Ising model and total positivity

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Joint work with Pavlo Pylyavskyy

`arXiv:1807.03282`

Part 1: Ising model

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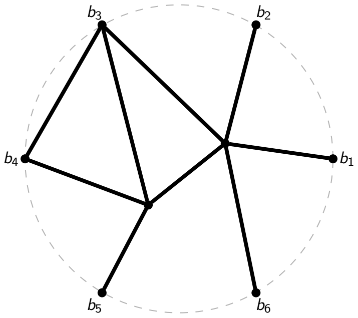

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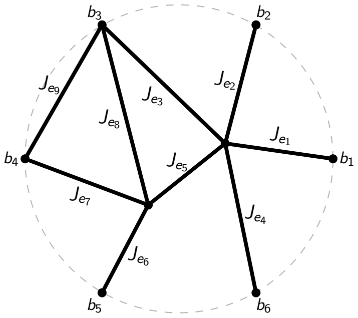

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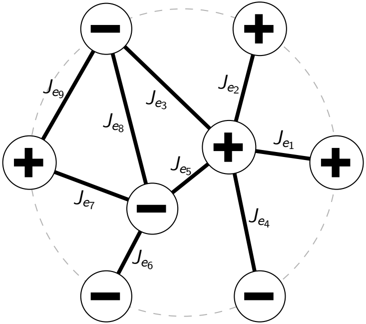

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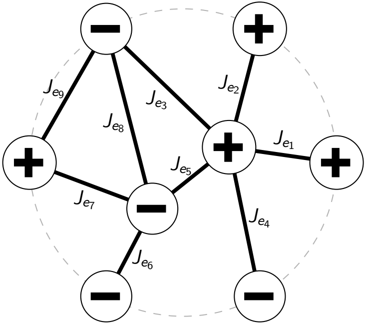

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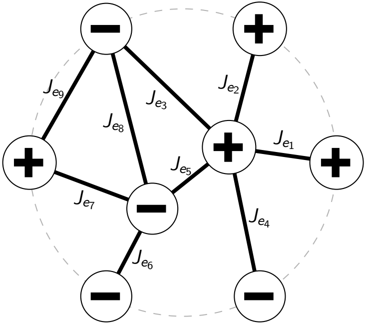

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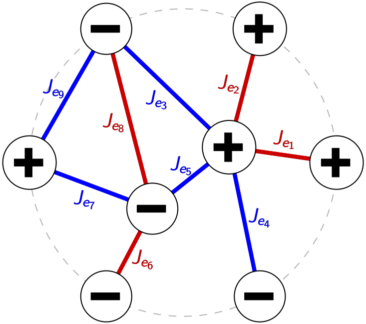

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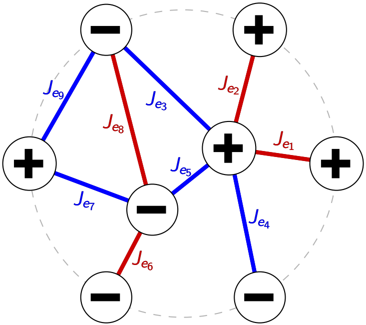

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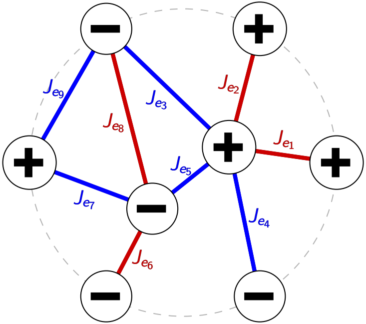

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$$\mathrm{Prob}(\sigma) := \frac{\mathrm{wt}(\sigma)}{Z}$$

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Theorem (G.–Pylyavskyy (2018))

Describe boundary correlations of the planar Ising model by inequalities.

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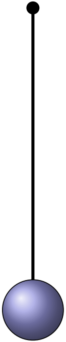

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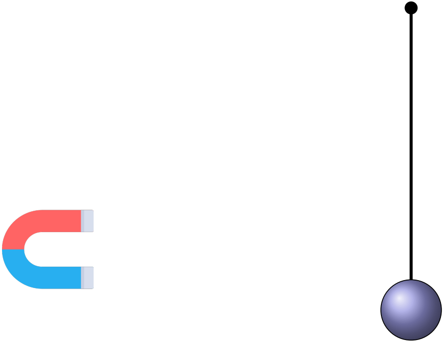

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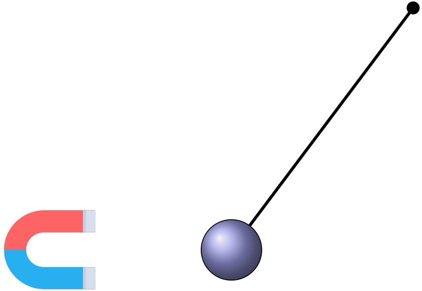

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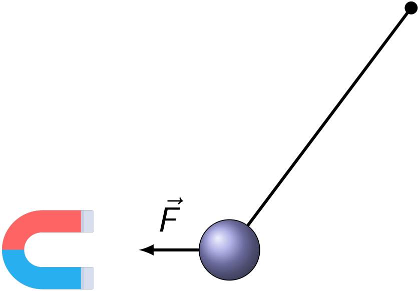

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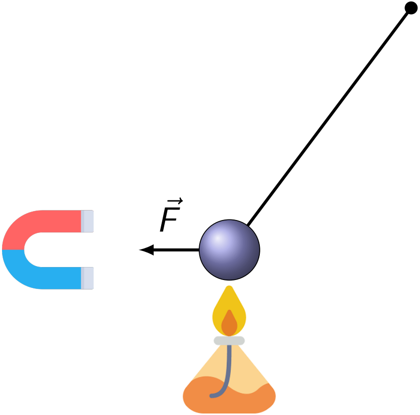

