Ising model and total positivity

Pavel Galashin

MIT
galashin@mit.edu

University of Michigan, October 19, 2018

Joint work with Pavlo Pylyavskyy

arXiv:1807.03282

Part 1: Ising model

Ising model: definition

Definition

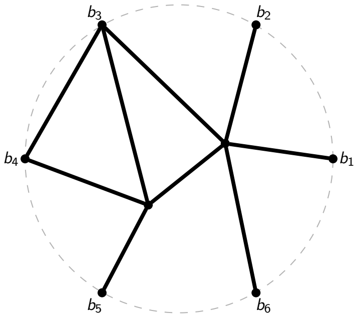
A planar Ising network is a pair (G, J) where:

Ising model: definition

Definition

A planar Ising network is a pair (G, J) where:

- $G = (V, E)$ is a planar graph embedded in a disk

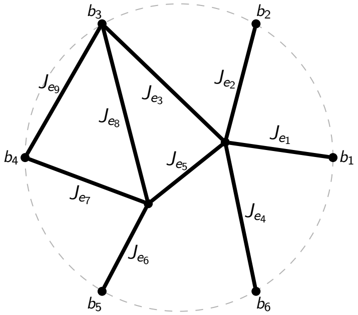


Ising model: definition

Definition

A planar Ising network is a pair (G, J) where:

- $G = (V, E)$ is a planar graph embedded in a disk
- $J \colon E \to \mathbb{R}_{>0}$ is a function

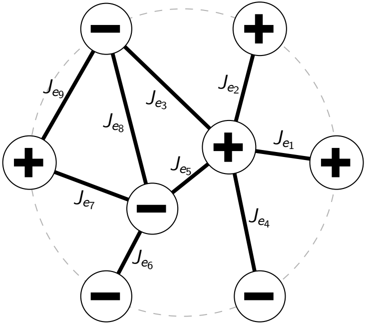


Ising model: definition

Definition

A planar Ising network is a pair (G, J) where:

- $G = (V, E)$ is a planar graph embedded in a disk
- J: $E \to \mathbb{R}_{>0}$ is a function



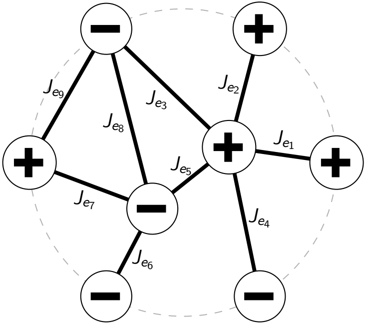
Spin configuration: a map $\sigma : V \to \{\pm 1\}$

Ising model: definition

Definition

A planar Ising network is a pair (G, J) where:

- $G = (V, E)$ is a planar graph embedded in a disk
- $J\colon E \to \mathbb{R}_{>0}$ is a function



Spin configuration: a map $\sigma : V \to \{\pm 1\}$

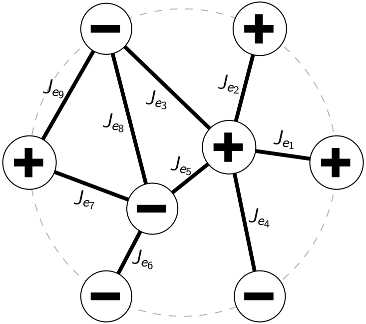
Ising model: probability measure on $\{\pm 1\}^V$

Ising model: definition

Definition

A planar Ising network is a pair (G, J) where:

- $G = (V, E)$ is a planar graph embedded in a disk
- J: $E \to \mathbb{R}_{>0}$ is a function



Spin configuration: a map $\sigma : V \to \{\pm 1\}$
Ising model: probability measure on $\{\pm 1\}^V$

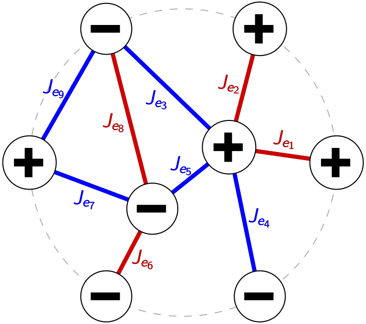
$$\mathrm{wt}(\sigma) := \prod_{\{u,v\} \in E} \exp\left(J_{\{u,v\}} \sigma_u \sigma_v\right)$$

Ising model: definition

Definition

A planar Ising network is a pair (G, J) where:

- $G = (V, E)$ is a planar graph embedded in a disk
- J: $E \to \mathbb{R}_{>0}$ is a function



Spin configuration: a map $\sigma : V \to \{\pm 1\}$
Ising model: probability measure on $\{\pm 1\}^V$

$$\mathrm{wt}(\sigma) := \prod_{\{u,v\} \in E} \exp\left(J_{\{u,v\}} \sigma_u \sigma_v\right)$$

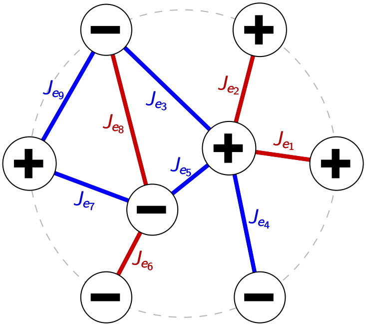
$$\mathrm{wt}(\sigma) = \frac{\exp\left(J_{e_1} + J_{e_2} + J_{e_6} + J_{e_8}\right)}{\exp\left(J_{e_3} + J_{e_4} + J_{e_5} + J_{e_7} + J_{e_9}\right)}$$

Ising model: definition

Definition

A planar Ising network is a pair (G, J) where:

- $G = (V, E)$ is a planar graph embedded in a disk
- J: $E \to \mathbb{R}_{>0}$ is a function



Spin configuration: a map $\sigma : V \to \{\pm 1\}$
Ising model: probability measure on $\{\pm 1\}^V$

$$\mathrm{wt}(\sigma) := \prod_{\{u,v\} \in E} \exp\left(J_{\{u,v\}} \sigma_u \sigma_v\right)$$

Partition function: $Z := \sum_{\sigma \in \{\pm 1\}^V} \mathrm{wt}(\sigma)$

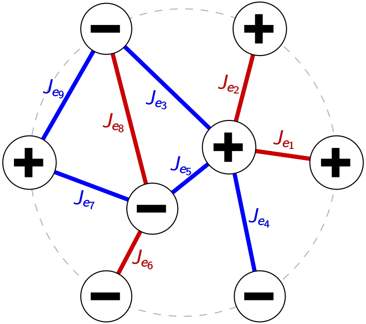
$$\mathrm{wt}(\sigma) = \frac{\exp\left(J_{e_1} + J_{e_2} + J_{e_6} + J_{e_8}\right)}{\exp\left(J_{e_3} + J_{e_4} + J_{e_5} + J_{e_7} + J_{e_9}\right)}$$

Ising model: definition

Definition

A planar Ising network is a pair (G, J) where:

- $G = (V, E)$ is a planar graph embedded in a disk
- J: $E \to \mathbb{R}_{>0}$ is a function



Spin configuration: a map $\sigma : V \to \{\pm 1\}$
Ising model: probability measure on $\{\pm 1\}^V$

$$\mathrm{wt}(\sigma) := \prod_{\{u,v\}\in E} \exp\left(J_{\{u,v\}}\sigma_u\sigma_v\right)$$

Partition function: $$Z := \sum_{\sigma\in\{\pm1\}^V} \mathrm{wt}(\sigma)$$

$$\mathrm{wt}(\sigma) = \frac{\exp\left(J_{e_1} + J_{e_2} + J_{e_6} + J_{e_8}\right)}{\exp\left(J_{e_3} + J_{e_4} + J_{e_5} + J_{e_7} + J_{e_9}\right)}$$

$$\mathrm{Prob}(\sigma) := \frac{\mathrm{wt}(\sigma)}{Z}$$

Ising model: correlation functions

Definition

For $u, v \in V$, their correlation is $\langle \sigma_u \sigma_v \rangle := \text{Prob}(\sigma_u = \sigma_v) - \text{Prob}(\sigma_u \neq \sigma_v)$.

Ising model: correlation functions

Definition

For $u, v \in V$, their correlation is $\langle \sigma_u \sigma_v \rangle := \mathrm{Prob}(\sigma_u = \sigma_v) - \mathrm{Prob}(\sigma_u \neq \sigma_v)$.

Theorem (Griffiths (1967))

For $u, v \in V$, we have $\langle \sigma_u \sigma_v \rangle \geq 0$.

Ising model: correlation functions

Definition

For $u, v \in V$, their correlation is $\langle \sigma_u \sigma_v \rangle := \mathrm{Prob}(\sigma_u = \sigma_v) - \mathrm{Prob}(\sigma_u \neq \sigma_v)$.

Theorem (Griffiths (1967))

For $u, v \in V$, we have $\langle \sigma_u \sigma_v \rangle \geq 0$.

Theorem (Kelly–Sherman (1968))

For $u, v, w \in V$, we have $\langle \sigma_u \sigma_w \rangle \geq \langle \sigma_u \sigma_v \rangle \cdot \langle \sigma_v \sigma_w \rangle$.

Ising model: correlation functions

Definition

For $u, v \in V$, their correlation is $\langle \sigma_u \sigma_v \rangle := \mathrm{Prob}(\sigma_u = \sigma_v) - \mathrm{Prob}(\sigma_u \neq \sigma_v)$.

Theorem (Griffiths (1967))

For $u, v \in V$, we have $\langle \sigma_u \sigma_v \rangle \geq 0$.

Theorem (Kelly–Sherman (1968))

For $u, v, w \in V$, we have $\langle \sigma_u \sigma_w \rangle \geq \langle \sigma_u \sigma_v \rangle \cdot \langle \sigma_v \sigma_w \rangle$.

Question (Kelly–Sherman (1968))

Describe correlations of the Ising model by inequalities.

Ising model: correlation functions

Definition

For $u, v \in V$, their correlation is $\langle \sigma_u \sigma_v \rangle := \text{Prob}(\sigma_u = \sigma_v) - \text{Prob}(\sigma_u \neq \sigma_v)$.

Theorem (Griffiths (1967))

For $u, v \in V$, we have $\langle \sigma_u \sigma_v \rangle \geq 0$.

Theorem (Kelly–Sherman (1968))

For $u, v, w \in V$, we have $\langle \sigma_u \sigma_w \rangle \geq \langle \sigma_u \sigma_v \rangle \cdot \langle \sigma_v \sigma_w \rangle$.

Question (Kelly–Sherman (1968))

Describe correlations of the Ising model by inequalities.

M. Lis (2017): more inequalities using objects from total positivity

Ising model: correlation functions

Definition

For $u, v \in V$, their correlation is $\langle \sigma_u \sigma_v \rangle := \text{Prob}(\sigma_u = \sigma_v) - \text{Prob}(\sigma_u \neq \sigma_v)$.

Theorem (Griffiths (1967))

For $u, v \in V$, we have $\langle \sigma_u \sigma_v \rangle \geq 0$.

Theorem (Kelly–Sherman (1968))

For $u, v, w \in V$, we have $\langle \sigma_u \sigma_w \rangle \geq \langle \sigma_u \sigma_v \rangle \cdot \langle \sigma_v \sigma_w \rangle$.

Question (Kelly–Sherman (1968))

Describe correlations of the Ising model by inequalities.

M. Lis (2017): more inequalities using objects from total positivity

Theorem (G.–Pylyavskyy (2018))

Describe boundary correlations of the planar Ising model by inequalities.

Ising model: history

Ising model: history

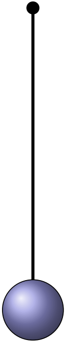
- Suggested by by W. Lenz to his student E. Ising in 1920

Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): not a good model for ferromagnetism

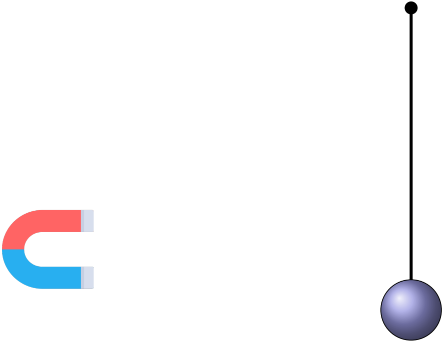
Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): not a good model for ferromagnetism



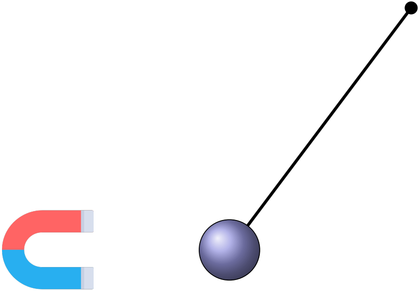
Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): not a good model for ferromagnetism



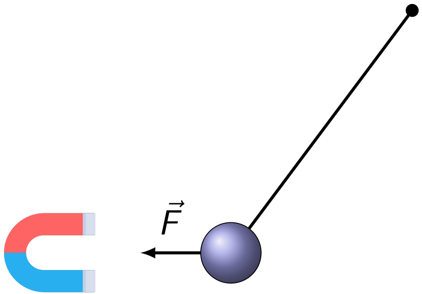
Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): not a good model for ferromagnetism



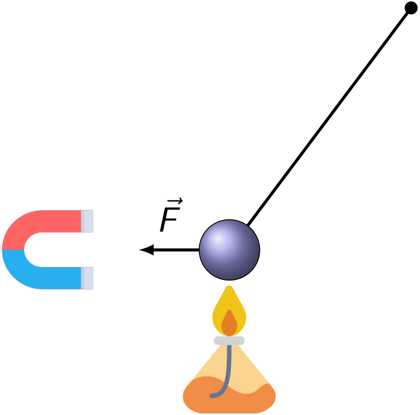
Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): not a good model for ferromagnetism



Ising model: history

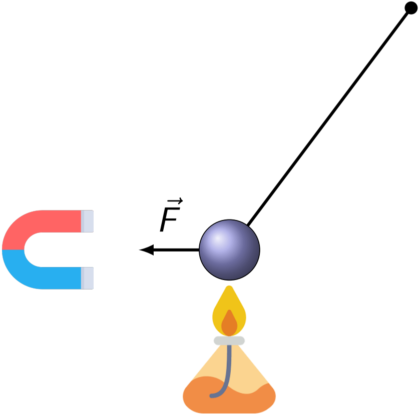
- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): not a good model for ferromagnetism



Ising model: history

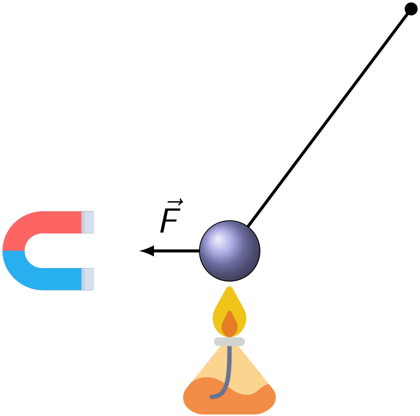
- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): not a good model for ferromagnetism

Q: how does $|\vec{F}|$ depend on T°?

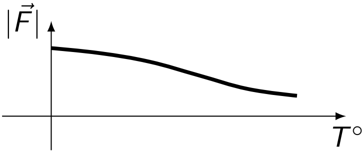


Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): not a good model for ferromagnetism

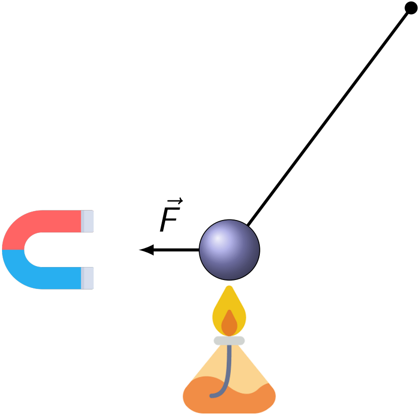


Q: how does $|\vec{F}|$ depend on T°?

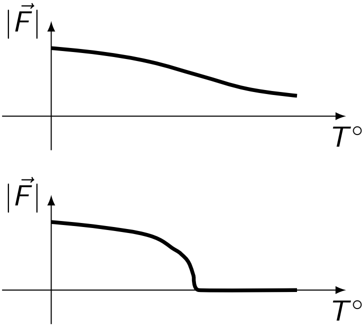


Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): not a good model for ferromagnetism

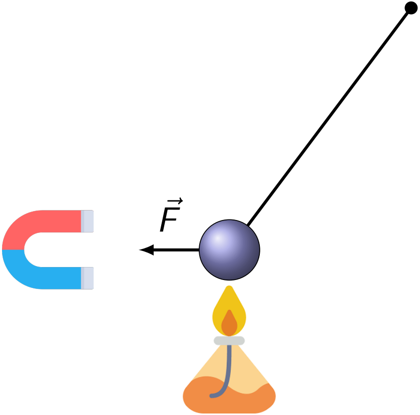


Q: how does $|\vec{F}|$ depend on T°?

