Ising model and total positivity

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Joint work with Pavlo Pylyavskyy

`arXiv:1807.03282`

Part 1: Ising model

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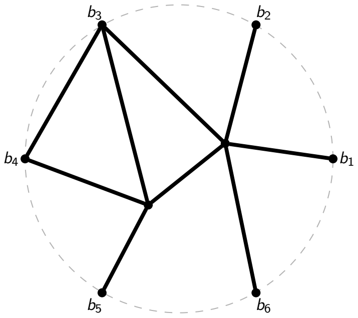

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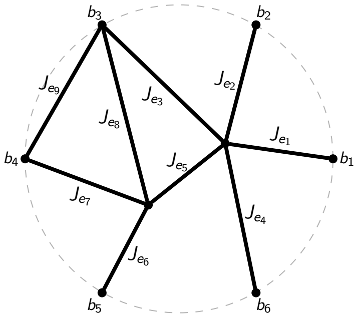

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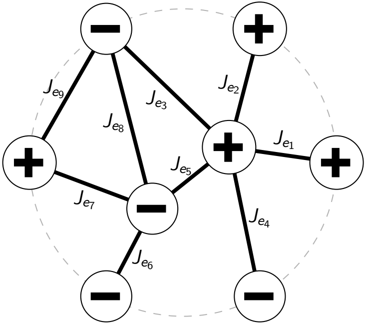

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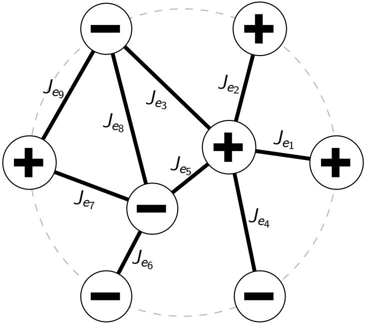

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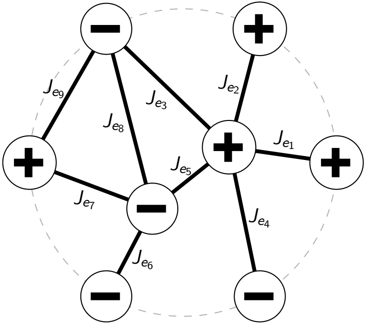

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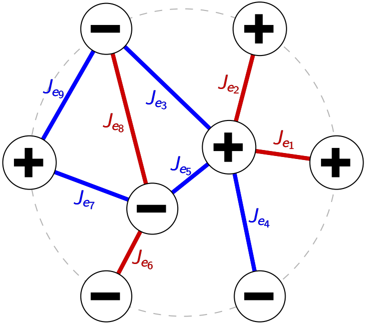

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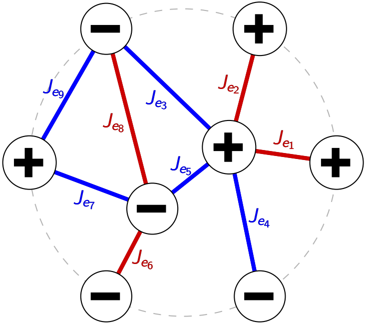

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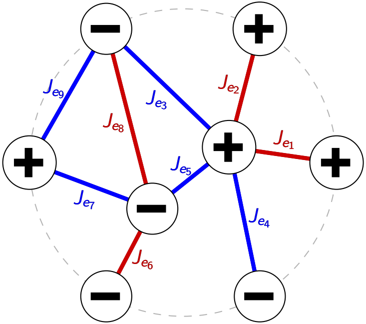

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$$\mathrm{Prob}(\sigma) := \frac{\mathrm{wt}(\sigma)}{Z}$$

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Theorem (G.–Pylyavskyy (2018))

Describe boundary correlations of the planar Ising model by inequalities.