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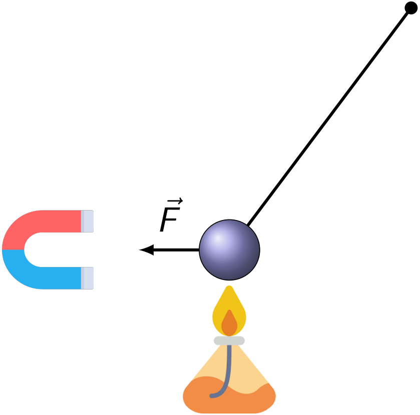

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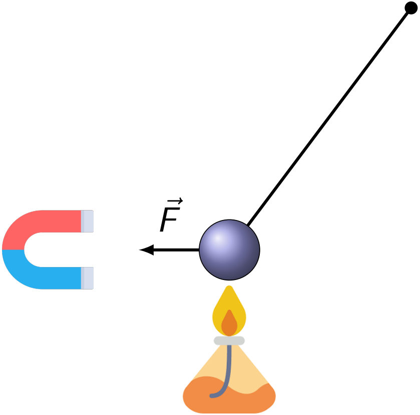

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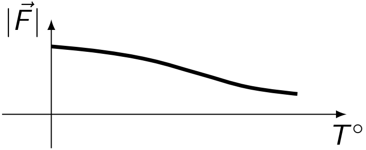

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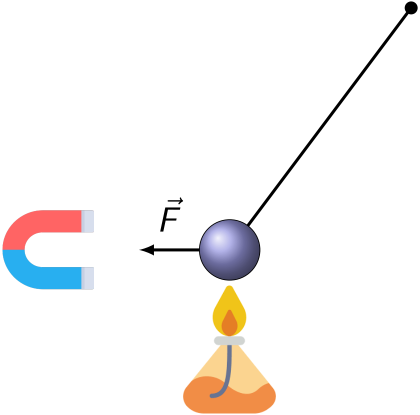

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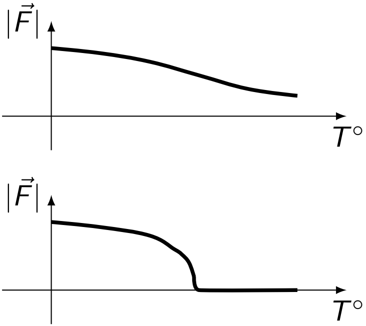

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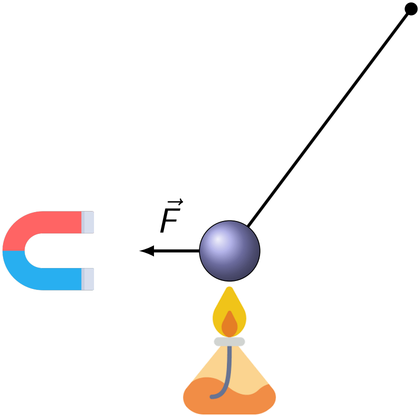

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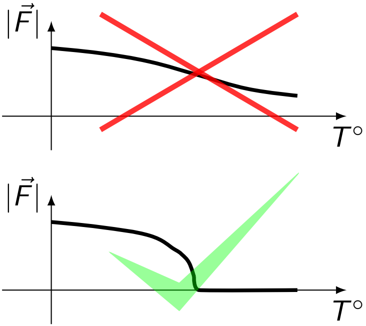

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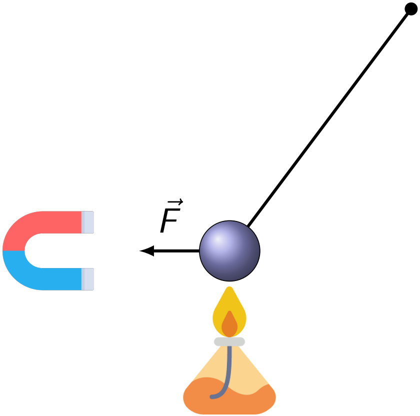

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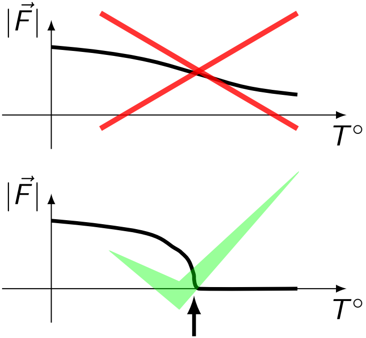

Curie point (P. Curie, 1895)

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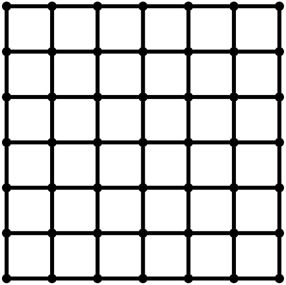

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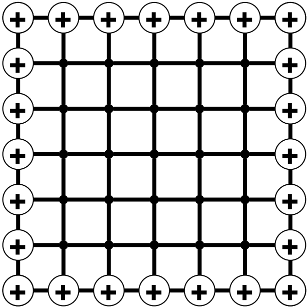

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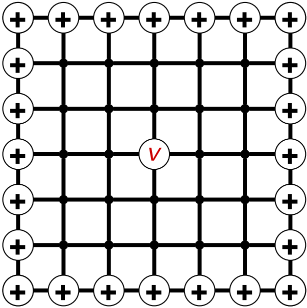

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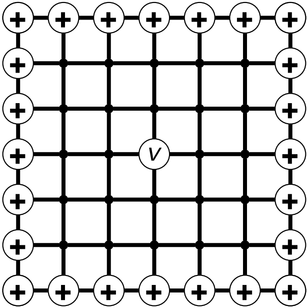
$$M(T) := \lim_{N \to \infty} (\mathrm{Prob}(\sigma_v = +1) - \mathrm{Prob}(\sigma_v = -1))$$