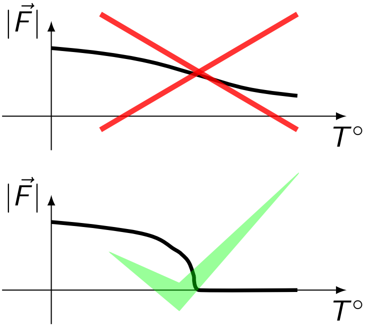


Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): not a good model for ferromagnetism

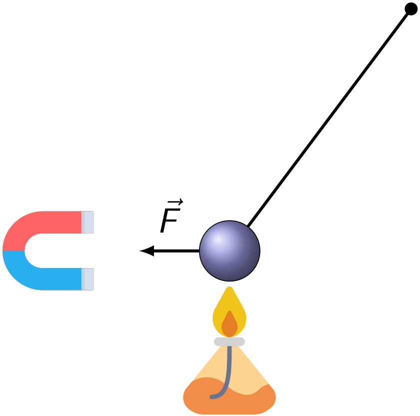


Q: how does $|\vec{F}|$ depend on T°?

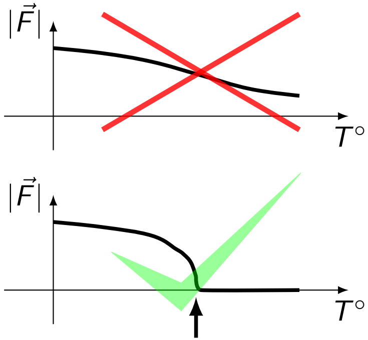


Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): not a good model for ferromagnetism



Q: how does $|\vec{F}|$ depend on T°?



Curie point (P. Curie, 1895)

Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): not a good model for ferromagnetism

Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): no phase transition in 1D $\Longrightarrow$ not a good model for ferromagnetism

Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): no phase transition in 1D $\Longrightarrow$ not a good model for ferromagnetism

Historically, we let $G := \mathbb{Z}^d \cap \Omega$ for some $\Omega \subset \mathbb{R}^d$ and set all $J_e := \frac{1}{T}$ for some temperature $T \in \mathbb{R}_{>0}$.

Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): no phase transition in 1D $\Longrightarrow$ not a good model for ferromagnetism

Historically, we let $G := \mathbb{Z}^d \cap \Omega$ for some $\Omega \subset \mathbb{R}^d$
and set all $J_e := \frac{1}{T}$ for some temperature $T \in \mathbb{R}_{>0}$.

- Peierls (1937): phase transition in $\mathbb{Z}^d$ for $d \geq 2$

Ising model: history

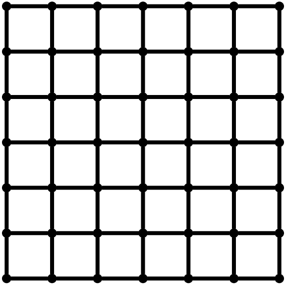
- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): no phase transition in 1D $\Longrightarrow$ not a good model for ferromagnetism

Historically, we let $G := \mathbb{Z}^d \cap \Omega$ for some $\Omega \subset \mathbb{R}^d$
and set all $J_e := \frac{1}{T}$ for some temperature $T \in \mathbb{R}_{>0}$.

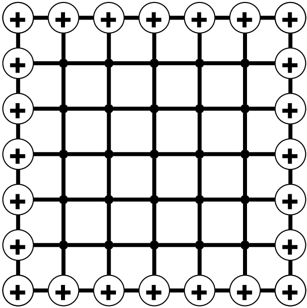
- Peierls (1937): phase transition in $\mathbb{Z}^d$ for $d \geq 2$
- Kramers–Wannier (1941): critical temperature $\frac{1}{T_c} = \frac{1}{2}\log\left(\sqrt{2}+1\right)$ for $\mathbb{Z}^2$

Ising model: phase transition

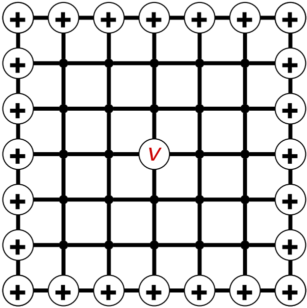
Let $G \subset \mathbb{Z}^d$ be a $(2N+1) \times (2N+1)$ square and $J_e = \frac{1}{T}$ for some fixed $T \in \mathbb{R}_{>0}$.



Let $G \subset \mathbb{Z}^d$ be a $(2N+1) \times (2N+1)$ square and $J_e = \frac{1}{T}$ for some fixed $T \in \mathbb{R}_{>0}$. Suppose that $\sigma_u = +1$ for all $u \in \partial G$.



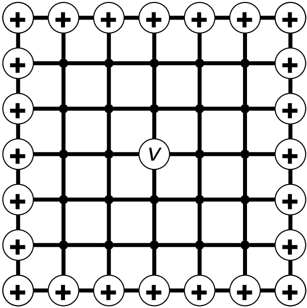
Let $G \subset \mathbb{Z}^d$ be a $(2N+1) \times (2N+1)$ square and $J_e = \frac{1}{T}$ for some fixed $T \in \mathbb{R}_{>0}$.
Suppose that $\sigma_u = +1$ for all $u \in \partial G$.
Let v be the vertex in the middle of the square.



Ising model: phase transition

Let $G \subset \mathbb{Z}^d$ be a $(2N+1) \times (2N+1)$ square and $J_e = \frac{1}{T}$ for some fixed $T \in \mathbb{R}_{>0}$.
Suppose that $\sigma_u = +1$ for all $u \in \partial G$.
Let v be the vertex in the middle of the square.
Define the spontaneous magnetization

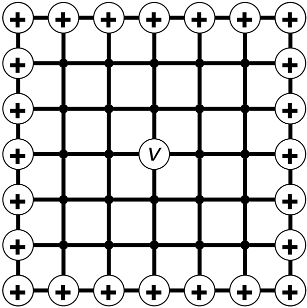
$$M(T) := \lim_{N \to \infty} \left(\text{Prob}(\sigma_v = +1) - \text{Prob}(\sigma_v = -1)\right)$$



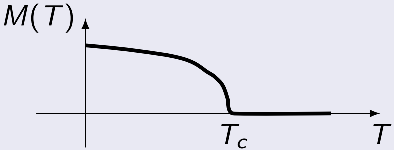
Ising model: phase transition

Let $G \subset \mathbb{Z}^d$ be a $(2N+1) \times (2N+1)$ square and $J_e = \frac{1}{T}$ for some fixed $T \in \mathbb{R}_{>0}$.
Suppose that $\sigma_u = +1$ for all $u \in \partial G$.
Let v be the vertex in the middle of the square.
Define the spontaneous magnetization

$$M(T) := \lim_{N \to \infty} (\mathrm{Prob}(\sigma_v = +1) - \mathrm{Prob}(\sigma_v = -1))$$

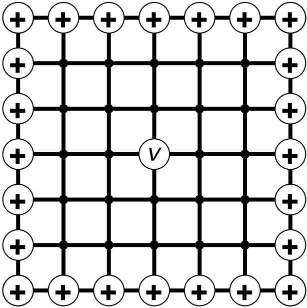


Theorem (Onsager (1944), Onsager–Kaufman (1949), Yang (1952))

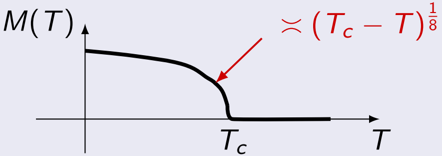


Ising model: phase transition

Let $G \subset \mathbb{Z}^d$ be a $(2N+1) \times (2N+1)$ square and $J_e = \frac{1}{T}$ for some fixed $T \in \mathbb{R}_{>0}$.
Suppose that $\sigma_u = +1$ for all $u \in \partial G$.
Let v be the vertex in the middle of the square.
Define the spontaneous magnetization
$M(T) := \lim_{N\to\infty} (\mathrm{Prob}(\sigma_v = +1) - \mathrm{Prob}(\sigma_v = -1))$



Theorem (Onsager (1944),Onsager–Kaufman (1949), Yang (1952))



Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): no phase transition in 1D $\Longrightarrow$ not a good model for ferromagnetism

Historically, we let $G := \mathbb{Z}^d \cap \Omega$ for some $\Omega \subset \mathbb{R}^d$ and set all $J_e := \frac{1}{T}$ for some temperature $T \in \mathbb{R}_{>0}$.

- Peierls (1937): phase transition in $\mathbb{Z}^d$ for $d \geq 2$
- Kramers–Wannier (1941): critical temperature $\frac{1}{T_c} = \frac{1}{2}\log\left(\sqrt{2}+1\right)$ for $\mathbb{Z}^2$

Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): no phase transition in 1D $\Longrightarrow$ not a good model for ferromagnetism

Historically, we let $G := \mathbb{Z}^d \cap \Omega$ for some $\Omega \subset \mathbb{R}^d$ and set all $J_e := \frac{1}{T}$ for some temperature $T \in \mathbb{R}_{>0}$.

- Peierls (1937): phase transition in $\mathbb{Z}^d$ for $d \geq 2$
- Kramers–Wannier (1941): critical temperature $\frac{1}{T_c} = \frac{1}{2}\log\left(\sqrt{2}+1\right)$ for $\mathbb{Z}^2$
- Onsager, Kaufman, Yang (1944–1952): exact expressions for the free energy and spontaneous magnetization

Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): no phase transition in 1D $\Longrightarrow$ not a good model for ferromagnetism

Historically, we let $G := \mathbb{Z}^d \cap \Omega$ for some $\Omega \subset \mathbb{R}^d$
and set all $J_e := \frac{1}{T}$ for some temperature $T \in \mathbb{R}_{>0}$.

- Peierls (1937): phase transition in $\mathbb{Z}^d$ for $d \geq 2$
- Kramers–Wannier (1941): critical temperature $\frac{1}{T_c} = \frac{1}{2} \log\left(\sqrt{2}+1\right)$ for $\mathbb{Z}^2$
- Onsager, Kaufman, Yang (1944–1952): exact expressions for the free energy and spontaneous magnetization
- Belavin–Polyakov–Zamolodchikov (1984): conjectured conformal invariance of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$

Ising model: history

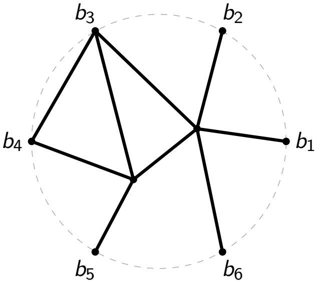
- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): no phase transition in 1D $\Longrightarrow$ not a good model for ferromagnetism

Historically, we let $G := \mathbb{Z}^d \cap \Omega$ for some $\Omega \subset \mathbb{R}^d$
and set all $J_e := \frac{1}{T}$ for some temperature $T \in \mathbb{R}_{>0}$.

- Peierls (1937): phase transition in $\mathbb{Z}^d$ for $d \geq 2$
- Kramers–Wannier (1941): critical temperature $\frac{1}{T_c} = \frac{1}{2}\log\left(\sqrt{2}+1\right)$ for $\mathbb{Z}^2$
- Onsager, Kaufman, Yang (1944–1952): exact expressions for the free energy and spontaneous magnetization
- Belavin–Polyakov–Zamolodchikov (1984): conjectured conformal invariance of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$
- Smirnov, Chelkak, Hongler, Izyurov, ... (2010–2015): proved conformal invariance and universality of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$

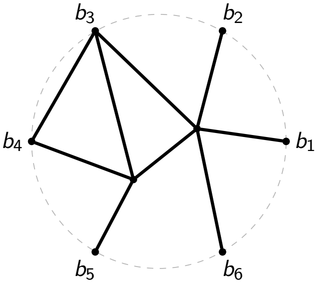
Ising model: boundary correlations

Recall: G is embedded in a disk. Let $b_1, \dots, b_n$ be the boundary vertices.



Ising model: boundary correlations

Recall: G is embedded in a disk. Let $b_1, \dots, b_n$ be the boundary vertices.
Correlation: $\langle \sigma_u \sigma_v \rangle := \text{Prob}(\sigma_u = \sigma_v) - \text{Prob}(\sigma_u \neq \sigma_v)$.

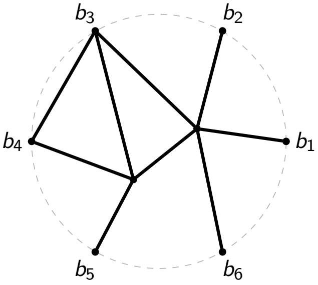


Ising model: boundary correlations

Recall: G is embedded in a disk. Let $b_1, \dots, b_n$ be the boundary vertices.
Correlation: $\langle \sigma_u \sigma_v \rangle := \text{Prob}(\sigma_u = \sigma_v) - \text{Prob}(\sigma_u \neq \sigma_v)$.

Definition

Boundary correlation matrix: $M(G, J) = (m_{ij})_{i,j=1}^n$, where $m_{ij} := \langle \sigma_{b_i} \sigma_{b_j} \rangle$.

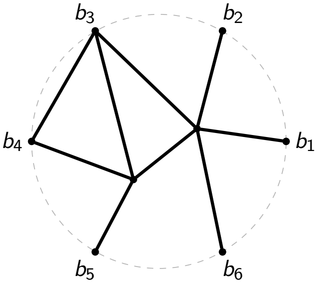


Ising model: boundary correlations

Recall: G is embedded in a disk. Let $b_1, \dots, b_n$ be the boundary vertices.
Correlation: $\langle \sigma_u \sigma_v \rangle := \text{Prob}(\sigma_u = \sigma_v) - \text{Prob}(\sigma_u \neq \sigma_v)$.

Definition

Boundary correlation matrix: $M(G, J) = (m_{ij})_{i,j=1}^n$, where $m_{ij} := \langle \sigma_{b_i} \sigma_{b_j} \rangle$.



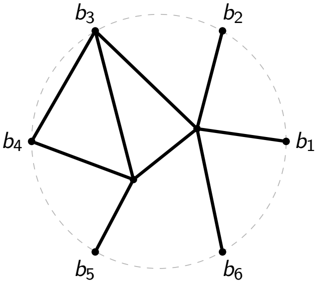
$M(G, J)$ is a symmetric matrix

Ising model: boundary correlations

Recall: G is embedded in a disk. Let $b_1, \ldots, b_n$ be the boundary vertices.
Correlation: $\langle \sigma_u \sigma_v \rangle := \text{Prob}(\sigma_u = \sigma_v) - \text{Prob}(\sigma_u \neq \sigma_v)$.

Definition

Boundary correlation matrix: $M(G, J) = (m_{ij})_{i,j=1}^{n}$, where $m_{ij} := \langle \sigma_{b_i} \sigma_{b_j} \rangle$.



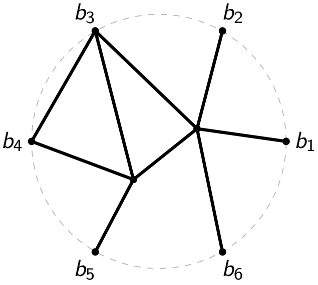
$M(G, J)$ is a symmetric matrix with 1's on the diagonal

Ising model: boundary correlations

Recall: G is embedded in a disk. Let $b_1, \dots, b_n$ be the boundary vertices.
Correlation: $\langle \sigma_u \sigma_v \rangle := \text{Prob}(\sigma_u = \sigma_v) - \text{Prob}(\sigma_u \neq \sigma_v)$.

Definition

Boundary correlation matrix: $M(G, J) = (m_{ij})_{i,j=1}^n$, where $m_{ij} := \langle \sigma_{b_i} \sigma_{b_j} \rangle$.



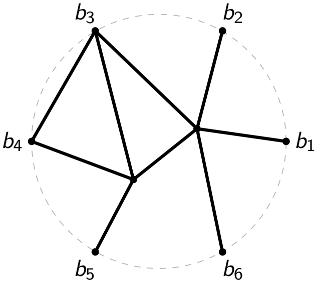
$M(G, J)$ is a symmetric matrix with 1's on the diagonal and nonnegative entries

Ising model: boundary correlations

Recall: G is embedded in a disk. Let $b_1, \dots, b_n$ be the boundary vertices.
Correlation: $\langle \sigma_u \sigma_v \rangle := \mathrm{Prob}(\sigma_u = \sigma_v) - \mathrm{Prob}(\sigma_u \neq \sigma_v)$.

Definition

Boundary correlation matrix: $M(G, J) = (m_{ij})_{i,j=1}^{n}$, where $m_{ij} := \langle \sigma_{b_i} \sigma_{b_j} \rangle$.



$M(G, J)$ is a symmetric matrix with 1's on the diagonal and nonnegative entries

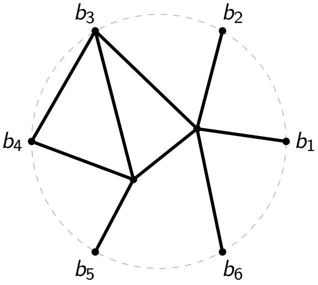
Lives inside $\mathbb{R}^{\binom{n}{2}}$

Ising model: boundary correlations

Recall: G is embedded in a disk. Let $b_1, \dots, b_n$ be the boundary vertices.
Correlation: $\langle \sigma_u \sigma_v \rangle := \text{Prob}(\sigma_u = \sigma_v) - \text{Prob}(\sigma_u \neq \sigma_v)$.