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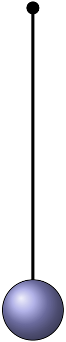

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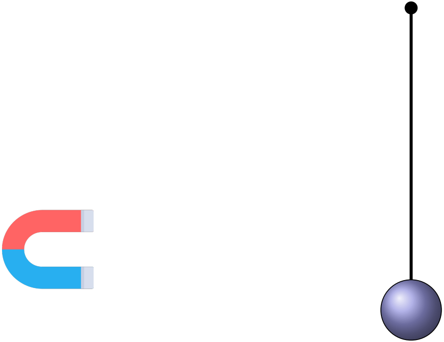

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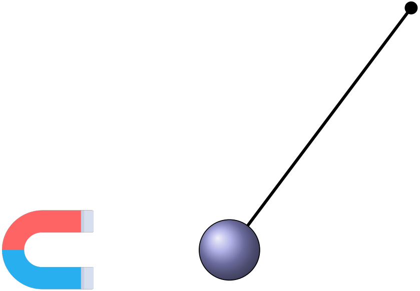

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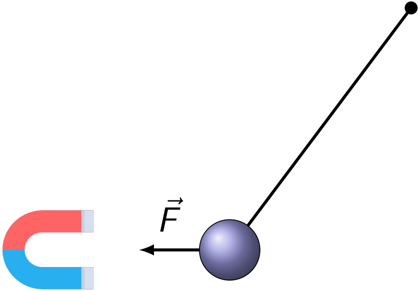

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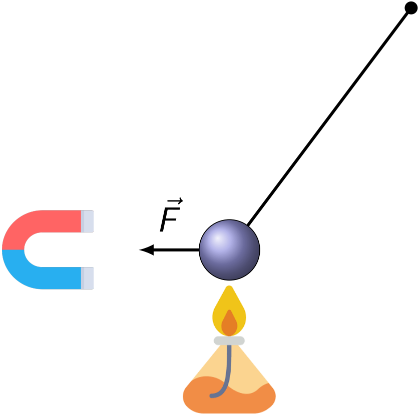

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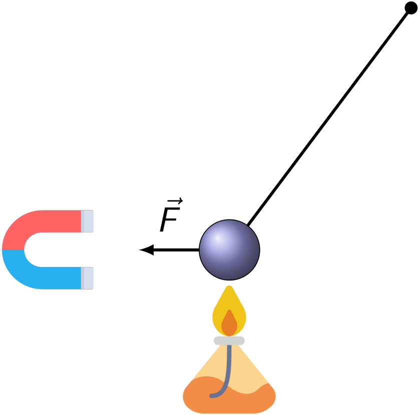

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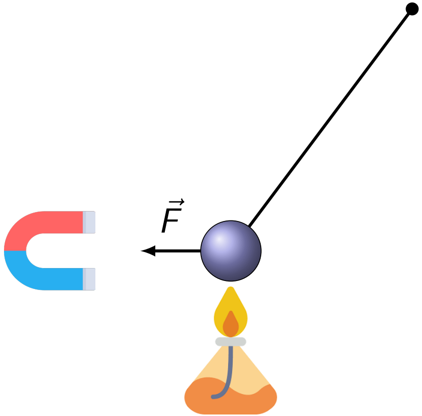

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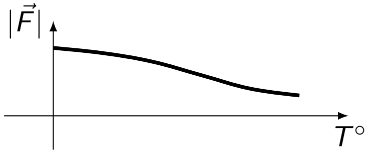

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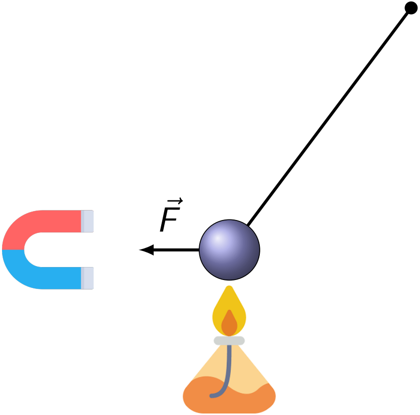

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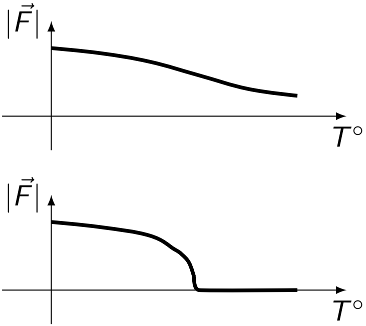

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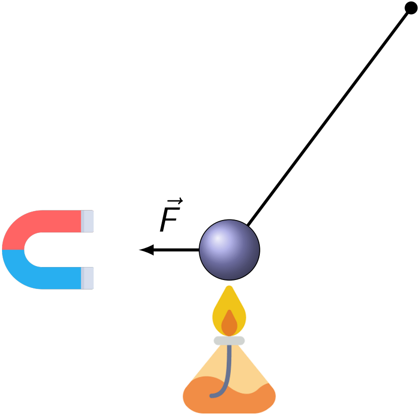