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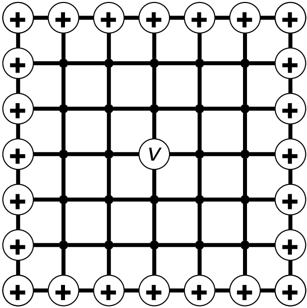

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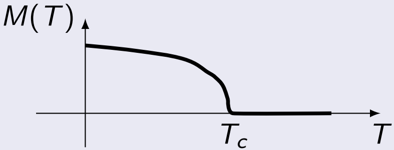

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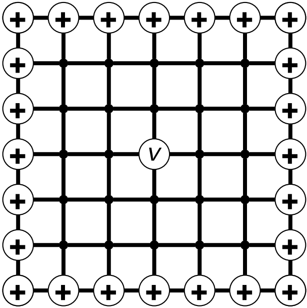

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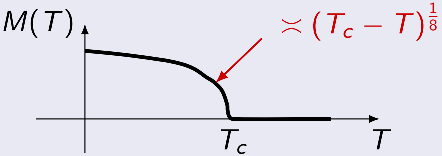

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Historically, we let $G := \mathbb{Z}^d \cap \Omega$ for some $\Omega \subset \mathbb{R}^d$ and set all $J_e := \frac{1}{T}$ for some temperature $T \in \mathbb{R}_{>0}$.

- Peierls (1937): phase transition in $\mathbb{Z}^d$ for $d \geq 2$
- Kramers–Wannier (1941): critical temperature $\frac{1}{T_c} = \frac{1}{2}\log\left(\sqrt{2}+1\right)$ for $\mathbb{Z}^2$
- Onsager, Kaufman, Yang (1944–1952): exact expressions for the free energy and spontaneous magnetization
- Belavin–Polyakov–Zamolodchikov (1984): conjectured conformal invariance of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$
- Smirnov, Chelkak, Hongler, Izyurov, ... (2010–2015): proved conformal invariance and universality of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$

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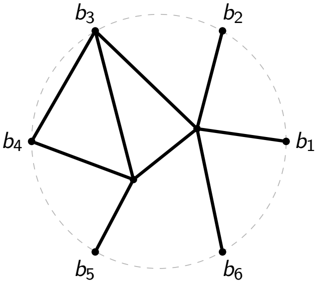

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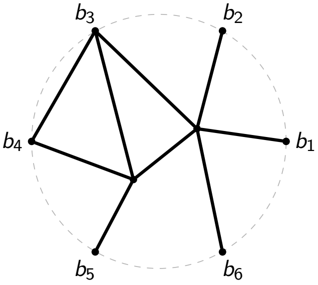

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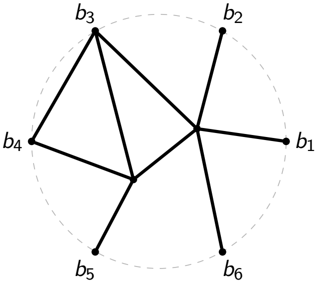

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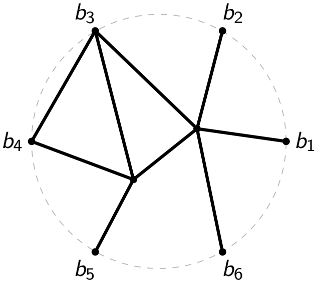

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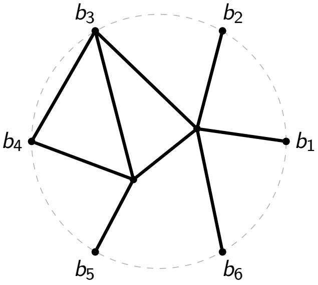

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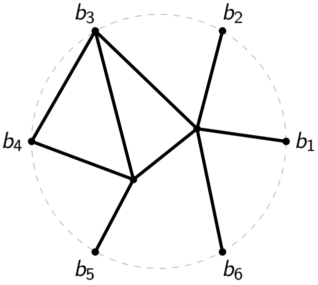

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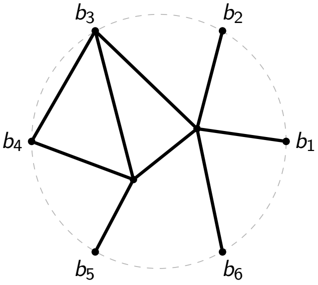

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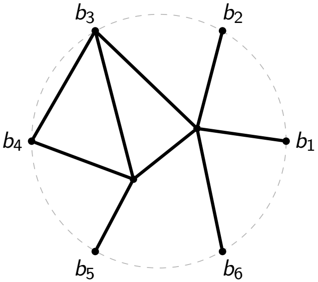

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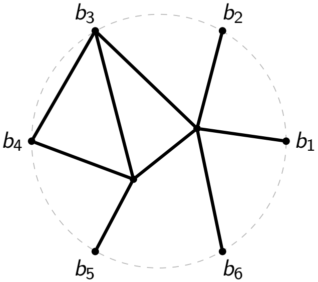

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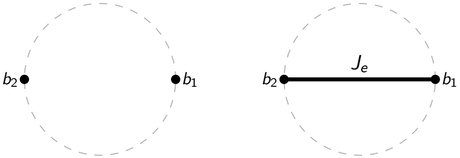

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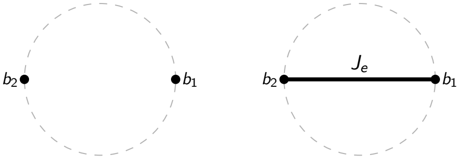

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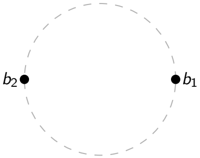

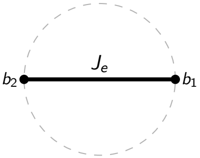

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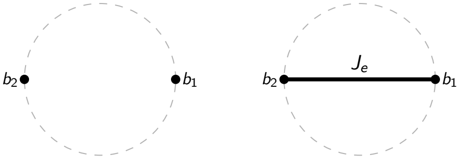