Definition

Boundary correlation matrix: $M(G,J) = (m_{ij})_{i,j=1}^n$, where $m_{ij} := \langle \sigma_{b_i} \sigma_{b_j} \rangle$.



$M(G,J)$ is a symmetric matrix with 1's on the diagonal and nonnegative entries

Lives inside $\mathbb{R}^{\binom{n}{2}}$

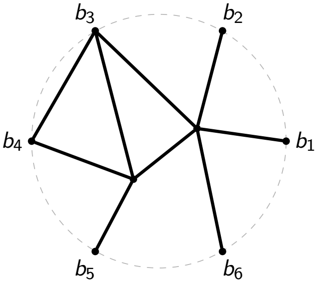
$\mathcal{X}_n := \{M(G,J) \mid (G,J) \text{ is a planar Ising network with } n \text{ boundary vertices}\}$

Ising model: boundary correlations

Recall: G is embedded in a disk. Let $b_1, \dots, b_n$ be the boundary vertices.
Correlation: $\langle \sigma_u \sigma_v \rangle := \mathrm{Prob}(\sigma_u = \sigma_v) - \mathrm{Prob}(\sigma_u \neq \sigma_v)$.

Definition

Boundary correlation matrix: $M(G, J) = (m_{ij})_{i,j=1}^n$, where $m_{ij} := \langle \sigma_{b_i} \sigma_{b_j} \rangle$.



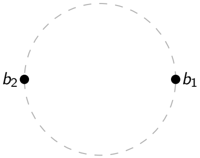
$M(G, J)$ is a symmetric matrix with 1's on the diagonal and nonnegative entries

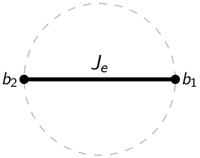
Lives inside $\mathbb{R}^{\binom{n}{2}}$

$\mathcal{X}_n := \{M(G, J) \mid (G, J) \text{ is a planar Ising network with } n \text{ boundary vertices}\}$

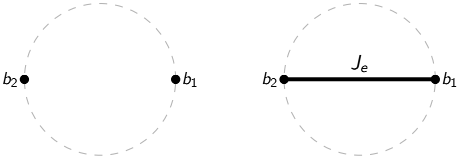
$\overline{\mathcal{X}}_n :=$ closure of $\mathcal{X}_n$ inside the space of $n \times n$ matrices.

Boundary correlations: an example for $n = 2$



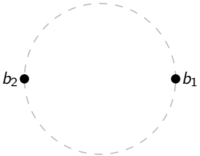


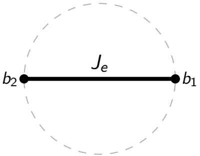
Boundary correlations: an example for $n = 2$



$$M(G,J) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}$$

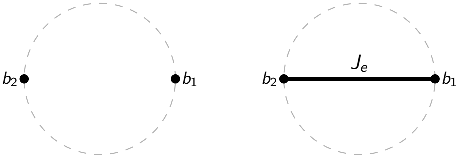
Boundary correlations: an example for $n = 2$





$$M(G, J) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}, \qquad m_{12} = \langle \sigma_1 \sigma_2 \rangle = \frac{2\exp(J_e) - 2\exp(-J_e)}{2\exp(J_e) + 2\exp(-J_e)}$$

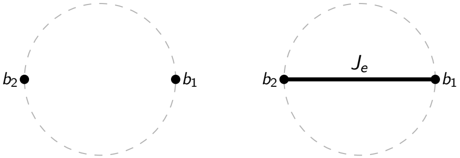
Boundary correlations: an example for $n = 2$



$$M(G, J) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}, \qquad m_{12} = \langle \sigma_1 \sigma_2 \rangle = \frac{2\exp(J_e) - 2\exp(-J_e)}{2\exp(J_e) + 2\exp(-J_e)}$$

$J_e = 0$	$J_e \in (0, \infty)$	$J_e = \infty$
$m_{12} = 0$	$m_{12} \in (0, 1)$	$m_{12} = 1$

Boundary correlations: an example for $n = 2$

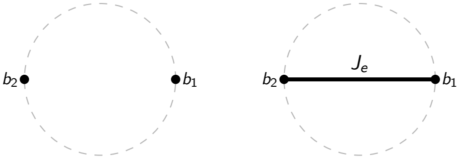


$$M(G, J) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}, \qquad m_{12} = \langle \sigma_1 \sigma_2 \rangle = \frac{2\exp(J_e) - 2\exp(-J_e)}{2\exp(J_e) + 2\exp(-J_e)}$$

$J_e = 0$	$J_e \in (0, \infty)$	$J_e = \infty$
$m_{12} = 0$	$m_{12} \in (0, 1)$	$m_{12} = 1$

- We have $\mathcal{X}_2 \cong [0, 1)$ and $\overline{\mathcal{X}}_2 \cong [0, 1]$.

Boundary correlations: an example for $n = 2$

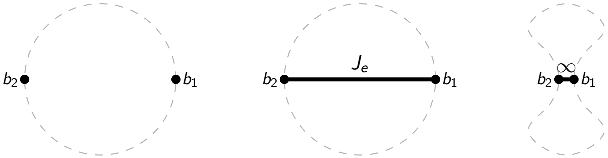


$$M(G, J) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}, \qquad m_{12} = \langle \sigma_1 \sigma_2 \rangle = \frac{2\exp(J_e) - 2\exp(-J_e)}{2\exp(J_e) + 2\exp(-J_e)}$$

$J_e = 0$	$J_e \in (0, \infty)$	$J_e = \infty$
$m_{12} = 0$	$m_{12} \in (0, 1)$	$m_{12} = 1$

- We have $\mathcal{X}_2 \cong [0, 1)$ and $\overline{\mathcal{X}}_2 \cong [0, 1]$.
- $\mathcal{X}_n$ is neither open nor closed inside $\mathbb{R}^{\binom{n}{2}}$.

Boundary correlations: an example for $n = 2$



$$M(G, J) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}, \qquad m_{12} = \langle \sigma_1 \sigma_2 \rangle = \frac{2\exp(J_e) - 2\exp(-J_e)}{2\exp(J_e) + 2\exp(-J_e)}$$

$J_e = 0$	$J_e \in (0, \infty)$	$J_e = \infty$
$m_{12} = 0$	$m_{12} \in (0, 1)$	$m_{12} = 1$

- We have $\mathcal{X}_2 \cong [0, 1)$ and $\overline{\mathcal{X}}_2 \cong [0, 1]$.
- $\mathcal{X}_n$ is neither open nor closed inside $\mathbb{R}^{\binom{n}{2}}$.
- $\overline{\mathcal{X}}_n$ is obtained from $\mathcal{X}_n$ by allowing $J_e = \infty$ (i.e., contracting edges).

Part 2: Total positivity

The totally nonnegative (TNN) Grassmannian

$$\mathrm{Gr}(k,n) := \{W \subset \mathbb{R}^n \mid \dim(W) = k\}.$$

The totally nonnegative (TNN) Grassmannian

$\mathrm{Gr}(k, n) := \{W \subset \mathbb{R}^n \mid \dim(W) = k\}$.
$\mathrm{Gr}(k, n) := \{k \times n$ matrices of rank $k\}/(\text{row operations})$.

The totally nonnegative (TNN) Grassmannian

$$\mathrm{Gr}(k,n) := \{W \subset \mathbb{R}^n \mid \dim(W) = k\}.$$
$$\mathrm{Gr}(k,n) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \in \mathrm{Gr}(2,4)$$

The totally nonnegative (TNN) Grassmannian

$$\mathrm{Gr}(k,n) := \{W \subset \mathbb{R}^n \mid \dim(W) = k\}.$$
$$\mathrm{Gr}(k,n) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \in \mathrm{Gr}(2,4)$$

Plücker coordinates: for $I \subset [n] := \{1, 2, \dots, n\}$ of size k,

The totally nonnegative (TNN) Grassmannian

$$\mathrm{Gr}(k,n) := \{W \subset \mathbb{R}^n \mid \dim(W) = k\}.$$
$$\mathrm{Gr}(k,n) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \in \mathrm{Gr}(2,4)$$

Plücker coordinates: for $I \subset [n] := \{1, 2, \dots, n\}$ of size k,

$$\Delta_I := k \times k \text{ minor with column set } I.$$

The totally nonnegative (TNN) Grassmannian

$$\mathrm{Gr}(k,n) := \{W \subset \mathbb{R}^n \mid \dim(W) = k\}.$$
$$\mathrm{Gr}(k,n) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \in \mathrm{Gr}(2,4) \qquad \begin{matrix} \Delta_{13} = 1 & \Delta_{12} = 2 & \Delta_{14} = 1 \\ \Delta_{24} = 3 & \Delta_{34} = 1 & \Delta_{23} = 1. \end{matrix}$$

Plücker coordinates: for $I \subset [n] := \{1, 2, \dots, n\}$ of size k,

$$\Delta_I := k \times k \text{ minor with column set } I.$$

The totally nonnegative (TNN) Grassmannian

$$\mathrm{Gr}(k,n) := \{W \subset \mathbb{R}^n \mid \dim(W) = k\}.$$
$$\mathrm{Gr}(k,n) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \in \mathrm{Gr}(2,4)$$

$\Delta_{13} = 1$ $\Delta_{12} = 2$ $\Delta_{14} = 1$
$\Delta_{24} = 3$ $\Delta_{34} = 1$ $\Delta_{23} = 1.$

Plücker coordinates: for $I \subset [n] := \{1, 2, \dots, n\}$ of size k,

$$\Delta_I := k \times k \text{ minor with column set } I.$$

The totally nonnegative (TNN) Grassmannian

$$\mathrm{Gr}(k,n) := \{W \subset \mathbb{R}^n \mid \dim(W) = k\}.$$
$$\mathrm{Gr}(k,n) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \in \mathrm{Gr}(2,4)$$

$$\Delta_{13} = 1 \;\; \Delta_{12} = 2 \;\; \Delta_{14} = 1$$
$$\Delta_{24} = 3 \;\; \Delta_{34} = 1 \;\; \Delta_{23} = 1.$$

Plücker coordinates: for $I \subset [n] := \{1, 2, \dots, n\}$ of size k,

$$\Delta_I := k \times k \text{ minor with column set } I.$$

Definition (Postnikov (2006))

The totally nonnegative Grassmannian is

$$\mathrm{Gr}_{\geq 0}(k,n) := \{W \in \mathrm{Gr}(k,n) \mid \Delta_I(W) \geq 0 \text{ for all } I\}.$$

The totally nonnegative (TNN) Grassmannian

$$\mathrm{Gr}(k,n) := \{W \subset \mathbb{R}^n \mid \dim(W) = k\}.$$
$$\mathrm{Gr}(k,n) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \in \mathrm{Gr}_{\geq 0}(2,4) \qquad \begin{matrix} \Delta_{13} = 1 & \Delta_{12} = 2 & \Delta_{14} = 1 \\ \Delta_{24} = 3 & \Delta_{34} = 1 & \Delta_{23} = 1. \end{matrix}$$

Plücker coordinates: for $I \subset [n] := \{1, 2, \dots, n\}$ of size k,

$$\Delta_I := k \times k \text{ minor with column set } I.$$

Definition (Postnikov (2006))

The totally nonnegative Grassmannian is

$$\mathrm{Gr}_{\geq 0}(k,n) := \{W \in \mathrm{Gr}(k,n) \mid \Delta_I(W) \geq 0 \text{ for all } I\}.$$

Example: $\mathrm{Gr}_{\geq 0}(2,4)$

$$\mathrm{RowSpan}\begin{pmatrix}1 & 1 & 0 & -1\\ 0 & 2 & 1 & 1\end{pmatrix} \in \mathrm{Gr}_{\geq 0}(2,4)$$

$$\Delta_{13} = 1, \quad \Delta_{24} = 3, \quad \Delta_{12} = 2, \quad \Delta_{34} = 1, \quad \Delta_{14} = 1, \quad \Delta_{23} = 1.$$

Example: $\mathrm{Gr}_{\geq 0}(2,4)$

$$\mathrm{RowSpan}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \in \mathrm{Gr}_{\geq 0}(2,4)$$

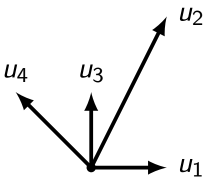
$$u_1 \quad u_2 \quad u_3 \quad u_4$$

$$\Delta_{13} = 1, \quad \Delta_{24} = 3, \quad \Delta_{12} = 2, \quad \Delta_{34} = 1, \quad \Delta_{14} = 1, \quad \Delta_{23} = 1.$$

Example: $\mathrm{Gr}_{\geq 0}(2,4)$

$$\mathrm{RowSpan}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \in \mathrm{Gr}_{\geq 0}(2,4)$$

$$\begin{matrix} u_1 & u_2 & u_3 & u_4 \end{matrix}$$

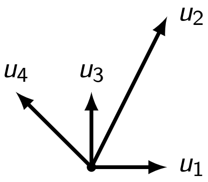


$$\Delta_{13} = 1, \quad \Delta_{24} = 3, \quad \Delta_{12} = 2, \quad \Delta_{34} = 1, \quad \Delta_{14} = 1, \quad \Delta_{23} = 1.$$

Example: $\mathrm{Gr}_{\geq 0}(2,4)$

$$\mathrm{RowSpan}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \in \mathrm{Gr}_{\geq 0}(2,4)$$

$u_1 \quad u_2 \quad u_3 \quad u_4$



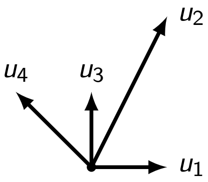
$$\Delta_{13} = 1, \quad \Delta_{24} = 3, \quad \Delta_{12} = 2, \quad \Delta_{34} = 1, \quad \Delta_{14} = 1, \quad \Delta_{23} = 1.$$

In $\mathrm{Gr}(2,4)$, we have a Plücker relation: $\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$.

Example: $\mathrm{Gr}_{\geq 0}(2,4)$

$$\mathrm{RowSpan}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \in \mathrm{Gr}_{\geq 0}(2,4)$$

$u_1 \quad u_2 \quad u_3 \quad u_4$



$$\Delta_{13} = 1, \quad \Delta_{24} = 3, \quad \Delta_{12} = 2, \quad \Delta_{34} = 1, \quad \Delta_{14} = 1, \quad \Delta_{23} = 1.$$

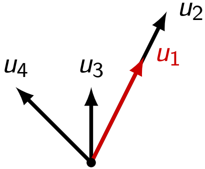
In $\mathrm{Gr}(2,4)$, we have a Plücker relation: $\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$.