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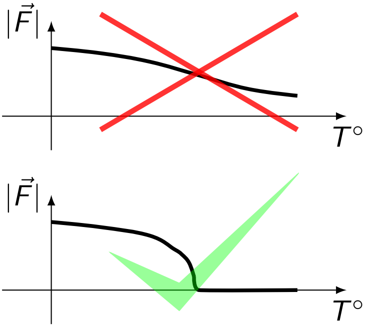

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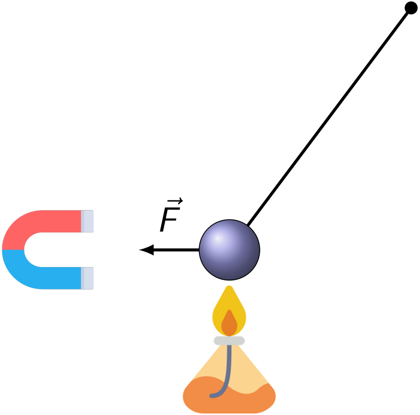

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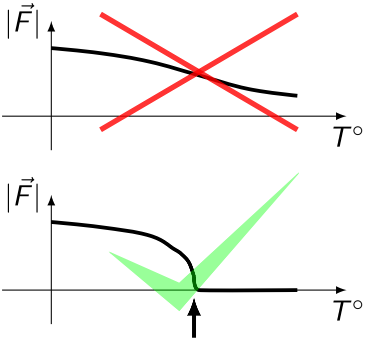

Curie point (P. Curie, 1895)

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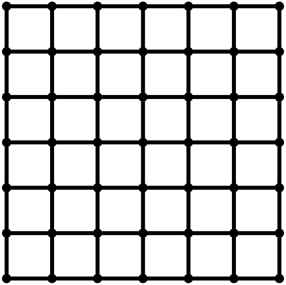

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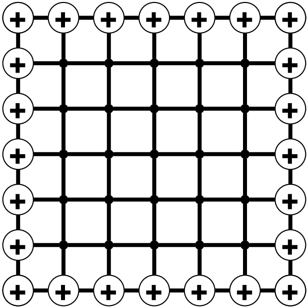

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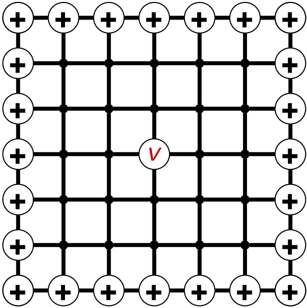

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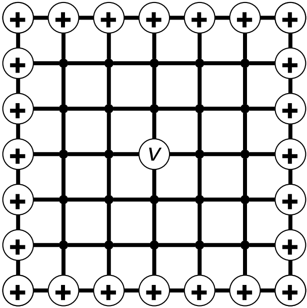
Define the spontaneous magnetization
$$M(T) := \lim_{N \to \infty} (\mathrm{Prob}(\sigma_v = +1) - \mathrm{Prob}(\sigma_v = -1))$$



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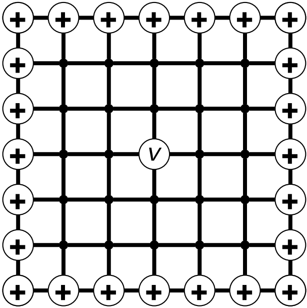

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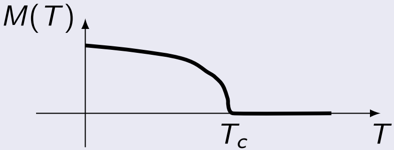

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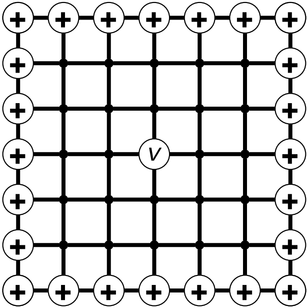

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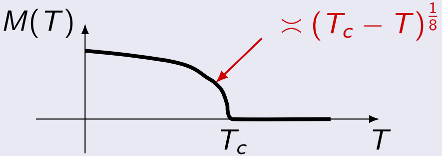

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- Belavin–Polyakov–Zamolodchikov (1984): conjectured conformal invariance of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$

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Historically, we let $G := \mathbb{Z}^d \cap \Omega$ for some $\Omega \subset \mathbb{R}^d$ and set all $J_e := \frac{1}{T}$ for some temperature $T \in \mathbb{R}_{>0}$.