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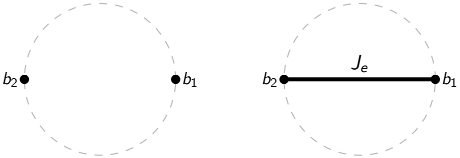

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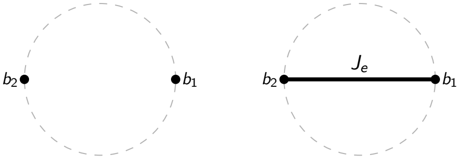

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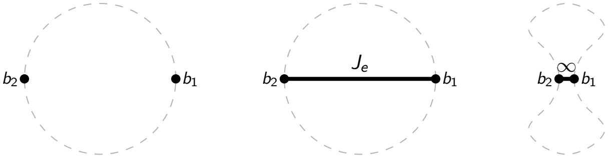

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Part 2: Total positivity

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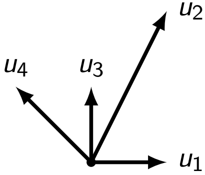

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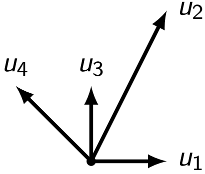

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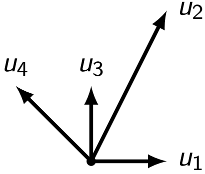

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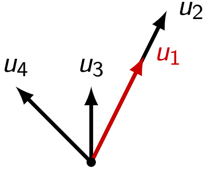

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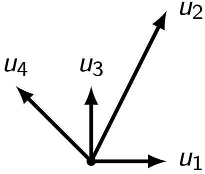

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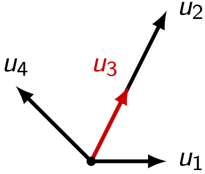

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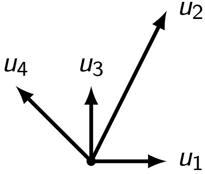

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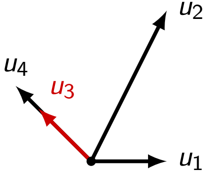

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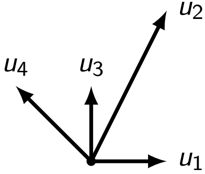

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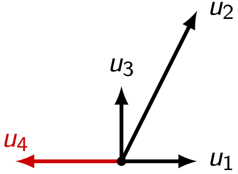

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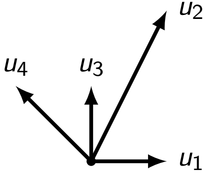

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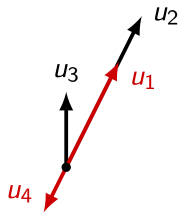

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Let C be a compact contractible topological manifold whose boundary is homeomorphic to a sphere. Then C is homeomorphic to a closed ball.

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Theorem (G.–Pylyavskyy (2018))

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Main result

$\mathcal{X}_n := \{M(G,J) \mid (G,J) \text{ is a planar Ising network with } n \text{ boundary vertices}\}$
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We have $\mathcal{X}_n, \overline{\mathcal{X}}_n \subset \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R},1) := \left\{ \begin{array}{c} \text{symmetric } n \times n \text{ matrices} \\ \text{with 1's on the diagonal} \end{array} \right\}$.

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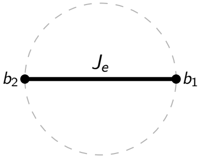

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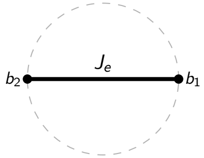

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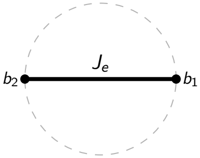

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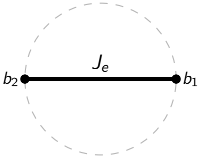

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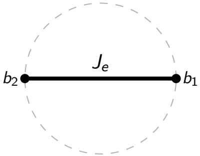

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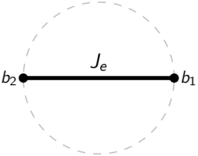

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Kramers–Wannier's duality

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Historically, we let $G := \mathbb{Z}^d \cap \Omega$ for some $\Omega \subset \mathbb{R}^d$
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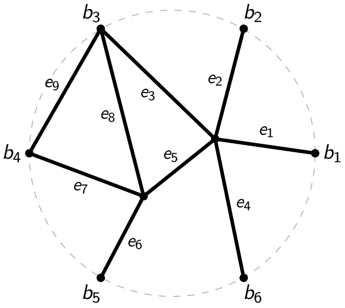

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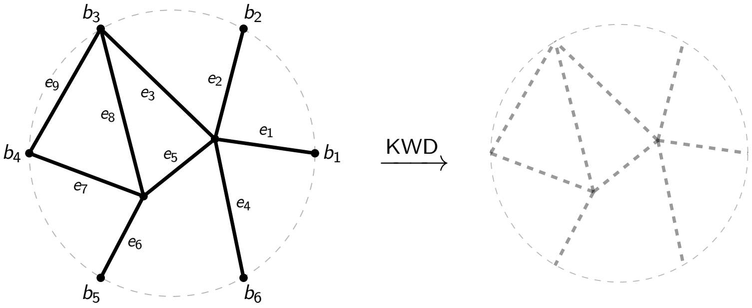