Top cell: $\Delta_{13}, \Delta_{24}, \Delta_{12}, \Delta_{34}, \Delta_{14}, \Delta_{23} > 0$

Example: $\mathrm{Gr}_{\geq 0}(2,4)$

$$\mathrm{RowSpan}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \in \mathrm{Gr}_{\geq 0}(2,4)$$

$u_1 \quad u_2 \quad u_3 \quad u_4$



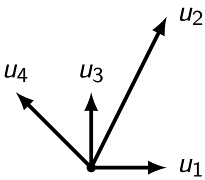
$$\Delta_{13} = 1, \quad \Delta_{24} = 3, \quad \Delta_{12} = 2, \quad \Delta_{34} = 1, \quad \Delta_{14} = 1, \quad \Delta_{23} = 1.$$

In $\mathrm{Gr}(2,4)$, we have a Plücker relation: $\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$.
Top cell: $\Delta_{13}, \Delta_{24}, \Delta_{12}, \Delta_{34}, \Delta_{14}, \Delta_{23} > 0$
Codimension 1 cells: $\Delta_{12} = 0$

Example: $\mathrm{Gr}_{\geq 0}(2,4)$

$$\mathrm{RowSpan}\begin{pmatrix}1 & 1 & 0 & -1\\ 0 & 2 & 1 & 1\end{pmatrix} \in \mathrm{Gr}_{\geq 0}(2,4)$$

$u_1 \quad u_2 \quad u_3 \quad u_4$



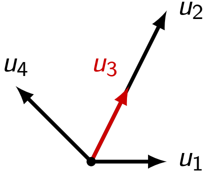
$$\Delta_{13} = 1, \quad \Delta_{24} = 3, \quad \Delta_{12} = 2, \quad \Delta_{34} = 1, \quad \Delta_{14} = 1, \quad \Delta_{23} = 1.$$

In $\mathrm{Gr}(2,4)$, we have a Plücker relation: $\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$.
Top cell: $\Delta_{13}, \Delta_{24}, \Delta_{12}, \Delta_{34}, \Delta_{14}, \Delta_{23} > 0$
Codimension 1 cells: $\Delta_{12} = 0$

Example: $\mathrm{Gr}_{\geq 0}(2,4)$

$$\mathrm{RowSpan}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \in \mathrm{Gr}_{\geq 0}(2,4)$$

$$\quad u_1 \;\; u_2 \;\; u_3 \;\; u_4$$



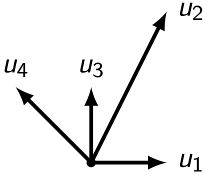
$$\Delta_{13} = 1, \quad \Delta_{24} = 3, \quad \Delta_{12} = 2, \quad \Delta_{34} = 1, \quad \Delta_{14} = 1, \quad \Delta_{23} = 1.$$

In $\mathrm{Gr}(2,4)$, we have a Plücker relation: $\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$.
Top cell: $\Delta_{13}, \Delta_{24}, \Delta_{12}, \Delta_{34}, \Delta_{14}, \Delta_{23} > 0$
Codimension 1 cells: $\Delta_{12} = 0$, $\Delta_{23} = 0$

Example: $\mathrm{Gr}_{\geq 0}(2,4)$

$$\mathrm{RowSpan}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \in \mathrm{Gr}_{\geq 0}(2,4)$$

$u_1 \ u_2 \ u_3 \ u_4$

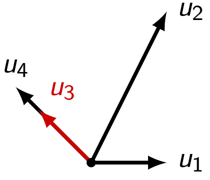


$$\Delta_{13} = 1, \quad \Delta_{24} = 3, \quad \Delta_{12} = 2, \quad \Delta_{34} = 1, \quad \Delta_{14} = 1, \quad \Delta_{23} = 1.$$

In $\mathrm{Gr}(2,4)$, we have a Plücker relation: $\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$.
Top cell: $\Delta_{13}, \Delta_{24}, \Delta_{12}, \Delta_{34}, \Delta_{14}, \Delta_{23} > 0$
Codimension 1 cells: $\Delta_{12} = 0$, $\Delta_{23} = 0$

Example: $\mathrm{Gr}_{\geq 0}(2,4)$

$$\mathrm{RowSpan}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \in \mathrm{Gr}_{\geq 0}(2,4)$$
$$\quad u_1 \;\; u_2 \;\; u_3 \;\; u_4$$



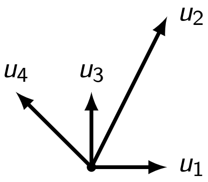
$$\Delta_{13} = 1, \quad \Delta_{24} = 3, \quad \Delta_{12} = 2, \quad \Delta_{34} = 1, \quad \Delta_{14} = 1, \quad \Delta_{23} = 1.$$

In $\mathrm{Gr}(2,4)$, we have a Plücker relation: $\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$.
Top cell: $\Delta_{13}, \Delta_{24}, \Delta_{12}, \Delta_{34}, \Delta_{14}, \Delta_{23} > 0$
Codimension 1 cells: $\Delta_{12} = 0$, $\Delta_{23} = 0$, $\Delta_{34} = 0$

Example: $\mathrm{Gr}_{\geq 0}(2,4)$

$$\mathrm{RowSpan}\begin{pmatrix}1 & 1 & 0 & -1\\ 0 & 2 & 1 & 1\end{pmatrix} \in \mathrm{Gr}_{\geq 0}(2,4)$$

$$\quad u_1 \quad u_2 \quad u_3 \quad u_4$$



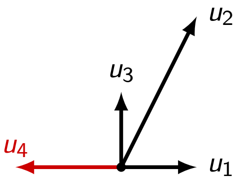
$$\Delta_{13} = 1, \quad \Delta_{24} = 3, \quad \Delta_{12} = 2, \quad \Delta_{34} = 1, \quad \Delta_{14} = 1, \quad \Delta_{23} = 1.$$

In $\mathrm{Gr}(2,4)$, we have a Plücker relation: $\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$.
Top cell: $\Delta_{13}, \Delta_{24}, \Delta_{12}, \Delta_{34}, \Delta_{14}, \Delta_{23} > 0$
Codimension 1 cells: $\Delta_{12} = 0$, $\Delta_{23} = 0$, $\Delta_{34} = 0$

Example: $\mathrm{Gr}_{\geq 0}(2,4)$

$$\mathrm{RowSpan}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \in \mathrm{Gr}_{\geq 0}(2,4)$$

$$\quad u_1 \quad u_2 \quad u_3 \quad u_4$$



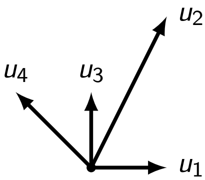
$$\Delta_{13} = 1, \quad \Delta_{24} = 3, \quad \Delta_{12} = 2, \quad \Delta_{34} = 1, \quad \Delta_{14} = 1, \quad \Delta_{23} = 1.$$

In $\mathrm{Gr}(2,4)$, we have a Plücker relation: $\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$.
Top cell: $\Delta_{13}, \Delta_{24}, \Delta_{12}, \Delta_{34}, \Delta_{14}, \Delta_{23} > 0$
Codimension 1 cells: $\Delta_{12} = 0$, $\Delta_{23} = 0$, $\Delta_{34} = 0$, $\Delta_{14} = 0$.

Example: $\mathrm{Gr}_{\geq 0}(2,4)$

$$\mathrm{RowSpan}\begin{pmatrix}1 & 1 & 0 & -1\\ 0 & 2 & 1 & 1\end{pmatrix} \in \mathrm{Gr}_{\geq 0}(2,4)$$

$u_1 \quad u_2 \quad u_3 \quad u_4$



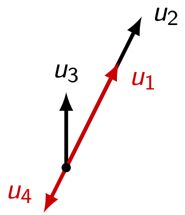
$$\Delta_{13}=1, \quad \Delta_{24}=3, \quad \Delta_{12}=2, \quad \Delta_{34}=1, \quad \Delta_{14}=1, \quad \Delta_{23}=1.$$

In $\mathrm{Gr}(2,4)$, we have a Plücker relation: $\Delta_{13}\Delta_{24}=\Delta_{12}\Delta_{34}+\Delta_{14}\Delta_{23}$.
Top cell: $\Delta_{13},\Delta_{24},\Delta_{12},\Delta_{34},\Delta_{14},\Delta_{23}>0$
Codimension 1 cells: $\Delta_{12}=0$, $\Delta_{23}=0$, $\Delta_{34}=0$, $\Delta_{14}=0$.

Example: $\text{Gr}_{\geq 0}(2,4)$

$$\text{RowSpan}\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \in \text{Gr}_{\geq 0}(2,4)$$

$u_1 \; u_2 \; u_3 \; u_4$



$$\Delta_{13} = 1, \quad \Delta_{24} = 3, \quad \Delta_{12} = 2, \quad \Delta_{34} = 1, \quad \Delta_{14} = 1, \quad \Delta_{23} = 1.$$

In $\text{Gr}(2,4)$, we have a Plücker relation: $\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$.
Top cell: $\Delta_{13}, \Delta_{24}, \Delta_{12}, \Delta_{34}, \Delta_{14}, \Delta_{23} > 0$
Codimension 1 cells: $\Delta_{12} = 0$, $\Delta_{23} = 0$, $\Delta_{34} = 0$, $\Delta_{14} = 0$.
Codimension 2 cell: $\Delta_{12} = \Delta_{14} = \Delta_{24} = 0$.

The topology of $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (Postnikov (2006))

Each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

The topology of $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (Postnikov (2006))

Each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

Conjecture (Postnikov (2006))

The closure of each boundary cell is homeomorphic to a closed ball.

The topology of $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (Postnikov (2006))

Each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

Conjecture (Postnikov (2006))

The closure of each boundary cell is homeomorphic to a closed ball.

$\mathrm{Gr}_{\geq 0}(k, n)$ *is a regular CW complex homeomorphic to a closed ball.*

The topology of $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (Postnikov (2006))

Each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

Conjecture (Postnikov (2006))

The closure of each boundary cell is homeomorphic to a closed ball.
$\mathrm{Gr}_{\geq 0}(k, n)$ *is a regular CW complex homeomorphic to a closed ball.*

Partial results:

The topology of $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (Postnikov (2006))

Each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

Conjecture (Postnikov (2006))

The closure of each boundary cell is homeomorphic to a closed ball. $\mathrm{Gr}_{\geq 0}(k, n)$ *is a regular CW complex homeomorphic to a closed ball.*

Partial results: Williams (2007),

The topology of $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (Postnikov (2006))

Each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

Conjecture (Postnikov (2006))

The closure of each boundary cell is homeomorphic to a closed ball.
$\mathrm{Gr}_{\geq 0}(k, n)$ *is a regular CW complex homeomorphic to a closed ball.*

Partial results: Williams (2007), Postnikov–Speyer–Williams (2009),

The topology of $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (Postnikov (2006))

Each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

Conjecture (Postnikov (2006))

The closure of each boundary cell is homeomorphic to a closed ball.
$\mathrm{Gr}_{\geq 0}(k, n)$ *is a regular CW complex homeomorphic to a closed ball.*

Partial results: Williams (2007), Postnikov–Speyer–Williams (2009), Rietsch–Williams (2010).

The topology of $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (Postnikov (2006))

Each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

Conjecture (Postnikov (2006))

The closure of each boundary cell is homeomorphic to a closed ball.
$\mathrm{Gr}_{\geq 0}(k, n)$ is a regular CW complex homeomorphic to a closed ball.

Partial results: Williams (2007), Postnikov–Speyer–Williams (2009), Rietsch–Williams (2010).

Analogous story:

The topology of $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (Postnikov (2006))

Each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

Conjecture (Postnikov (2006))

The closure of each boundary cell is homeomorphic to a closed ball.
$\mathrm{Gr}_{\geq 0}(k, n)$ *is a regular CW complex homeomorphic to a closed ball.*

Partial results: Williams (2007), Postnikov–Speyer–Williams (2009), Rietsch–Williams (2010).

Analogous story: Fomin–Shapiro (2000),

The topology of $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (Postnikov (2006))

Each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

Conjecture (Postnikov (2006))

The closure of each boundary cell is homeomorphic to a closed ball.
$\mathrm{Gr}_{\geq 0}(k, n)$ is a regular CW complex homeomorphic to a closed ball.

Partial results: Williams (2007), Postnikov–Speyer–Williams (2009), Rietsch–Williams (2010).

Analogous story: Fomin–Shapiro (2000), Hersh (2014)

The topology of $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (Postnikov (2006))

Each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

Conjecture (Postnikov (2006))

The closure of each boundary cell is homeomorphic to a closed ball.
$\mathrm{Gr}_{\geq 0}(k, n)$ is a regular CW complex homeomorphic to a closed ball.

Partial results: Williams (2007), Postnikov–Speyer–Williams (2009), Rietsch–Williams (2010).

Analogous story: Fomin–Shapiro (2000), Hersh (2014)

Theorem (G.–Karp–Lam (2017))

$\mathrm{Gr}_{\geq 0}(k, n)$ is homeomorphic to a $k(n-k)$-dimensional closed ball.

The topology of $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (Postnikov (2006))

Each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

Conjecture (Postnikov (2006))

The closure of each boundary cell is homeomorphic to a closed ball.
$\mathrm{Gr}_{\geq 0}(k, n)$ is a regular CW complex homeomorphic to a closed ball.

Partial results: Williams (2007), Postnikov–Speyer–Williams (2009), Rietsch–Williams (2010).

Analogous story: Fomin–Shapiro (2000), Hersh (2014)

Theorem (G.–Karp–Lam (2017))

$\mathrm{Gr}_{\geq 0}(k, n)$ is homeomorphic to a $k(n-k)$-dimensional closed ball.

Theorem (G.–Karp–Lam (2018+))

$\mathrm{Gr}_{\geq 0}(k, n)$ is a regular CW complex.

The totally nonnegative orthogonal Grassmannian

Recall: $\mathrm{Gr}_{\geq 0}(k,n) := \{W \in \mathrm{Gr}(k,n) \mid \Delta_I(W) \geq 0 \text{ for all } I\}$.

The totally nonnegative orthogonal Grassmannian

Recall: $\mathrm{Gr}_{\geq 0}(k,n) := \{W \in \mathrm{Gr}(k,n) \mid \Delta_I(W) \geq 0 \text{ for all } I\}$.
The orthogonal Grassmannian:
$\mathrm{OG}(n,2n) := \{W \in \mathrm{Gr}(n,2n) \mid \Delta_I(W) = \Delta_{[2n]\setminus I}(W) \text{ for all } I\}$.

The totally nonnegative orthogonal Grassmannian

Recall: $\mathrm{Gr}_{\geq 0}(k,n) := \{W \in \mathrm{Gr}(k,n) \mid \Delta_I(W) \geq 0 \text{ for all } I\}$.
The orthogonal Grassmannian:
$\mathrm{OG}(n,2n) := \{W \in \mathrm{Gr}(n,2n) \mid \Delta_I(W) = \Delta_{[2n]\setminus I}(W) \text{ for all } I\}$.

Definition (Huang–Wen (2013))

The totally nonnegative orthogonal Grassmannian:
$\mathrm{OG}_{\geq 0}(n,2n) := \mathrm{OG}(n,2n) \cap \mathrm{Gr}_{\geq 0}(n,2n)$

The totally nonnegative orthogonal Grassmannian

Recall: $\mathrm{Gr}_{\geq 0}(k,n) := \{W \in \mathrm{Gr}(k,n) \mid \Delta_I(W) \geq 0 \text{ for all } I\}$.
The orthogonal Grassmannian:
$\mathrm{OG}(n,2n) := \{W \in \mathrm{Gr}(n,2n) \mid \Delta_I(W) = \Delta_{[2n]\setminus I}(W) \text{ for all } I\}$.

Definition (Huang–Wen (2013))

The totally nonnegative orthogonal Grassmannian:
$\mathrm{OG}_{\geq 0}(n,2n) := \mathrm{OG}(n,2n) \cap \mathrm{Gr}_{\geq 0}(n,2n)$, i.e.,
$\mathrm{OG}_{\geq 0}(n,2n) := \{W \in \mathrm{Gr}(n,2n) \mid \Delta_I(W) = \Delta_{[2n]\setminus I}(W) \geq 0 \text{ for all } I\}$.

The totally nonnegative orthogonal Grassmannian

Recall: $\mathrm{Gr}_{\geq 0}(k,n) := \{W \in \mathrm{Gr}(k,n) \mid \Delta_I(W) \geq 0 \text{ for all } I\}$.
The orthogonal Grassmannian:
$\mathrm{OG}(n,2n) := \{W \in \mathrm{Gr}(n,2n) \mid \Delta_I(W) = \Delta_{[2n]\setminus I}(W) \text{ for all } I\}$.

Definition (Huang–Wen (2013))

The totally nonnegative orthogonal Grassmannian:
$\mathrm{OG}_{\geq 0}(n,2n) := \mathrm{OG}(n,2n) \cap \mathrm{Gr}_{\geq 0}(n,2n)$, i.e.,
$\mathrm{OG}_{\geq 0}(n,2n) := \{W \in \mathrm{Gr}(n,2n) \mid \Delta_I(W) = \Delta_{[2n]\setminus I}(W) \geq 0 \text{ for all } I\}$.

- $\dim(\mathrm{Gr}_{\geq 0}(n,2n)) = n^2$

The totally nonnegative orthogonal Grassmannian

Recall: $\mathrm{Gr}_{\geq 0}(k, n) := \{W \in \mathrm{Gr}(k, n) \mid \Delta_I(W) \geq 0 \text{ for all } I\}$.
The orthogonal Grassmannian:
$\mathrm{OG}(n, 2n) := \{W \in \mathrm{Gr}(n, 2n) \mid \Delta_I(W) = \Delta_{[2n]\setminus I}(W) \text{ for all } I\}$.

Definition (Huang–Wen (2013))

The totally nonnegative orthogonal Grassmannian:
$\mathrm{OG}_{\geq 0}(n, 2n) := \mathrm{OG}(n, 2n) \cap \mathrm{Gr}_{\geq 0}(n, 2n)$, i.e.,
$\mathrm{OG}_{\geq 0}(n, 2n) := \{W \in \mathrm{Gr}(n, 2n) \mid \Delta_I(W) = \Delta_{[2n]\setminus I}(W) \geq 0 \text{ for all } I\}$.

- $\dim(\mathrm{Gr}_{\geq 0}(n, 2n)) = n^2$
- $\dim(\mathrm{OG}_{\geq 0}(n, 2n)) = \binom{n}{2} = \frac{n(n-1)}{2}$

The totally nonnegative orthogonal Grassmannian

Recall: $\mathrm{Gr}_{\geq 0}(k,n) := \{W \in \mathrm{Gr}(k,n) \mid \Delta_I(W) \geq 0 \text{ for all } I\}$.
The orthogonal Grassmannian:
$\mathrm{OG}(n,2n) := \{W \in \mathrm{Gr}(n,2n) \mid \Delta_I(W) = \Delta_{[2n]\setminus I}(W) \text{ for all } I\}$.

