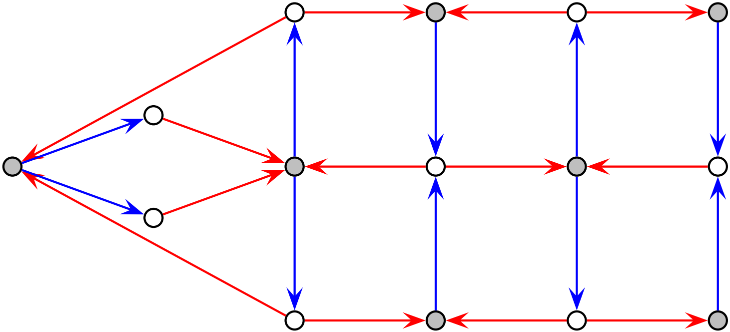


$A_5 \times A_5$.

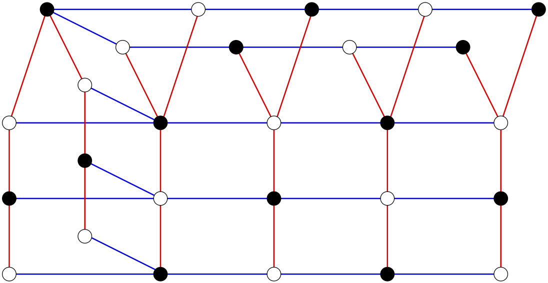
The case $(A^{m-1}D)_n$



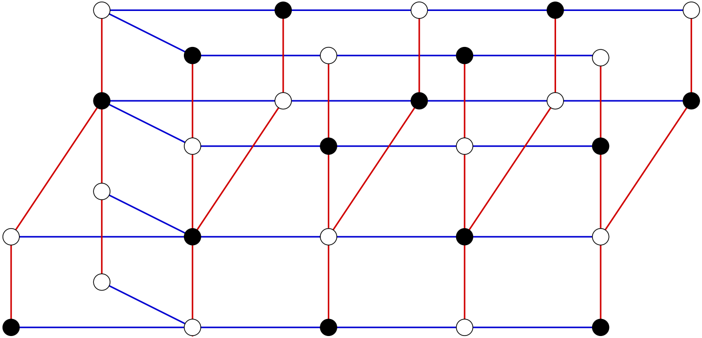
The case $(A^{m-1}D)_n$



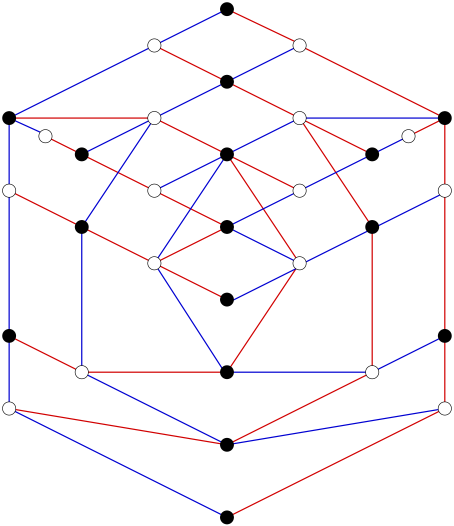
The case $(AD^{m-1})_n$



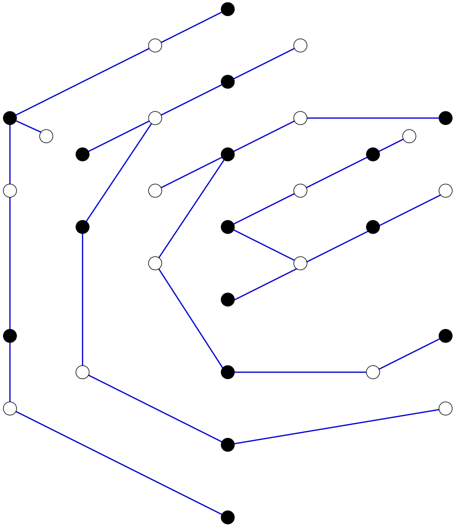
The case EE^n



11 exceptional bigraphs



11 exceptional bigraphs