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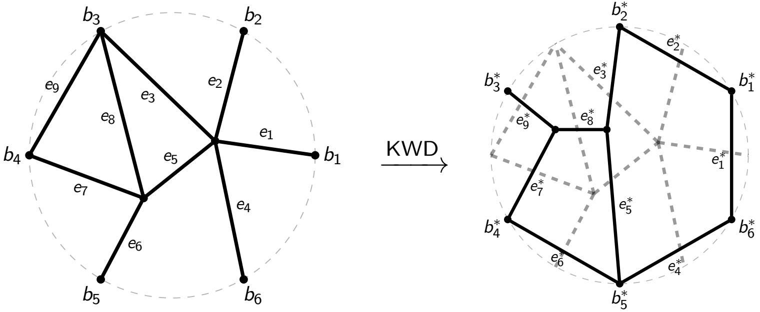

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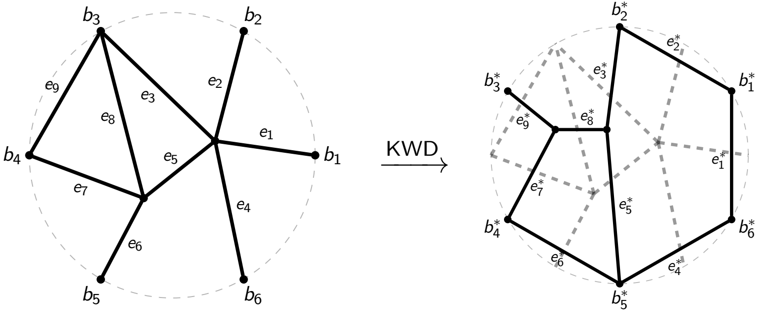

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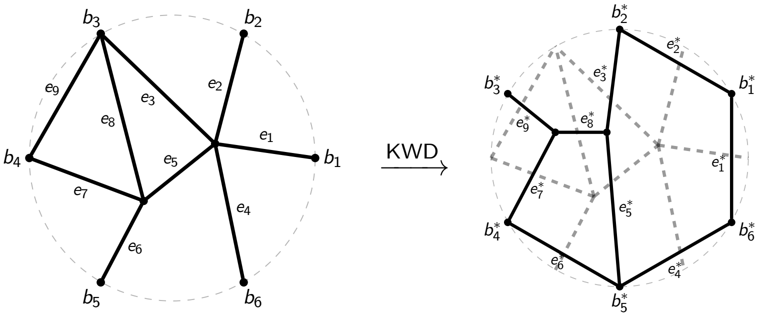

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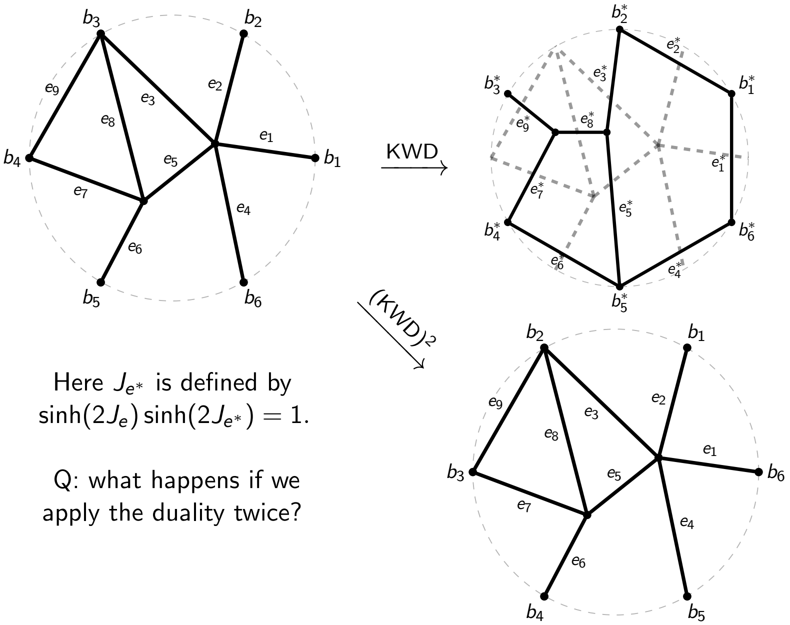

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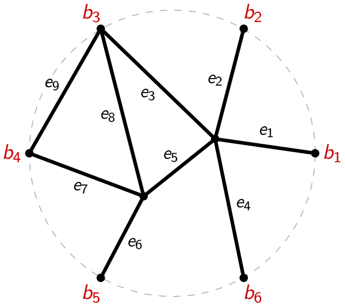

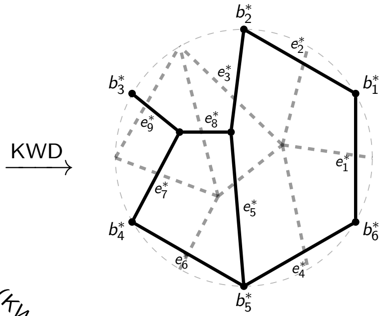

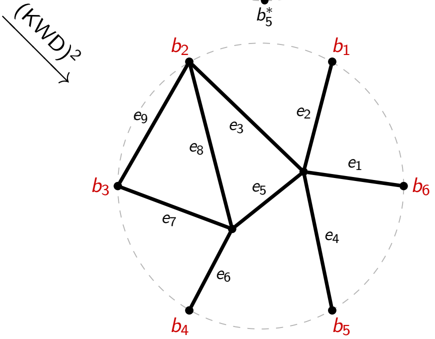

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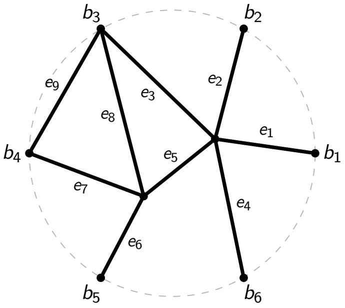

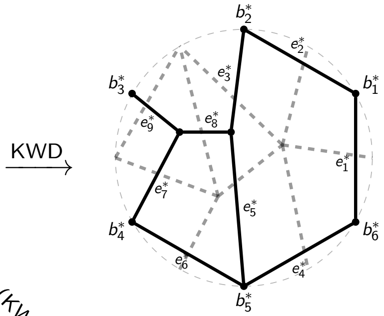

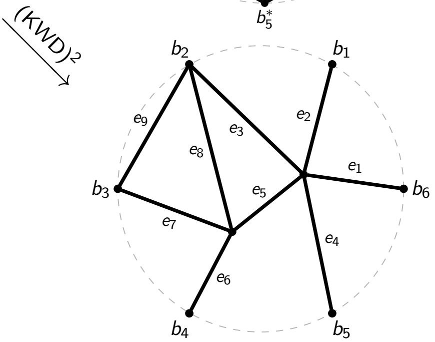

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$(\mathrm{KWD})^2 =$ cyclic shift!