Definition (Huang–Wen (2013))

The totally nonnegative orthogonal Grassmannian:
$\mathrm{OG}_{\geq 0}(n,2n) := \mathrm{OG}(n,2n) \cap \mathrm{Gr}_{\geq 0}(n,2n)$, i.e.,
$\mathrm{OG}_{\geq 0}(n,2n) := \{W \in \mathrm{Gr}(n,2n) \mid \Delta_I(W) = \Delta_{[2n]\setminus I}(W) \geq 0 \text{ for all } I\}$.

- $\dim(\mathrm{Gr}_{\geq 0}(n,2n)) = n^2$
- $\dim(\mathrm{OG}_{\geq 0}(n,2n)) = \binom{n}{2} = \frac{n(n-1)}{2}$
- boundary cells of $\mathrm{Gr}_{\geq 0}(n,2n)$ are indexed by permutations*

The totally nonnegative orthogonal Grassmannian

Recall: $\mathrm{Gr}_{\geq 0}(k,n) := \{W \in \mathrm{Gr}(k,n) \mid \Delta_I(W) \geq 0 \text{ for all } I\}$.
The orthogonal Grassmannian:
$\mathrm{OG}(n,2n) := \{W \in \mathrm{Gr}(n,2n) \mid \Delta_I(W) = \Delta_{[2n]\setminus I}(W) \text{ for all } I\}$.

Definition (Huang–Wen (2013))

The totally nonnegative orthogonal Grassmannian:
$\mathrm{OG}_{\geq 0}(n,2n) := \mathrm{OG}(n,2n) \cap \mathrm{Gr}_{\geq 0}(n,2n)$, i.e.,
$\mathrm{OG}_{\geq 0}(n,2n) := \{W \in \mathrm{Gr}(n,2n) \mid \Delta_I(W) = \Delta_{[2n]\setminus I}(W) \geq 0 \text{ for all } I\}$.

- $\dim(\mathrm{Gr}_{\geq 0}(n,2n)) = n^2$
- $\dim(\mathrm{OG}_{\geq 0}(n,2n)) = \binom{n}{2} = \frac{n(n-1)}{2}$
- boundary cells of $\mathrm{Gr}_{\geq 0}(n,2n)$ are indexed by permutations*
- boundary cells of $\mathrm{OG}_{\geq 0}(n,2n)$ are indexed by fixed-point free involutions

Main result

$\mathcal{X}_n := \{M(G,J) \mid (G,J)$ is a planar Ising network with n boundary vertices$\}$
$\overline{\mathcal{X}}_n :=$ closure of $\mathcal{X}_n$ inside the space of $n \times n$ matrices.

Main result

$\mathcal{X}_n := \{M(G,J) \mid (G,J)$ is a planar Ising network with n boundary vertices$\}$
$\overline{\mathcal{X}}_n :=$ closure of $\mathcal{X}_n$ inside the space of $n \times n$ matrices.
We have $\mathcal{X}_n, \overline{\mathcal{X}}_n \subset \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) := \left\{ \begin{array}{c} \text{symmetric } n \times n \text{ matrices} \\ \text{with 1's on the diagonal} \end{array} \right\}$.

Main result

$\mathcal{X}_n := \{M(G,J) \mid (G,J)$ is a planar Ising network with n boundary vertices$\}$
$\overline{\mathcal{X}}_n :=$ closure of $\mathcal{X}_n$ inside the space of $n \times n$ matrices.

We have $\mathcal{X}_n, \overline{\mathcal{X}}_n \subset \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) := \left\{ \begin{array}{c} \text{symmetric } n \times n \text{ matrices} \\ \text{with 1's on the diagonal} \end{array} \right\}$.

Definition

The doubling map ϕ:

$$\begin{pmatrix} 1 & m_{12} & m_{13} & m_{14} \\ m_{12} & 1 & m_{23} & m_{24} \\ m_{13} & m_{23} & 1 & m_{34} \\ m_{14} & m_{24} & m_{34} & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & m_{12} & -m_{12} & -m_{13} & m_{13} & m_{14} & -m_{14} \\ -m_{12} & m_{12} & 1 & 1 & m_{23} & -m_{23} & -m_{24} & m_{24} \\ m_{13} & -m_{13} & -m_{23} & m_{23} & 1 & 1 & m_{34} & -m_{34} \\ -m_{14} & m_{14} & m_{24} & -m_{24} & -m_{34} & m_{34} & 1 & 1 \end{pmatrix}$$

Main result

$\mathcal{X}_n := \{M(G,J) \mid (G,J)$ is a planar Ising network with n boundary vertices$\}$
$\overline{\mathcal{X}}_n :=$ closure of $\mathcal{X}_n$ inside the space of $n \times n$ matrices.

We have $\mathcal{X}_n, \overline{\mathcal{X}}_n \subset \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) := \left\{ \begin{array}{c} \text{symmetric } n \times n \text{ matrices} \\ \text{with 1's on the diagonal} \end{array} \right\}$.

Definition

The doubling map ϕ:

$$\begin{pmatrix} 1 & m_{12} & m_{13} & m_{14} \\ m_{12} & 1 & m_{23} & m_{24} \\ m_{13} & m_{23} & 1 & m_{34} \\ m_{14} & m_{24} & m_{34} & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & m_{12} & -m_{12} & -m_{13} & m_{13} & m_{14} & -m_{14} \\ -m_{12} & m_{12} & 1 & 1 & m_{23} & -m_{23} & -m_{24} & m_{24} \\ m_{13} & -m_{13} & -m_{23} & m_{23} & 1 & 1 & m_{34} & -m_{34} \\ -m_{14} & m_{14} & m_{24} & -m_{24} & -m_{34} & m_{34} & 1 & 1 \end{pmatrix}$$

Main result

$\mathcal{X}_n := \{M(G,J) \mid (G,J)$ is a planar Ising network with n boundary vertices$\}$
$\overline{\mathcal{X}}_n :=$ closure of $\mathcal{X}_n$ inside the space of $n \times n$ matrices.
We have $\mathcal{X}_n, \overline{\mathcal{X}}_n \subset \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R},1) := \left\{ \begin{array}{c} \text{symmetric } n \times n \text{ matrices} \\ \text{with 1's on the diagonal} \end{array} \right\}$.

Definition

The doubling map ϕ:

$$\begin{pmatrix} 1 & m_{12} & m_{13} & m_{14} \\ m_{12} & 1 & m_{23} & m_{24} \\ m_{13} & m_{23} & 1 & m_{34} \\ m_{14} & m_{24} & m_{34} & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & m_{12} & -m_{12} & -m_{13} & m_{13} & m_{14} & -m_{14} \\ -m_{12} & m_{12} & 1 & 1 & m_{23} & -m_{23} & -m_{24} & m_{24} \\ m_{13} & -m_{13} & -m_{23} & m_{23} & 1 & 1 & m_{34} & -m_{34} \\ -m_{14} & m_{14} & m_{24} & -m_{24} & -m_{34} & m_{34} & 1 & 1 \end{pmatrix}$$

Main result

$\mathcal{X}_n := \{M(G,J) \mid (G,J) \text{ is a planar Ising network with } n \text{ boundary vertices}\}$
$\overline{\mathcal{X}}_n :=$ closure of $\mathcal{X}_n$ inside the space of $n \times n$ matrices.

We have $\mathcal{X}_n, \overline{\mathcal{X}}_n \subset \operatorname{Mat}_n^{\mathrm{sym}}(\mathbb{R},1) := \left\{ \begin{array}{c} \text{symmetric } n \times n \text{ matrices} \\ \text{with 1's on the diagonal} \end{array} \right\}$.

Definition

The doubling map ϕ:

$$\begin{pmatrix} 1 & m_{12} & m_{13} & m_{14} \\ m_{12} & 1 & m_{23} & m_{24} \\ m_{13} & m_{23} & 1 & m_{34} \\ m_{14} & m_{24} & m_{34} & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & m_{12} & -m_{12} & -m_{13} & m_{13} & m_{14} & -m_{14} \\ -m_{12} & m_{12} & 1 & 1 & m_{23} & -m_{23} & -m_{24} & m_{24} \\ m_{13} & -m_{13} & -m_{23} & m_{23} & 1 & 1 & m_{34} & -m_{34} \\ -m_{14} & m_{14} & m_{24} & -m_{24} & -m_{34} & m_{34} & 1 & 1 \end{pmatrix}$$

Main result

$\mathcal{X}_n := \{M(G,J) \mid (G,J) \text{ is a planar Ising network with } n \text{ boundary vertices}\}$
$\overline{\mathcal{X}}_n :=$ closure of $\mathcal{X}_n$ inside the space of $n \times n$ matrices.

We have $\mathcal{X}_n, \overline{\mathcal{X}}_n \subset \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R},1) := \left\{ \begin{array}{c} \text{symmetric } n \times n \text{ matrices} \\ \text{with 1's on the diagonal} \end{array} \right\}$.

Definition

The doubling map ϕ:

$$\begin{pmatrix} 1 & m_{12} & m_{13} & m_{14} \\ m_{12} & 1 & m_{23} & m_{24} \\ m_{13} & m_{23} & 1 & m_{34} \\ m_{14} & m_{24} & m_{34} & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & m_{12} & -m_{12} & -m_{13} & m_{13} & m_{14} & -m_{14} \\ -m_{12} & m_{12} & 1 & 1 & m_{23} & -m_{23} & -m_{24} & m_{24} \\ m_{13} & -m_{13} & -m_{23} & m_{23} & 1 & 1 & m_{34} & -m_{34} \\ -m_{14} & m_{14} & m_{24} & -m_{24} & -m_{34} & m_{34} & 1 & 1 \end{pmatrix}$$

Main result

$\mathcal{X}_n := \{M(G,J) \mid (G,J) \text{ is a planar Ising network with } n \text{ boundary vertices}\}$
$\overline{\mathcal{X}}_n :=$ closure of $\mathcal{X}_n$ inside the space of $n \times n$ matrices.

We have $\mathcal{X}_n, \overline{\mathcal{X}}_n \subset \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R},1) := \left\{ \begin{array}{c} \text{symmetric } n \times n \text{ matrices} \\ \text{with 1's on the diagonal} \end{array} \right\}$.

Definition

The doubling map ϕ:

$$\begin{pmatrix} 1 & m_{12} & m_{13} & m_{14} \\ m_{12} & 1 & m_{23} & m_{24} \\ m_{13} & m_{23} & 1 & m_{34} \\ m_{14} & m_{24} & m_{34} & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & m_{12} & -m_{12} & -m_{13} & m_{13} & m_{14} & -m_{14} \\ -m_{12} & m_{12} & 1 & 1 & m_{23} & -m_{23} & -m_{24} & m_{24} \\ m_{13} & -m_{13} & -m_{23} & m_{23} & 1 & 1 & m_{34} & -m_{34} \\ -m_{14} & m_{14} & m_{24} & -m_{24} & -m_{34} & m_{34} & 1 & 1 \end{pmatrix}$$

Theorem (G.–Pylyavskyy (2018))

$$\begin{array}{ccc} \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R},1) & \xrightarrow[\phi]{} & \mathrm{OG}(n,2n) \\ \uparrow & & \uparrow \\ \overline{\mathcal{X}}_n & & \mathrm{OG}_{\geq 0}(n,2n) \end{array}$$

Main result

$\mathcal{X}_n := \{M(G,J) \mid (G,J) \text{ is a planar Ising network with } n \text{ boundary vertices}\}$
$\overline{\mathcal{X}}_n :=$ closure of $\mathcal{X}_n$ inside the space of $n \times n$ matrices.
We have $\mathcal{X}_n, \overline{\mathcal{X}}_n \subset \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R},1) := \left\{ \begin{array}{c} \text{symmetric } n \times n \text{ matrices} \\ \text{with 1's on the diagonal} \end{array} \right\}$.

Definition

The doubling map ϕ:

$$\begin{pmatrix} 1 & m_{12} & m_{13} & m_{14} \\ m_{12} & 1 & m_{23} & m_{24} \\ m_{13} & m_{23} & 1 & m_{34} \\ m_{14} & m_{24} & m_{34} & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & m_{12} & -m_{12} & -m_{13} & m_{13} & m_{14} & -m_{14} \\ -m_{12} & m_{12} & 1 & 1 & m_{23} & -m_{23} & -m_{24} & m_{24} \\ m_{13} & -m_{13} & -m_{23} & m_{23} & 1 & 1 & m_{34} & -m_{34} \\ -m_{14} & m_{14} & m_{24} & -m_{24} & -m_{34} & m_{34} & 1 & 1 \end{pmatrix}$$

Theorem (G.–Pylyavskyy (2018))

- *The map ϕ restricts to a homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n,2n)$.*

$$\begin{array}{ccc} \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R},1) & \xhookrightarrow[\phi]{} & \mathrm{OG}(n,2n) \\ \uparrow & & \uparrow \\ \overline{\mathcal{X}}_n & \xrightarrow[\phi]{\sim} & \mathrm{OG}_{\geq 0}(n,2n) \end{array}$$

Main result

$\mathcal{X}_n := \{M(G,J) \mid (G,J)$ is a planar Ising network with n boundary vertices$\}$
$\overline{\mathcal{X}}_n :=$ closure of $\mathcal{X}_n$ inside the space of $n \times n$ matrices.
We have $\mathcal{X}_n, \overline{\mathcal{X}}_n \subset \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) := \left\{ \begin{array}{c} \text{symmetric } n \times n \text{ matrices} \\ \text{with 1's on the diagonal} \end{array} \right\}.$

Definition

The doubling map ϕ:

$$\begin{pmatrix} 1 & m_{12} & m_{13} & m_{14} \\ m_{12} & 1 & m_{23} & m_{24} \\ m_{13} & m_{23} & 1 & m_{34} \\ m_{14} & m_{24} & m_{34} & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & m_{12} & -m_{12} & -m_{13} & m_{13} & m_{14} & -m_{14} \\ -m_{12} & m_{12} & 1 & 1 & m_{23} & -m_{23} & -m_{24} & m_{24} \\ m_{13} & -m_{13} & -m_{23} & m_{23} & 1 & 1 & m_{34} & -m_{34} \\ -m_{14} & m_{14} & m_{24} & -m_{24} & -m_{34} & m_{34} & 1 & 1 \end{pmatrix}$$

Theorem (G.–Pylyavskyy (2018))

- *The map ϕ restricts to a homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n, 2n)$.*
- *Each of the spaces is homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*

$$\begin{array}{ccc} \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) & \overset{\phi}{\hookrightarrow} & \mathrm{OG}(n, 2n) \\ \uparrow & & \uparrow \\ \overline{\mathcal{X}}_n & \xrightarrow[\phi]{\sim} & \mathrm{OG}_{\geq 0}(n, 2n) \end{array}$$

Example: $n = 2$

Theorem (G.–Pylyavskyy (2018))

- *The map* ϕ *restricts to a homeomorphism between* $\overline{\mathcal{X}}_n$ *and* $\mathrm{OG}_{\geq 0}(n, 2n)$*.*
- *Each of the spaces is homeomorphic to an* $\binom{n}{2}$*-dimensional closed ball.*

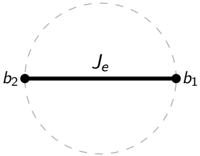
$$\begin{array}{ccc} \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) & \xhookrightarrow[\phi]{} & \mathrm{OG}(n, 2n) \\ \uparrow & & \uparrow \\ \overline{\mathcal{X}}_n & \xrightarrow[\phi]{\sim} & \mathrm{OG}_{\geq 0}(n, 2n) \end{array}$$

Example: $n = 2$

Theorem (G.–Pylyavskyy (2018))

- *The map ϕ restricts to a homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n, 2n)$.*
- *Each of the spaces is homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*

$$\begin{array}{ccc} \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) & \overset{}{\underset{\phi}{\hookrightarrow}} & \mathrm{OG}(n, 2n) \\ \uparrow & & \uparrow \\ \overline{\mathcal{X}}_n & \overset{\sim}{\underset{\phi}{\longrightarrow}} & \mathrm{OG}_{\geq 0}(n, 2n) \end{array}$$

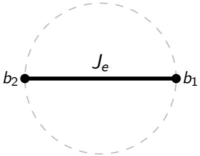


Example: $n = 2$

Theorem (G.–Pylyavskyy (2018))

- *The map ϕ restricts to a homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n, 2n)$.*
- *Each of the spaces is homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*

$$\begin{array}{ccc} \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) & \underset{\phi}{\hookrightarrow} & \mathrm{OG}(n, 2n) \\ \uparrow & & \uparrow \\ \overline{\mathcal{X}}_n & \underset{\phi}{\xrightarrow{\sim}} & \mathrm{OG}_{\geq 0}(n, 2n) \end{array}$$



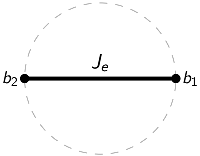
$$M(G, J) = \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix}$$

Example: $n = 2$

Theorem (G.–Pylyavskyy (2018))