- Peierls (1937): phase transition in $\mathbb{Z}^d$ for $d \geq 2$
- Kramers–Wannier (1941): critical temperature $\frac{1}{T_c} = \frac{1}{2}\log\left(\sqrt{2}+1\right)$ for $\mathbb{Z}^2$
- Onsager, Kaufman, Yang (1944–1952): exact expressions for the free energy and spontaneous magnetization
- Belavin–Polyakov–Zamolodchikov (1984): conjectured conformal invariance of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$
- Smirnov, Chelkak, Hongler, Izyurov, ... (2010–2015): proved conformal invariance and universality of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$

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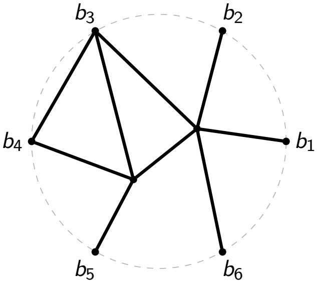

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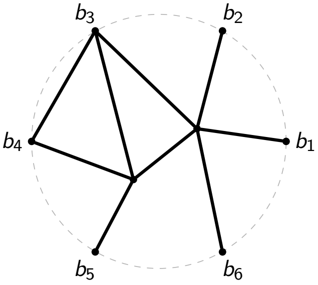

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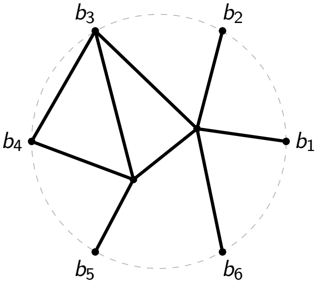

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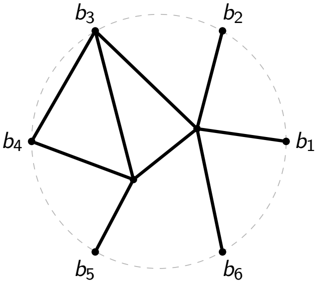

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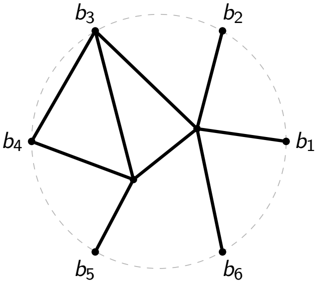

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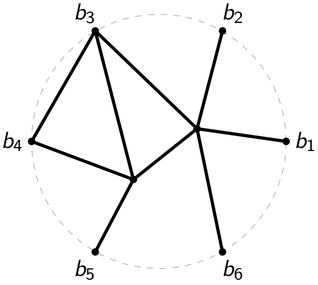

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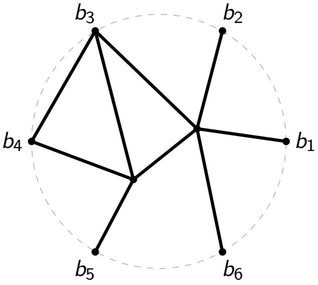

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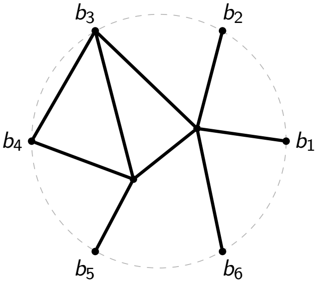

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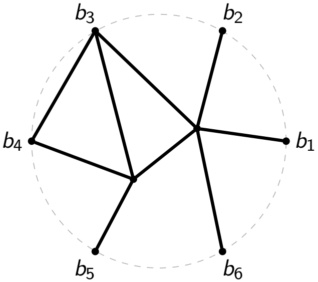

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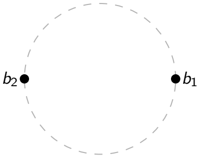