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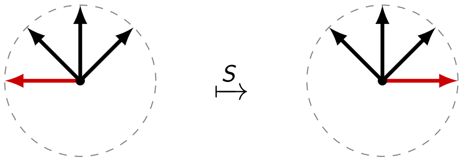

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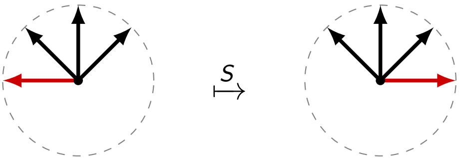
Example: For $\mathrm{Gr}_{\geq 0}(2,4)$, we have

$$X_0 = \begin{pmatrix} 1 & 0 & -1 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 & 0 \end{pmatrix} \overset{S}{\mapsto} \begin{pmatrix} \sqrt{2} & 1 & 0 & -1 \\ 0 & 1 & \sqrt{2} & 1 \end{pmatrix} = X_0 \in \mathrm{Gr}_{\geq 0}(2,4)$$



Cyclic shift on $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (G.–Karp–Lam (2017))

$\mathrm{Gr}_{\geq 0}(k, n)$ is homeomorphic to a $k(n-k)$-dimensional closed ball.

Our proof involves a flow that contracts the whole $\mathrm{Gr}_{\geq 0}(k, n)$ to the unique cyclically symmetric point $X_0 \in \mathrm{Gr}_{\geq 0}(k, n)$.

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Kramers–Wannier's duality vs. cyclic shift

Theorem (G.–Pylyavskyy (2018))

- *The map ϕ restricts to a homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n, 2n)$.*
- *Each of the spaces is homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*

$$\begin{array}{ccc} \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) & \underset{\phi}{\hookrightarrow} & \mathrm{OG}(n, 2n) \\ \big\uparrow & & \big\uparrow \\ \overline{\mathcal{X}}_n & \xrightarrow[\phi]{\sim} & \mathrm{OG}_{\geq 0}(n, 2n) \end{array}$$

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- *There exists a unique matrix $M_0 \in \overline{\mathcal{X}}_n$ that is fixed by KWD.*

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\Cup & & \Cup \\
M_0 & \longmapsto & X_0
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Kramers–Wannier's duality vs. cyclic shift

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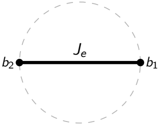
Example:



$$M_0 \leftrightarrow J_e = \frac{1}{2}\log(\sqrt{2}+1)$$

Kramers–Wannier's duality vs. cyclic shift

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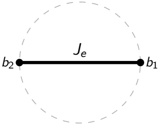

$$M_0 \leftrightarrow J_e = \frac{1}{2} \log(\sqrt{2} + 1)$$

Fixed point M_0 of KWD $\leftrightarrow$ Ising model at critical temperature $\leftrightarrow$ X_0?

Kramers–Wannier's duality vs. cyclic shift

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Kramers–Wannier's duality vs. cyclic shift

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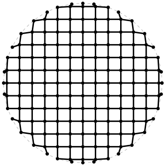
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

Kramers–Wannier's duality vs. cyclic shift

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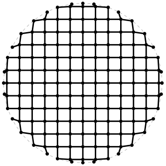
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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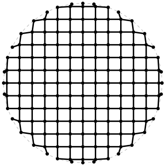
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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Proposition (G.–Pylyavskyy (2018))

The entries of $M_0 = (m_{ij})_{i,j=1}^{n}$ *are given by* $m_{ij} = \dfrac{\sum_I \Delta_I(X_0)}{\sum_{I'} \Delta_{I'}(X_0)}$.

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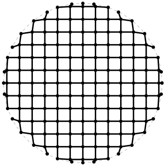
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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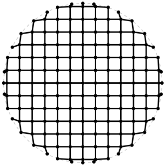
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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Question

How close is M_0 to $M(G,J)$?

Kramers–Wannier's duality vs. cyclic shift

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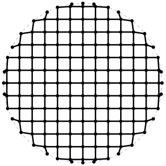
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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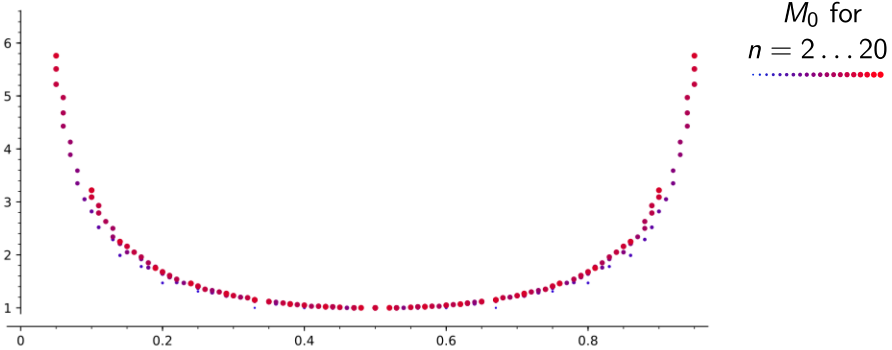
Question