- *The map ϕ restricts to a homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n, 2n)$.*
- *Each of the spaces is homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*

$$\begin{array}{ccc} \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) & \xhookrightarrow[\phi]{} & \mathrm{OG}(n, 2n) \\ \uparrow & & \uparrow \\ \overline{\mathcal{X}}_n & \xrightarrow[\phi]{\sim} & \mathrm{OG}_{\geq 0}(n, 2n) \end{array}$$



$$M(G, J) = \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix}$$

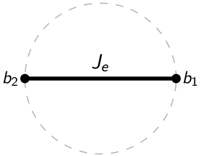
$$\overline{\mathcal{X}}_2 : \quad 0 \leq m \leq 1$$

Example: $n = 2$

Theorem (G.–Pylyavskyy (2018))

- *The map ϕ restricts to a homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n, 2n)$.*
- *Each of the spaces is homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*

$$\begin{array}{ccc} \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) & \xhookrightarrow[\phi]{} & \mathrm{OG}(n, 2n) \\ \uparrow & & \uparrow \\ \overline{\mathcal{X}}_n & \xrightarrow[\phi]{\sim} & \mathrm{OG}_{\geq 0}(n, 2n) \end{array}$$



$$M(G, J) = \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} 1 & 1 & m & -m \\ -m & m & 1 & 1 \end{pmatrix}$$

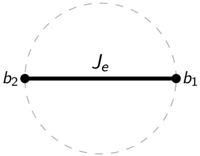
$\overline{\mathcal{X}}_2 : \quad 0 \leq m \leq 1$

Example: $n = 2$

Theorem (G.–Pylyavskyy (2018))

- *The map ϕ restricts to a homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n, 2n)$.*
- *Each of the spaces is homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*

$$\begin{array}{ccc} \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) & \overset{\phi}{\hookrightarrow} & \mathrm{OG}(n, 2n) \\ \uparrow & & \uparrow \\ \overline{\mathcal{X}}_n & \xrightarrow[\phi]{\sim} & \mathrm{OG}_{\geq 0}(n, 2n) \end{array}$$



$$M(G, J) = \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} 1 & 1 & m & -m \\ -m & m & 1 & 1 \end{pmatrix}$$

$\overline{\mathcal{X}}_2 : \quad 0 \leq m \leq 1$

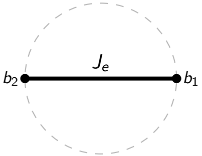
$$\Delta_{13} = 1 + m^2, \quad \Delta_{12} = 2m, \quad \Delta_{14} = 1 - m^2$$
$$\Delta_{24} = 1 + m^2, \quad \Delta_{34} = 2m, \quad \Delta_{23} = 1 - m^2$$

Example: $n = 2$

Theorem (G.–Pylyavskyy (2018))

- *The map ϕ restricts to a homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n, 2n)$.*
- *Each of the spaces is homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*

$$\begin{array}{ccc} \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) & \xhookrightarrow[\phi]{} & \mathrm{OG}(n, 2n) \\ \uparrow & & \uparrow \\ \overline{\mathcal{X}}_n & \xrightarrow[\phi]{\sim} & \mathrm{OG}_{\geq 0}(n, 2n) \end{array}$$



$$M(G, J) = \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} 1 & 1 & m & -m \\ -m & m & 1 & 1 \end{pmatrix} \in \mathrm{OG}_{\geq 0}(2, 4)$$

$\overline{\mathcal{X}}_2 : \quad 0 \leq m \leq 1$

$$\Delta_{13} = 1 + m^2, \quad \Delta_{12} = 2m, \quad \Delta_{14} = 1 - m^2$$
$$\Delta_{24} = 1 + m^2, \quad \Delta_{34} = 2m, \quad \Delta_{23} = 1 - m^2$$

Kramers–Wannier's duality

Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): no phase transition in 1D $\Longrightarrow$ not a good model for ferromagnetism

Historically, we let $G := \mathbb{Z}^d \cap \Omega$ for some $\Omega \subset \mathbb{R}^d$ and set all $J_e := \frac{1}{T}$ for some temperature $T \in \mathbb{R}_{>0}$.

- Peierls (1937): phase transition in $\mathbb{Z}^d$ for $d \geq 2$
- Kramers–Wannier (1941): critical temperature $\frac{1}{T_c} = \frac{1}{2} \log\left(\sqrt{2}+1\right)$ for $\mathbb{Z}^2$
- Onsager, Kaufman, Yang (1944–1952): exact expressions for the free energy and spontaneous magnetization
- Belavin–Polyakov–Zamolodchikov (1984): conjectured conformal invariance of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$
- Smirnov, Chelkak, Hongler, Izyurov, ... (2010–2015): proved conformal invariance and universality of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$

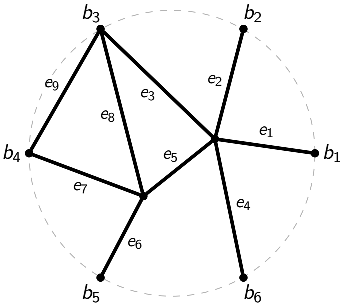
Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): no phase transition in 1D $\Longrightarrow$ not a good model for ferromagnetism

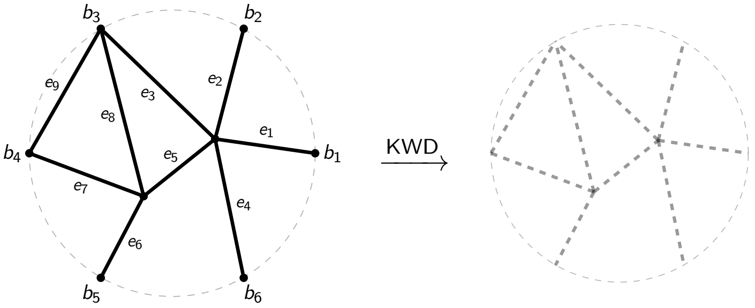
Historically, we let $G := \mathbb{Z}^d \cap \Omega$ for some $\Omega \subset \mathbb{R}^d$
and set all $J_e := \frac{1}{T}$ for some temperature $T \in \mathbb{R}_{>0}$.

- Peierls (1937): phase transition in $\mathbb{Z}^d$ for $d \geq 2$
- Kramers–Wannier (1941): critical temperature $\frac{1}{T_c} = \frac{1}{2}\log\left(\sqrt{2}+1\right)$ for $\mathbb{Z}^2$
- Onsager, Kaufman, Yang (1944–1952): exact expressions for the free energy and spontaneous magnetization
- Belavin–Polyakov–Zamolodchikov (1984): conjectured conformal invariance of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$
- Smirnov, Chelkak, Hongler, Izyurov, ... (2010–2015): proved conformal invariance and universality of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$

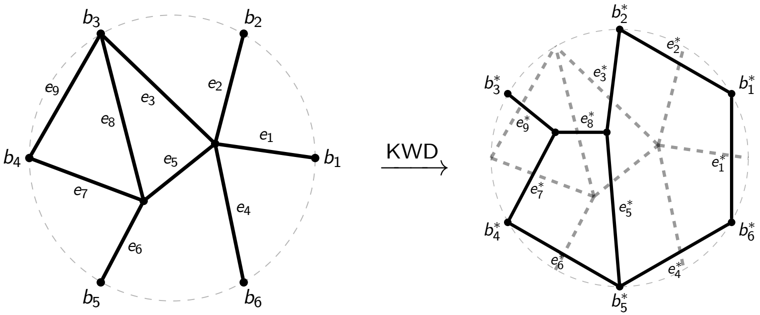
Kramers–Wannier's duality



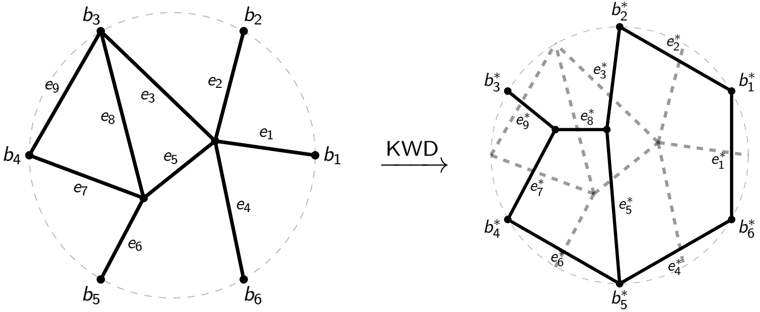
Kramers–Wannier's duality



Kramers–Wannier's duality

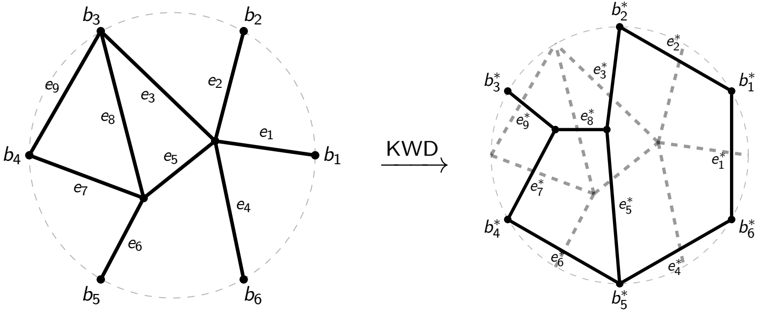


Kramers–Wannier's duality



Here J_{e^*} is defined by
$\sinh(2J_e)\sinh(2J_{e^*}) = 1.$

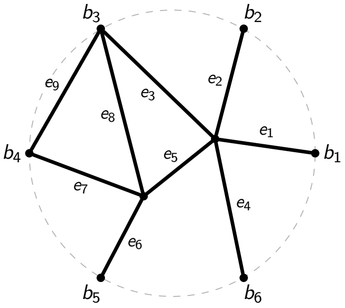
Kramers–Wannier's duality

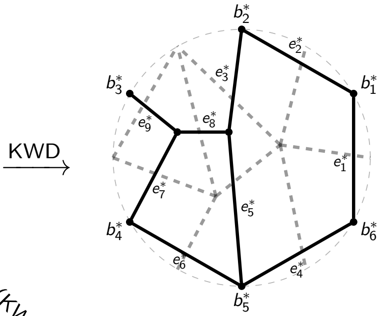


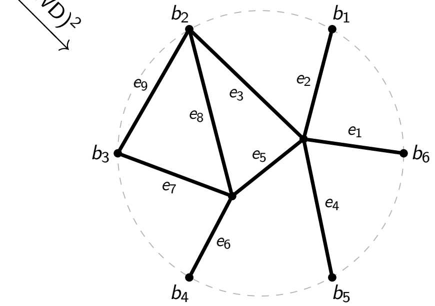
Here J_{e^*} is defined by
$\sinh(2J_e)\sinh(2J_{e^*}) = 1.$

Q: what happens if we apply the duality twice?

Kramers–Wannier's duality



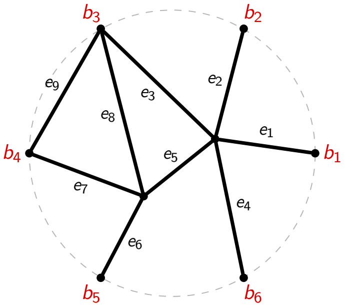


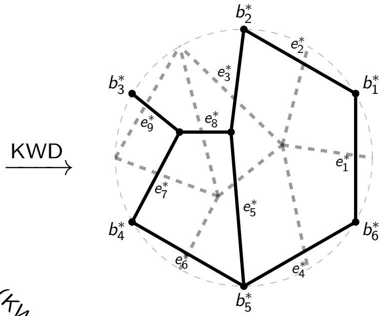


Here J_{e^*} is defined by $\sinh(2J_e)\sinh(2J_{e^*}) = 1.$

Q: what happens if we apply the duality twice?

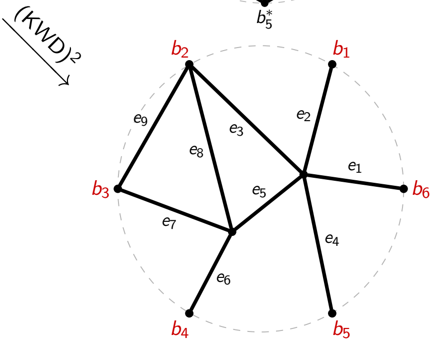
Kramers–Wannier's duality



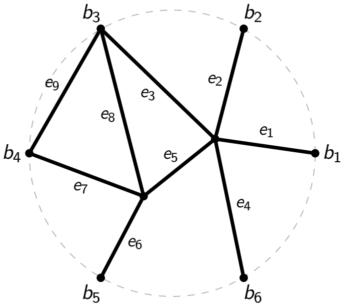


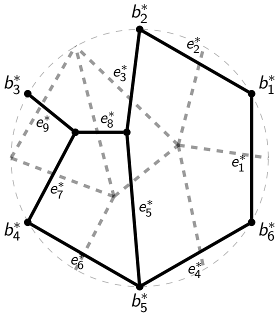
Here J_{e^*} is defined by $\sinh(2J_e)\sinh(2J_{e^*}) = 1.$

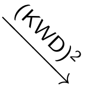
Q: what happens if we apply the duality twice?



Kramers–Wannier's duality



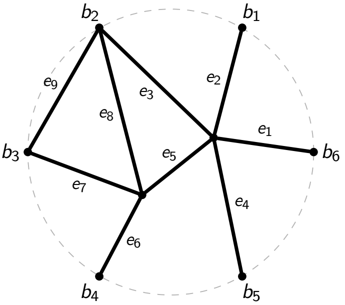




Here J_{e^*} is defined by $\sinh(2J_e)\sinh(2J_{e^*}) = 1$.

Q: what happens if we apply the duality twice?

$(\mathrm{KWD})^2$ = cyclic shift!



Kramers–Wannier's duality vs. critical temperature

Recall: J_{e^*} is defined by $\sinh(2J_e)\sinh(2J_{e^*}) = 1$.

Kramers–Wannier's duality vs. critical temperature

Recall: J_{e^*} is defined by $\sinh(2J_e)\sinh(2J_{e^*}) = 1$.

- Preserves partition function Z

Kramers–Wannier's duality vs. critical temperature

Recall: J_{e^*} is defined by $\sinh(2J_e)\sinh(2J_{e^*}) = 1$.

- Preserves partition function Z
- Switches between high and low temperature expansions for Z

Kramers–Wannier's duality vs. critical temperature

Recall: J_{e^*} is defined by $\sinh(2J_e)\sinh(2J_{e^*}) = 1$.

- Preserves partition function Z
- Switches between high and low temperature expansions for Z
- Takes correlations to "disorder variables"

Kramers–Wannier's duality vs. critical temperature

Recall: J_{e^*} is defined by $\sinh(2J_e)\sinh(2J_{e^*}) = 1$.

- Preserves partition function Z
- Switches between high and low temperature expansions for Z
- Takes correlations to "disorder variables"
- The unique solution to $\sinh(2J)\sinh(2J) = 1$ is given by

$$J = \frac{1}{2}\log(\sqrt{2} + 1)$$

Kramers–Wannier's duality vs. critical temperature

Recall: J_{e^*} is defined by $\sinh(2J_e)\sinh(2J_{e^*}) = 1$.

- Preserves partition function Z
- Switches between high and low temperature expansions for Z
- Takes correlations to "disorder variables"
- The unique solution to $\sinh(2J)\sinh(2J) = 1$ is given by

$$J = \frac{1}{2}\log(\sqrt{2}+1)$$

- Takes $G = \mathbb{Z}^2 \cap \Omega$ to $G^* \approx (\mathbb{Z}+\frac{1}{2})^2 \cap \Omega$

Kramers–Wannier's duality vs. critical temperature

Recall: J_{e^*} is defined by $\sinh(2J_e)\sinh(2J_{e^*}) = 1$.

- Preserves partition function Z
- Switches between high and low temperature expansions for Z
- Takes correlations to "disorder variables"
- The unique solution to $\sinh(2J)\sinh(2J) = 1$ is given by

$$J = \frac{1}{2}\log(\sqrt{2}+1)$$

- Takes $G = \mathbb{Z}^2 \cap \Omega$ to $G^* \approx (\mathbb{Z}+\frac{1}{2})^2 \cap \Omega$
- *Fixed point of* KWD $\leftrightarrow$ *Ising model at critical temperature*

Cyclic shift on $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (G.–Karp–Lam (2017))

$\mathrm{Gr}_{\geq 0}(k, n)$ *is homeomorphic to a* $k(n-k)$*-dimensional closed ball.*

Cyclic shift on $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (G.–Karp–Lam (2017))

$\mathrm{Gr}_{\geq 0}(k, n)$ *is homeomorphic to a* $k(n-k)$*-dimensional closed ball.*

Our proof involves a flow that contracts the whole $\mathrm{Gr}_{\geq 0}(k, n)$ to the unique cyclically symmetric point $X_0 \in \mathrm{Gr}_{\geq 0}(k, n)$.

Cyclic shift on $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (G.–Karp–Lam (2017))

$\mathrm{Gr}_{\geq 0}(k, n)$ *is homeomorphic to a* $k(n-k)$*-dimensional closed ball.*

Our proof involves a flow that contracts the whole $\mathrm{Gr}_{\geq 0}(k, n)$ to the unique cyclically symmetric point $X_0 \in \mathrm{Gr}_{\geq 0}(k, n)$.