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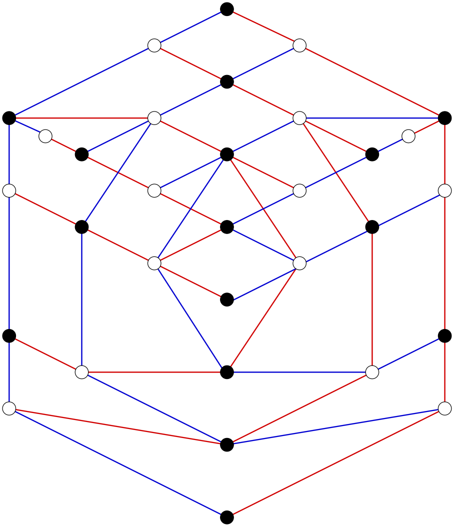
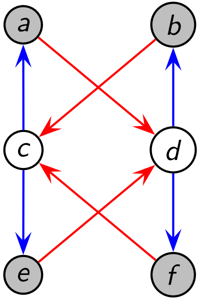
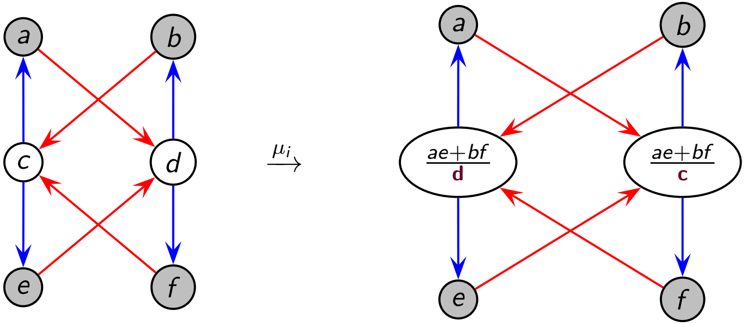


Part 3: Proof

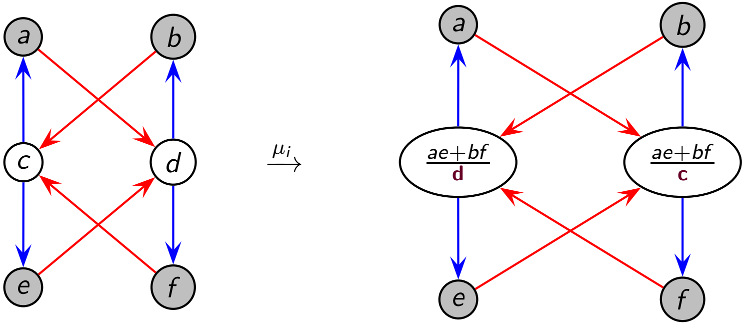
Twists



Twists

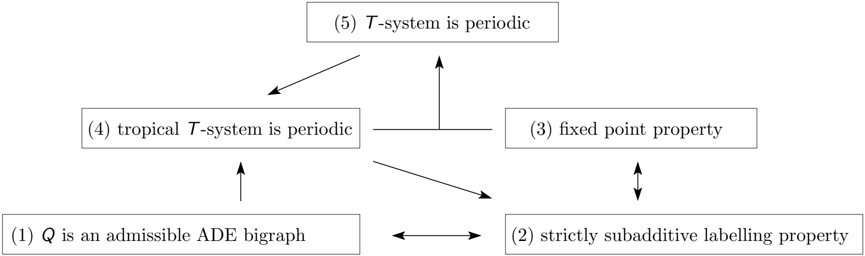

a
b
c
d
e
f
$\xrightarrow{\mu_i}$
a
b
$\frac{ae+bf}{\mathbf{d}}$
$\frac{ae+bf}{\mathbf{c}}$
e
f