How close is M_0 *to* $M(G, J)$*? Do they have the same scaling limit?*



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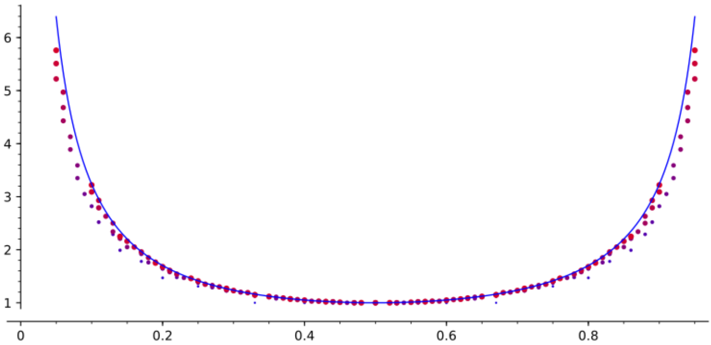
Kramers–Wannier's duality vs. cyclic shift



M_0 for $n = 2 \dots 20$

$M(G, J)$ as $n \to \infty$

$\sqrt{\frac{2}{1-\cos(t)}}$

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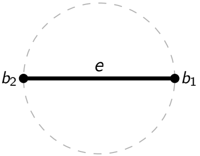
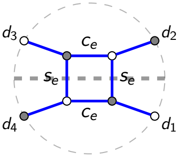
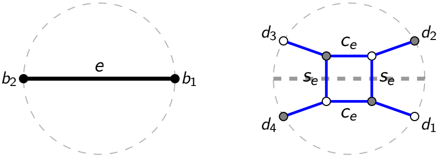
Open problems

Ising network → planar bipartite graph



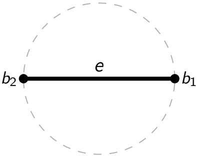


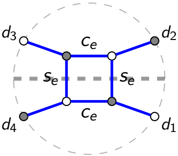
Ising network → planar bipartite graph



Here $s_e := \operatorname{sech}(2J_e)$, $c_e := \tanh(2J_e)$ so that $s_e^2 + c_e^2 = 1$.

Ising network → planar bipartite graph





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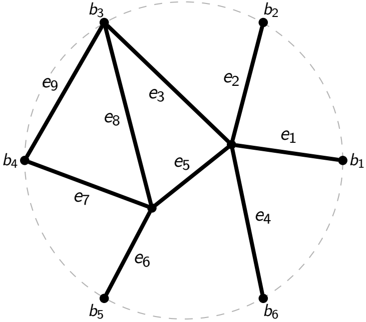
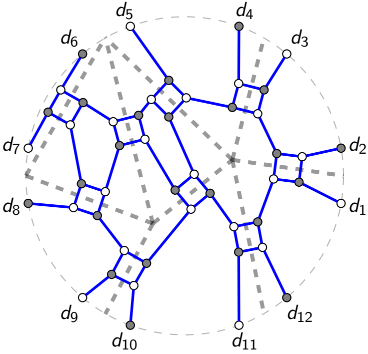
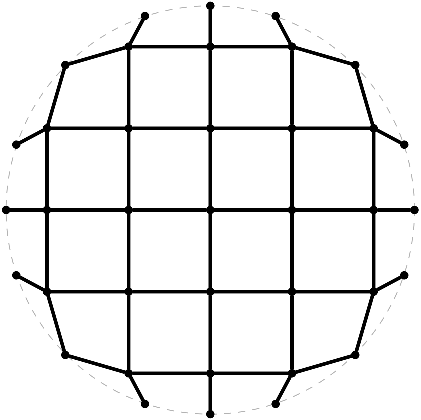




Random almost perfect matchings



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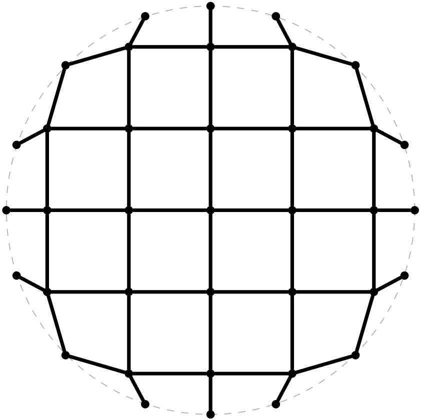

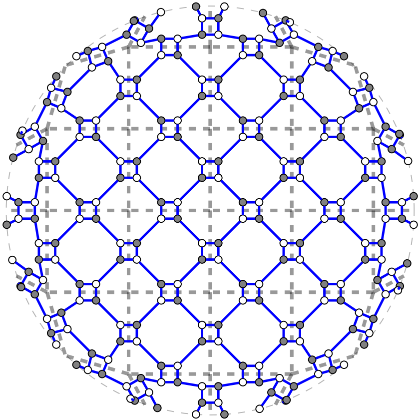

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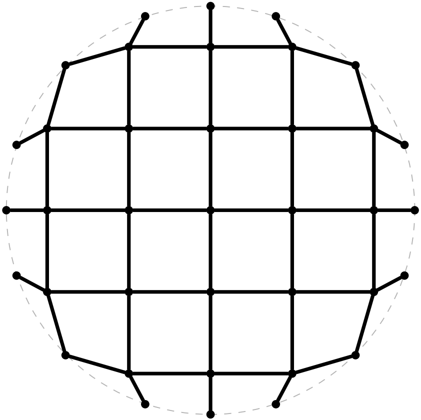
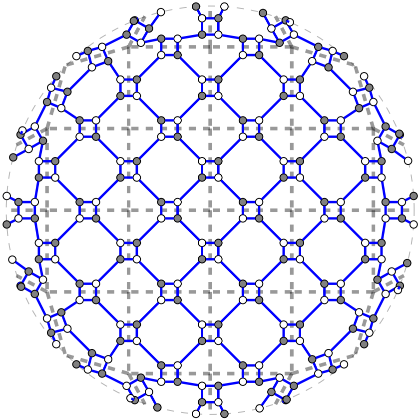
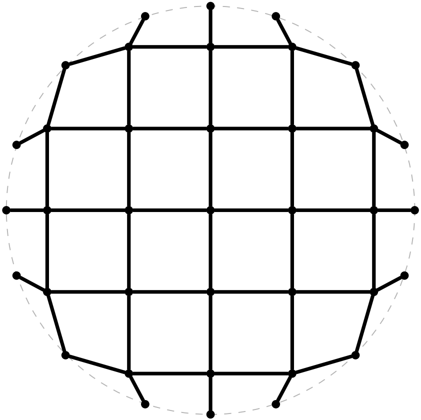