Cyclic shift $S : \mathrm{Gr}(k, n) \to \mathrm{Gr}(k, n)$, $\quad [w_1|w_2|\dots|w_n] \mapsto [(-1)^{k-1}w_n|w_1|\dots|w_{n-1}]$.

Cyclic shift on $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (G.–Karp–Lam (2017))

$\mathrm{Gr}_{\geq 0}(k, n)$ *is homeomorphic to a* $k(n-k)$*-dimensional closed ball.*

Our proof involves a flow that contracts the whole $\mathrm{Gr}_{\geq 0}(k, n)$ to the unique cyclically symmetric point $X_0 \in \mathrm{Gr}_{\geq 0}(k, n)$.

Cyclic shift $S : \mathrm{Gr}(k, n) \to \mathrm{Gr}(k, n), \quad [w_1|w_2|\dots|w_n] \mapsto [(-1)^{k-1}w_n|w_1|\dots|w_{n-1}]$.

This map preserves $\mathrm{Gr}_{\geq 0}(k, n)$.

Cyclic shift on $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (G.–Karp–Lam (2017))

$\mathrm{Gr}_{\geq 0}(k, n)$ *is homeomorphic to a* $k(n-k)$*-dimensional closed ball.*

Our proof involves a flow that contracts the whole $\mathrm{Gr}_{\geq 0}(k, n)$ to the unique cyclically symmetric point $X_0 \in \mathrm{Gr}_{\geq 0}(k, n)$.

Cyclic shift $S : \mathrm{Gr}(k, n) \to \mathrm{Gr}(k, n)$, $\quad [w_1 | w_2 | \dots | w_n] \mapsto [(-1)^{k-1} w_n | w_1 | \dots | w_{n-1}]$.

This map preserves $\mathrm{Gr}_{\geq 0}(k, n)$.

Example: For $\mathrm{Gr}_{\geq 0}(2, 4)$, we have

$$X_0 = \begin{pmatrix} 1 & 0 & -1 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 & 0 \end{pmatrix} \overset{S}{\mapsto} \begin{pmatrix} \sqrt{2} & 1 & 0 & -1 \\ 0 & 1 & \sqrt{2} & 1 \end{pmatrix}$$

Cyclic shift on $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (G.–Karp–Lam (2017))

$\mathrm{Gr}_{\geq 0}(k, n)$ is homeomorphic to a $k(n-k)$-dimensional closed ball.

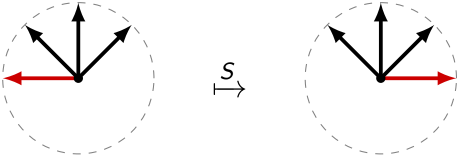
Our proof involves a flow that contracts the whole $\mathrm{Gr}_{\geq 0}(k, n)$ to the unique cyclically symmetric point $X_0 \in \mathrm{Gr}_{\geq 0}(k, n)$.

Cyclic shift $S : \mathrm{Gr}(k, n) \to \mathrm{Gr}(k, n)$, $\quad [w_1|w_2|\dots|w_n] \mapsto [(-1)^{k-1}w_n|w_1|\dots|w_{n-1}]$.

This map preserves $\mathrm{Gr}_{\geq 0}(k, n)$.

Example: For $\mathrm{Gr}_{\geq 0}(2, 4)$, we have

$$X_0 = \begin{pmatrix} 1 & 0 & -1 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 & 0 \end{pmatrix} \overset{S}{\mapsto} \begin{pmatrix} \sqrt{2} & 1 & 0 & -1 \\ 0 & 1 & \sqrt{2} & 1 \end{pmatrix}$$



Cyclic shift on $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (G.–Karp–Lam (2017))

$\mathrm{Gr}_{\geq 0}(k, n)$ *is homeomorphic to a* $k(n-k)$*-dimensional closed ball.*

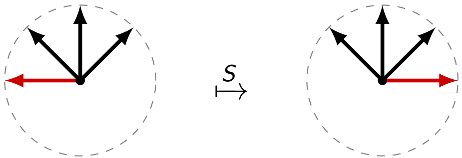
Our proof involves a flow that contracts the whole $\mathrm{Gr}_{\geq 0}(k, n)$ to the unique cyclically symmetric point $X_0 \in \mathrm{Gr}_{\geq 0}(k, n)$.

Cyclic shift $S : \mathrm{Gr}(k, n) \to \mathrm{Gr}(k, n)$, $\quad [w_1|w_2|\dots|w_n] \mapsto [(-1)^{k-1}w_n|w_1|\dots|w_{n-1}]$.

This map preserves $\mathrm{Gr}_{\geq 0}(k, n)$.

Example: For $\mathrm{Gr}_{\geq 0}(2, 4)$, we have

$$X_0 = \begin{pmatrix} 1 & 0 & -1 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 & 0 \end{pmatrix} \overset{S}{\mapsto} \begin{pmatrix} \sqrt{2} & 1 & 0 & -1 \\ 0 & 1 & \sqrt{2} & 1 \end{pmatrix} = X_0 \in \mathrm{Gr}_{\geq 0}(2, 4)$$



Cyclic shift on $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (G.–Karp–Lam (2017))

$\mathrm{Gr}_{\geq 0}(k, n)$ *is homeomorphic to a* $k(n-k)$*-dimensional closed ball.*

Our proof involves a flow that contracts the whole $\mathrm{Gr}_{\geq 0}(k, n)$ to the unique cyclically symmetric point $X_0 \in \mathrm{Gr}_{\geq 0}(k, n)$.

Cyclic shift $S : \mathrm{Gr}(k, n) \to \mathrm{Gr}(k, n)$, $\quad [w_1|w_2|\dots|w_n] \mapsto [(-1)^{k-1} w_n|w_1|\dots|w_{n-1}]$.

This map preserves $\mathrm{Gr}_{\geq 0}(k, n)$.

Example: For $\mathrm{Gr}_{\geq 0}(2, 4)$, we have

$$X_0 = \begin{pmatrix} 1 & 0 & -1 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 & 0 \end{pmatrix} \overset{S}{\mapsto} \begin{pmatrix} \sqrt{2} & 1 & 0 & -1 \\ 0 & 1 & \sqrt{2} & 1 \end{pmatrix} = X_0 \in \mathrm{Gr}_{\geq 0}(2, 4)$$

$$\begin{array}{lll} \Delta_{13} = 2 & \Delta_{12} = \sqrt{2} & \Delta_{14} = \sqrt{2} \\ \Delta_{24} = 2 & \Delta_{34} = \sqrt{2} & \Delta_{23} = \sqrt{2}. \end{array}$$

Cyclic shift on $\mathrm{Gr}_{\geq 0}(k,n)$

Theorem (G.–Karp–Lam (2017))

$\mathrm{Gr}_{\geq 0}(k,n)$ *is homeomorphic to a* $k(n-k)$*-dimensional closed ball.*

Our proof involves a flow that contracts the whole $\mathrm{Gr}_{\geq 0}(k,n)$ to the unique cyclically symmetric point $X_0 \in \mathrm{Gr}_{\geq 0}(k,n)$.

Cyclic shift $S : \mathrm{Gr}(k,n) \to \mathrm{Gr}(k,n)$, $\quad [w_1|w_2|\dots|w_n] \mapsto [(-1)^{k-1}w_n|w_1|\dots|w_{n-1}]$.

This map preserves $\mathrm{Gr}_{\geq 0}(k,n)$.

Example: For $\mathrm{Gr}_{\geq 0}(2,4)$, we have

$$X_0 = \begin{pmatrix} 1 & 0 & -1 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 & 0 \end{pmatrix} \overset{S}{\mapsto} \begin{pmatrix} \sqrt{2} & 1 & 0 & -1 \\ 0 & 1 & \sqrt{2} & 1 \end{pmatrix} = X_0 \in \mathrm{OG}_{\geq 0}(2,4)$$

$$\begin{array}{lll} \Delta_{13} = 2 & \Delta_{12} = \sqrt{2} & \Delta_{14} = \sqrt{2} \\ \Delta_{24} = 2 & \Delta_{34} = \sqrt{2} & \Delta_{23} = \sqrt{2}. \end{array}$$

Kramers–Wannier's duality vs. cyclic shift

Theorem (G.–Pylyavskyy (2018))

- *The map ϕ restricts to a homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n, 2n)$.*
- *Each of the spaces is homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*

$$\begin{array}{ccc} \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) & \xhookrightarrow[\phi]{} & \mathrm{OG}(n, 2n) \\ \uparrow & & \uparrow \\ \overline{\mathcal{X}}_n & \xrightarrow[\phi]{\sim} & \mathrm{OG}_{\geq 0}(n, 2n) \end{array}$$

Theorem (G.–Pylyavskyy (2018))

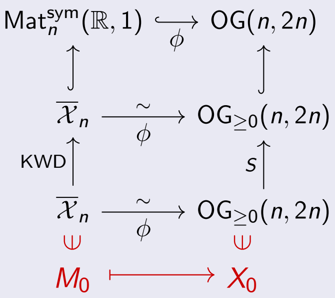
- *The map ϕ restricts to a homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n, 2n)$.*
- *Each of the spaces is homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*
- *ϕ translates $\mathrm{KWD} : \overline{\mathcal{X}}_n \to \overline{\mathcal{X}}_n$ into the cyclic shift S on $\mathrm{OG}_{\geq 0}(n, 2n)$*

$$\begin{array}{ccc}
\mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) & \xhookrightarrow[\phi]{} & \mathrm{OG}(n, 2n) \\
\big\uparrow & & \big\uparrow \\
\overline{\mathcal{X}}_n & \xrightarrow[\phi]{\sim} & \mathrm{OG}_{\geq 0}(n, 2n) \\
{\scriptstyle \mathrm{KWD}}\big\uparrow & & \big\uparrow{\scriptstyle S} \\
\overline{\mathcal{X}}_n & \xrightarrow[\phi]{\sim} & \mathrm{OG}_{\geq 0}(n, 2n)
\end{array}$$

Kramers–Wannier's duality vs. cyclic shift

Theorem (G.–Pylyavskyy (2018))

- *The map ϕ restricts to a homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n, 2n)$.*
- *Each of the spaces is homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*
- *ϕ translates $\mathrm{KWD} : \overline{\mathcal{X}}_n \to \overline{\mathcal{X}}_n$ into the cyclic shift S on $\mathrm{OG}_{\geq 0}(n, 2n)$*
- *There exists a unique matrix $M_0 \in \overline{\mathcal{X}}_n$ that is fixed by KWD.*



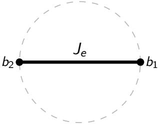
Kramers–Wannier's duality vs. cyclic shift

Theorem (G.–Pylyavskyy (2018))

- *The map ϕ restricts to a homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n, 2n)$.*
- *Each of the spaces is homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*
- *ϕ translates $\mathrm{KWD} : \overline{\mathcal{X}}_n \to \overline{\mathcal{X}}_n$ into the cyclic shift S on $\mathrm{OG}_{\geq 0}(n, 2n)$*
- *There exists a unique matrix $M_0 \in \overline{\mathcal{X}}_n$ that is fixed by* KWD.

$$\begin{array}{ccc}
\mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) & \overset{\phi}{\hookrightarrow} & \mathrm{OG}(n, 2n) \\
\uparrow & & \uparrow \\
\overline{\mathcal{X}}_n & \xrightarrow[\phi]{\sim} & \mathrm{OG}_{\geq 0}(n, 2n) \\
\mathrm{KWD} \uparrow & & \uparrow S \\
\overline{\mathcal{X}}_n & \xrightarrow[\phi]{\sim} & \mathrm{OG}_{\geq 0}(n, 2n) \\
\Cup & & \Cup \\
M_0 & \longmapsto & X_0
\end{array}$$

Example:



$M_0 \leftrightarrow J_e = \frac{1}{2} \log(\sqrt{2} + 1)$

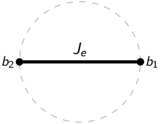
Kramers–Wannier's duality vs. cyclic shift

Theorem (G.–Pylyavskyy (2018))

- *The map ϕ restricts to a homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n,2n)$.*
- *Each of the spaces is homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*
- *ϕ translates KWD : $\overline{\mathcal{X}}_n \to \overline{\mathcal{X}}_n$ into the cyclic shift S on $\mathrm{OG}_{\geq 0}(n,2n)$*
- *There exists a unique matrix $M_0 \in \overline{\mathcal{X}}_n$ that is fixed by KWD.*

$$
\begin{array}{ccc}
\mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R},1) & \xhookrightarrow[\phi]{} & \mathrm{OG}(n,2n) \\
\uparrow & & \uparrow \\
\overline{\mathcal{X}}_n & \xrightarrow[\phi]{\sim} & \mathrm{OG}_{\geq 0}(n,2n) \\
\mathrm{KWD}\uparrow & & \uparrow S \\
\overline{\mathcal{X}}_n & \xrightarrow[\phi]{\sim} & \mathrm{OG}_{\geq 0}(n,2n) \\
\Cup & & \Cup \\
M_0 & \longmapsto & X_0
\end{array}
$$

Example:

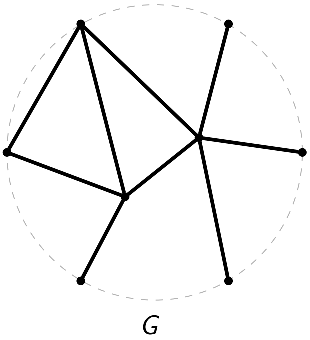


$M_0 \leftrightarrow J_e = \frac{1}{2}\log(\sqrt{2}+1)$

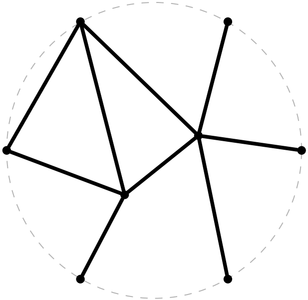
Fixed point M_0 of KWD $\leftrightarrow$ Ising model at critical temperature $\leftrightarrow$ X_0?

Boundary cells

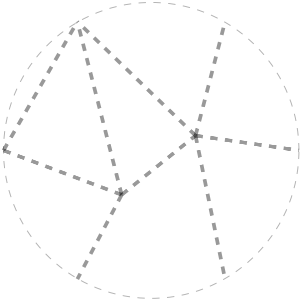
Medial graph



Medial graph

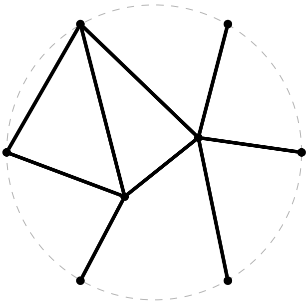


G

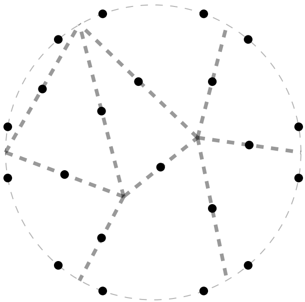


Medial graph of *G*

Medial graph

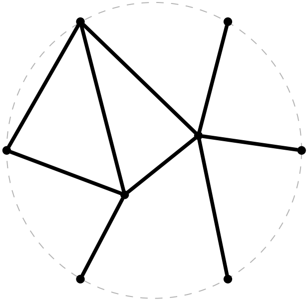


G

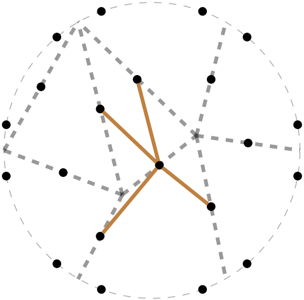


Medial graph of G

Medial graph

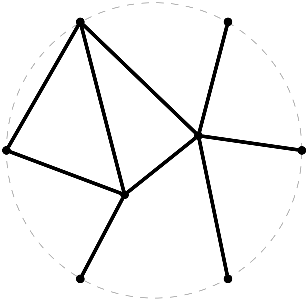


G

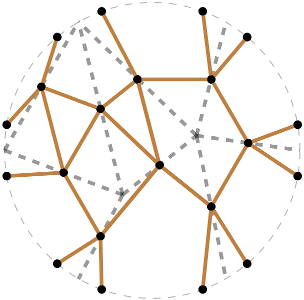


Medial graph of G

Medial graph

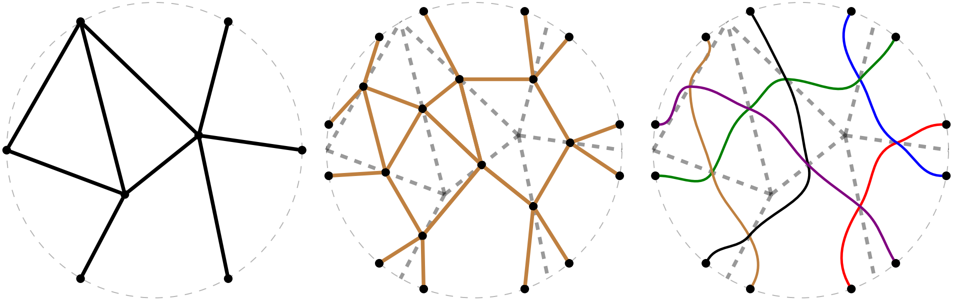


G



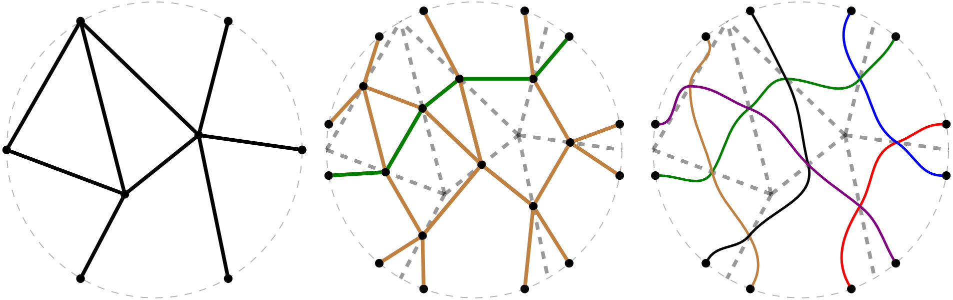
Medial graph of G

Medial graph



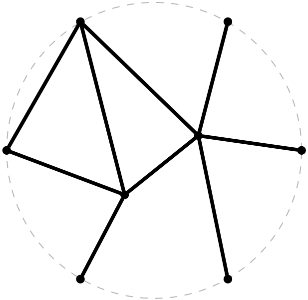
G — Medial graph of G — Medial pairing τ_G of G