Twists

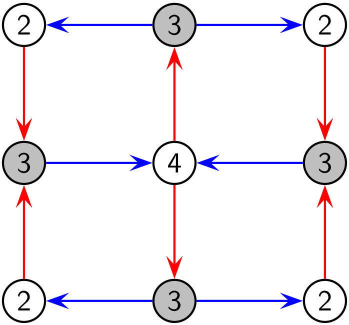


$$(\mu_i \mu_{i+1})^3 = id!$$

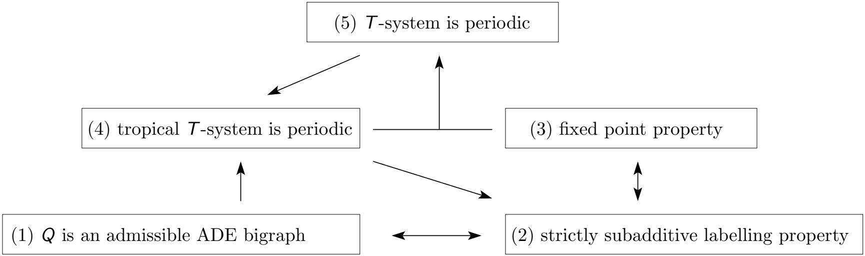
Plan of the proof



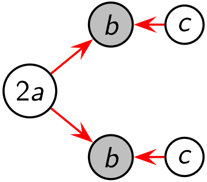
Subadditive function

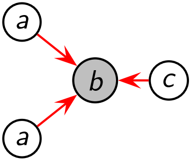


Plan of the proof



$A_{2n-1} \leftrightarrow D_{n+1}$ duality





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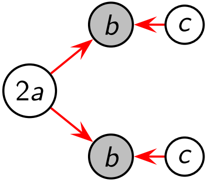
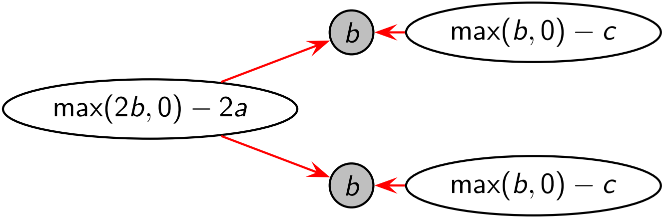
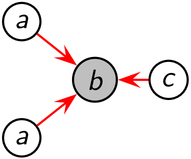
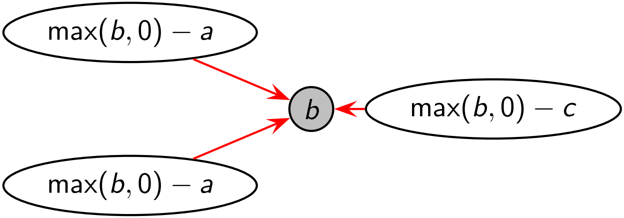
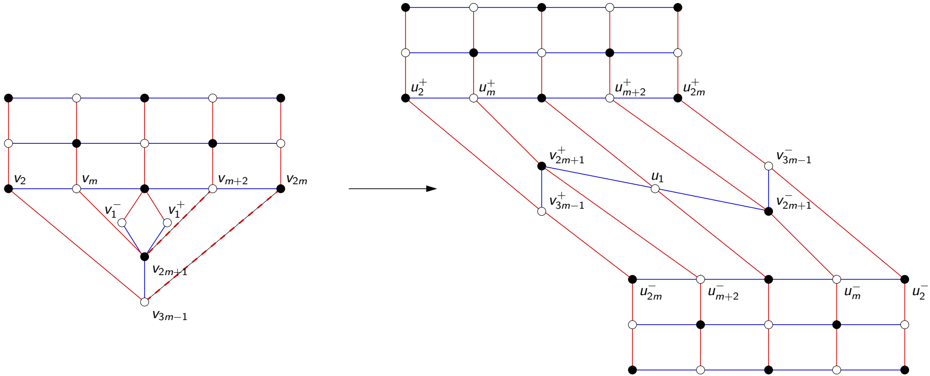








$(A^{m-1}D)_n \leftrightarrow A_{2n-1} \otimes D_{m+1}$ duality



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Thank you!