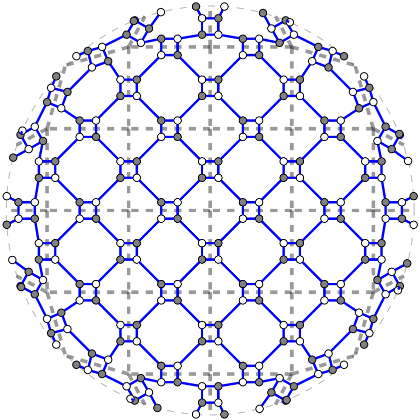


Question

- *What is the* *shape* *of a random almost perfect matching?*

Random almost perfect matchings

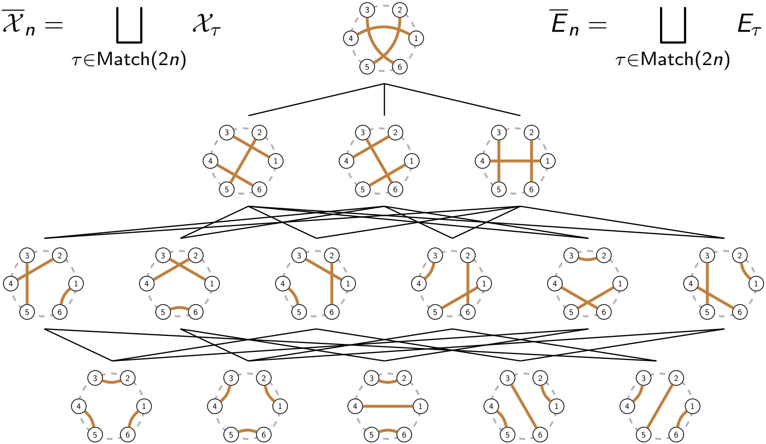




Question

- *What is the shape of a random almost perfect matching?*
- *Is there a phase transition?*

$$\overline{\mathcal{X}}_n = \bigsqcup_{\tau \in \text{Match}(2n)} \mathcal{X}_\tau \qquad \overline{E}_n = \bigsqcup_{\tau \in \text{Match}(2n)} E_\tau$$



Ising model vs. Electrical networks

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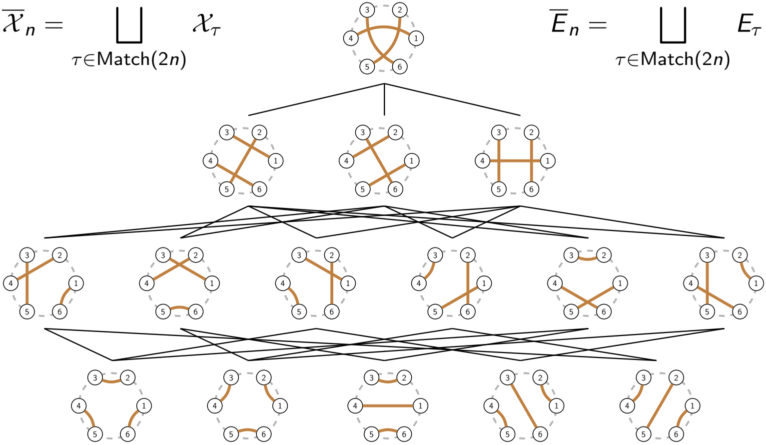

Problem

- *Construct a stratification-preserving homeomorphism between* $\overline{\mathcal{X}}_n$ *and* $\overline{E}_n$*.*

Ising model vs. Electrical networks

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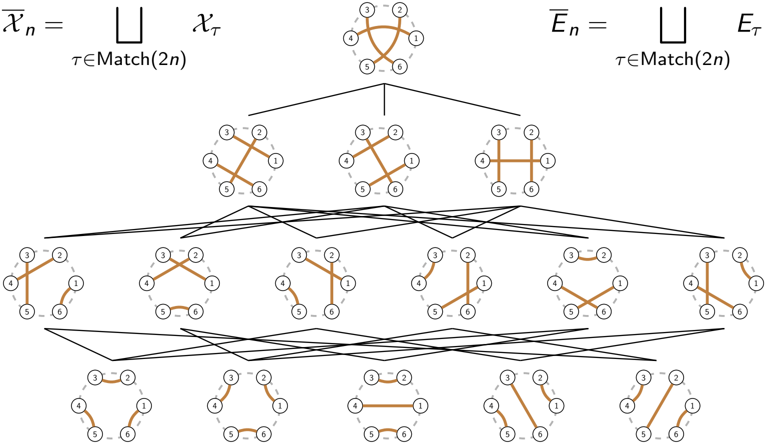

Problem

- *Construct a stratification-preserving homeomorphism between $\overline{\mathcal{X}}_n$ and $\overline{E}_n$.*
- *Show that the closure of $\mathcal{X}_\tau$ and of E_τ is a ball.*

Thank you!

Slides: http://math.mit.edu/~galashin/slides/uiuc_ising.pdf

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J. Stat. Phys., 166(1):72–89, 2017.

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Electroid varieties and a compactification of the space of electrical networks
Advances in Mathematics, 338 (2018): 549-600.