Medial graph

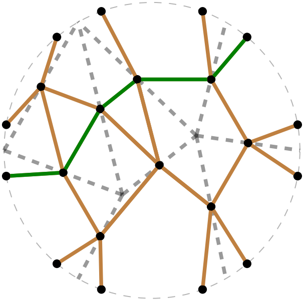


G — Medial graph of G — Medial pairing τ_G of G

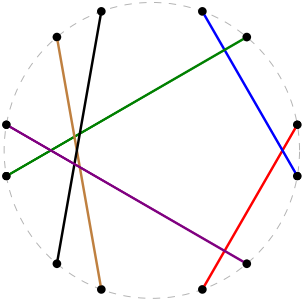
Medial graph



G

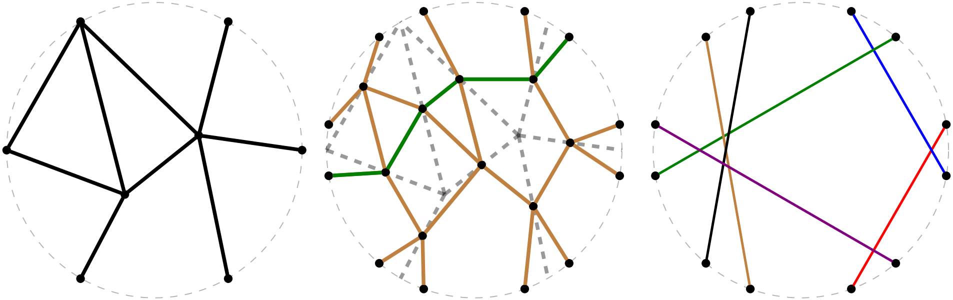


Medial graph of G



Medial pairing τ_G of G

Medial graph



G Medial graph of G Medial pairing τ_G of G

Definition

G is called reduced $\iff |E| = \text{xing}(\tau_G)$

Boundary cells

Let $\mathcal{X}_G := \{M(G,J) \mid J : E \to \mathbb{R}_{>0}\} \subset \overline{\mathcal{X}}_n$.

Boundary cells

Let $\mathcal{X}_G := \{M(G, J) \mid J : E \to \mathbb{R}_{>0}\} \subset \overline{\mathcal{X}}_n$.

Theorem (G.–Pylyavskyy (2018))

- *Every point in $\overline{\mathcal{X}}_n$ belongs to $\mathcal{X}_G$ for some reduced G*

Boundary cells

Let $\mathcal{X}_G := \{M(G,J) \mid J : E \to \mathbb{R}_{>0}\} \subset \overline{\mathcal{X}}_n$.

Theorem (G.–Pylyavskyy (2018))

- *Every point in $\overline{\mathcal{X}}_n$ belongs to $\mathcal{X}_G$ for some reduced G*
- *If G and G' are reduced then we have* $\begin{cases} \mathcal{X}_G = \mathcal{X}_{G'}, & \text{if } \tau_G = \tau_{G'} \\ \mathcal{X}_G \cap \mathcal{X}_{G'} = \emptyset, & \text{if } \tau_G \neq \tau_{G'} \end{cases}$

Boundary cells

Let $\mathcal{X}_G := \{M(G,J) \mid J : E \to \mathbb{R}_{>0}\} \subset \overline{\mathcal{X}}_n$.

Theorem (G.–Pylyavskyy (2018))

- *Every point in* $\overline{\mathcal{X}}_n$ *belongs to* $\mathcal{X}_G$ *for some reduced* G
- *If* G *and* G' *are reduced then we have* $\begin{cases} \mathcal{X}_G = \mathcal{X}_{G'}, & \text{if } \tau_G = \tau_{G'} \\ \mathcal{X}_G \cap \mathcal{X}_{G'} = \emptyset, & \text{if } \tau_G \neq \tau_{G'} \end{cases}$
- *We have a stratification* $\overline{\mathcal{X}}_n = \bigsqcup_{\tau \in \mathrm{Match}(2n)} \mathcal{X}_\tau.$

Boundary cells

Let $\mathcal{X}_G := \{M(G, J) \mid J : E \to \mathbb{R}_{>0}\} \subset \overline{\mathcal{X}}_n$.

Theorem (G.–Pylyavskyy (2018))

- *Every point in* $\overline{\mathcal{X}}_n$ *belongs to* $\mathcal{X}_G$ *for some reduced* G
- *If* G *and* G' *are reduced then we have* $\begin{cases} \mathcal{X}_G = \mathcal{X}_{G'}, & \text{if } \tau_G = \tau_{G'} \\ \mathcal{X}_G \cap \mathcal{X}_{G'} = \emptyset, & \text{if } \tau_G \neq \tau_{G'} \end{cases}$
- *We have a stratification* $\overline{\mathcal{X}}_n = \bigsqcup_{\tau \in \text{Match}(2n)} \mathcal{X}_\tau$.
- G *is reduced* $\iff$ *the map* $\mathbb{R}_{>0}^E \to \mathcal{X}_G$ *is injective*

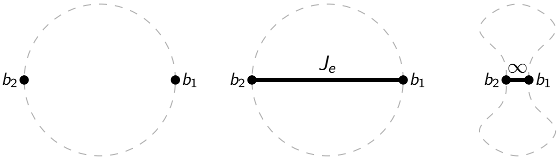
Boundary cells

Let $\mathcal{X}_G := \{M(G,J) \mid J : E \to \mathbb{R}_{>0}\} \subset \overline{\mathcal{X}}_n$.

Theorem (G.–Pylyavskyy (2018))

- *Every point in* $\overline{\mathcal{X}}_n$ *belongs to* $\mathcal{X}_G$ *for some reduced* G
- *If* G *and* G' *are reduced then we have* $\begin{cases} \mathcal{X}_G = \mathcal{X}_{G'}, & \text{if } \tau_G = \tau_{G'} \\ \mathcal{X}_G \cap \mathcal{X}_{G'} = \emptyset, & \text{if } \tau_G \neq \tau_{G'} \end{cases}$
- *We have a stratification* $\overline{\mathcal{X}}_n = \bigsqcup_{\tau \in \mathrm{Match}(2n)} \mathcal{X}_\tau.$
- G *is reduced* $\iff$ *the map* $\mathbb{R}_{>0}^E \to \mathcal{X}_G$ *is injective*
- *For reduced* G, G', *we have* $M(G,J) = M(G',J') \iff (G',J')$ *is obtained from* (G,J) *by a sequence of* $Y - \Delta$ *moves*

Boundary correlations: an example for $n = 2$



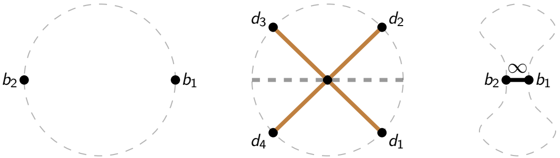
$$M(G, J) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}$$

$J_e = 0$, $m_{12} = 0$

$J_e \in (0, \infty)$, $m_{12} \in (0, 1)$

$J_e = \infty$, $m_{12} = 1$

Boundary correlations: an example for $n = 2$

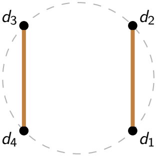


$$M(G, J) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}$$

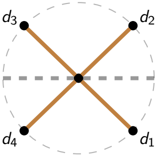
$J_e = 0$	$J_e \in (0, \infty)$	$J_e = \infty$
$m_{12} = 0$	$m_{12} \in (0, 1)$	$m_{12} = 1$

Boundary correlations: an example for $n = 2$

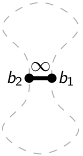
$$M(G,J) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}$$



$J_e = 0$
$m_{12} = 0$

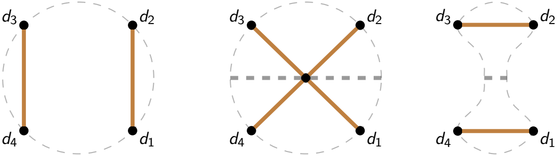


$J_e \in (0, \infty)$
$m_{12} \in (0, 1)$



$J_e = \infty$
$m_{12} = 1$

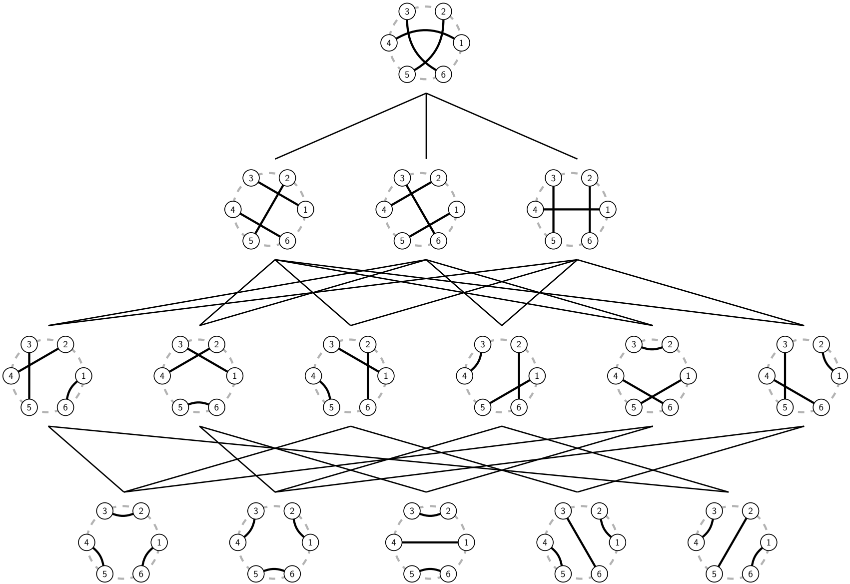
Boundary correlations: an example for $n = 2$



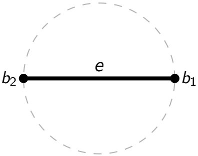
$$M(G, J) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}$$

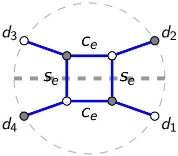
$J_e = 0$	$J_e \in (0, \infty)$	$J_e = \infty$
$m_{12} = 0$	$m_{12} \in (0, 1)$	$m_{12} = 1$

Matchings for $n = 3$

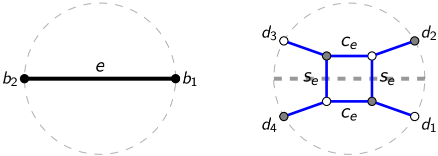


Plabic graphs



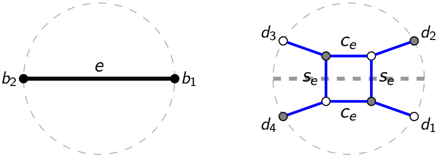


Plabic graphs

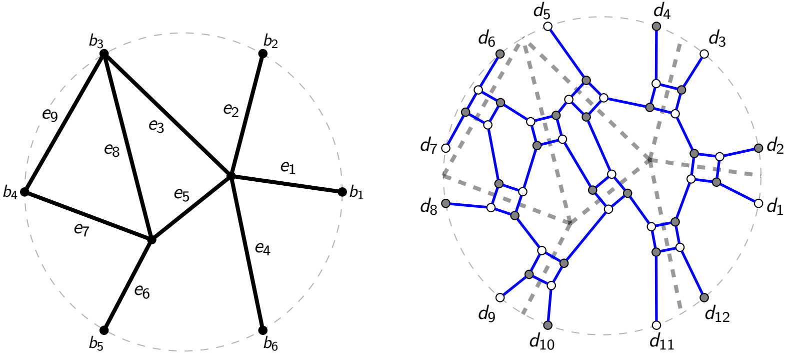


Here $s_e := \operatorname{sech}(2J_e)$, $c_e := \tanh(2J_e)$ so that $s_e^2 + c_e^2 = 1$.

Plabic graphs



Here $s_e := \operatorname{sech}(2J_e)$, $\quad c_e := \tanh(2J_e)$ so that $s_e^2 + c_e^2 = 1$.



Open problems

- Show that $\overline{\mathcal{X}}_n$ is a regular CW complex.

Open problems

- Show that $\overline{\mathcal{X}}_n$ is a regular CW complex. That is, show that the closure of $\mathcal{X}_\tau$ is a closed ball for any matching τ.

Open problems

- Show that $\overline{\mathcal{X}}_n$ is a regular CW complex. That is, show that the closure of $\mathcal{X}_\tau$ is a closed ball for any matching τ.
- Give a stratification-preserving homeomorphism between the space $\overline{\mathcal{X}}_n$ and Lam's compactification E_n of the space of electrical networks.

Open problems

- Show that $\overline{\mathcal{X}}_n$ is a regular CW complex. That is, show that the closure of $\mathcal{X}_\tau$ is a closed ball for any matching τ.
- Give a stratification-preserving homeomorphism between the space $\overline{\mathcal{X}}_n$ and Lam's compactification E_n of the space of electrical networks.
- What is the "scaling limit" of $X_0 \in \mathrm{OG}_{\geq 0}(n, 2n)$ as $n \to \infty$?

Open problems

- Show that $\overline{\mathcal{X}}_n$ is a regular CW complex. That is, show that the closure of $\mathcal{X}_\tau$ is a closed ball for any matching τ.
- Give a stratification-preserving homeomorphism between the space $\overline{\mathcal{X}}_n$ and Lam's compactification E_n of the space of electrical networks.
- What is the "scaling limit" of $X_0 \in \mathrm{OG}_{\geq 0}(n, 2n)$ as $n \to \infty$? In what sense is it "conformally invariant" or "universal"?

Open problems

- Show that $\overline{\mathcal{X}}_n$ is a regular CW complex. That is, show that the closure of $\mathcal{X}_\tau$ is a closed ball for any matching τ.
- Give a stratification-preserving homeomorphism between the space $\overline{\mathcal{X}}_n$ and Lam's compactification E_n of the space of electrical networks.
- What is the "scaling limit" of $X_0 \in \mathrm{OG}_{\geq 0}(n, 2n)$ as $n \to \infty$? In what sense is it "conformally invariant" or "universal"?
- Cluster algebra-like structure on $\mathrm{OG}_{\geq 0}(n, 2n)$?

Open problems

- Show that $\overline{\mathcal{X}}_n$ is a regular CW complex. That is, show that the closure of $\mathcal{X}_\tau$ is a closed ball for any matching τ.
- Give a stratification-preserving homeomorphism between the space $\overline{\mathcal{X}}_n$ and Lam's compactification E_n of the space of electrical networks.
- What is the "scaling limit" of $X_0 \in \mathrm{OG}_{\geq 0}(n, 2n)$ as $n \to \infty$? In what sense is it "conformally invariant" or "universal"?
- Cluster algebra-like structure on $\mathrm{OG}_{\geq 0}(n, 2n)$? Positivity tests?

Thank you!

Slides: http://math.mit.edu/~galashin/slides/umich_ising.pdf

Pavel Galashin and Pavlo Pylyavskyy.
Ising model and the positive orthogonal Grassmannian
arXiv:1807.03282.

Pavel Galashin, Steven N. Karp, and Thomas Lam.
The totally nonnegative Grassmannian is a ball.
arXiv:1707.02010.

Marcin Lis.
The planar Ising model and total positivity.
J. Stat. Phys., 166(1):72–89, 2017.

Alexander Postnikov.
Total positivity, Grassmannians, and networks.
arXiv:math/0609764.

Thomas Lam.
Electroid varieties and a compactification of the space of electrical networks
Advances in Mathematics, 338 (2018): 549-600.