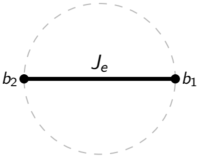

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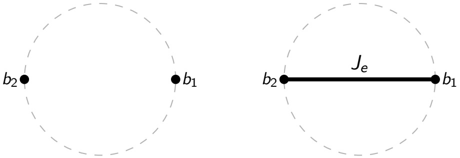

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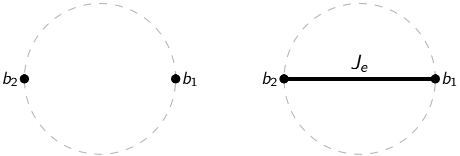

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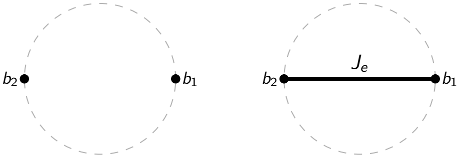

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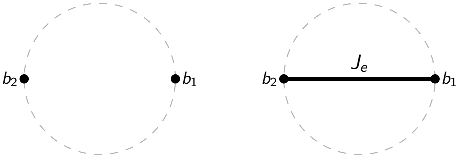

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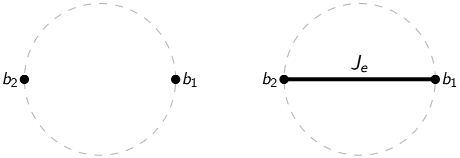

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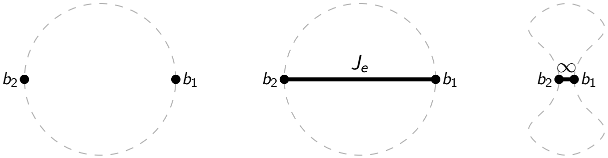

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Part 2: Total positivity

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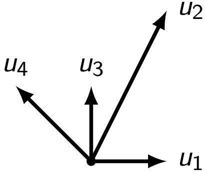

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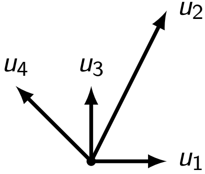

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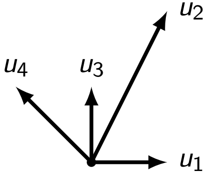

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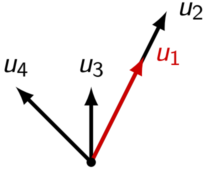

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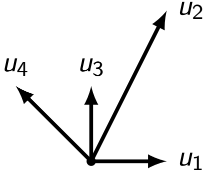

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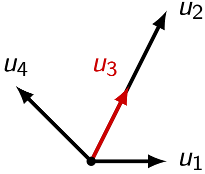

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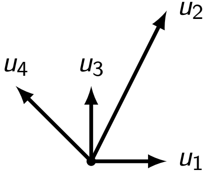

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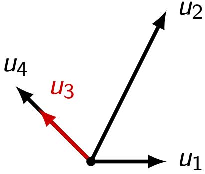

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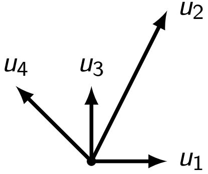

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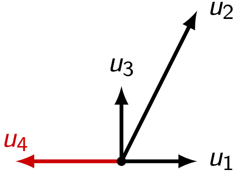

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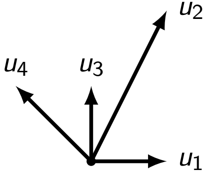

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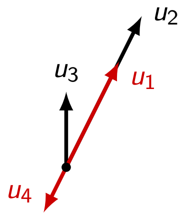

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Theorem (Smale (1960), Freedman (1982), Perelman (2003))

Let C be a compact contractible topological manifold whose boundary is homeomorphic to a sphere. Then C is homeomorphic to a closed ball.

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Main result

$\mathcal{X}_n := \{M(G,J) \mid (G,J)$ is a planar Ising network with n boundary vertices$\}$
$\overline{\mathcal{X}}_n :=$ closure of $\mathcal{X}_n$ inside the space of $n \times n$ matrices.
We have $\mathcal{X}_n, \overline{\mathcal{X}}_n \subset \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R},1) := \left\{ \begin{array}{c} \text{symmetric } n\times n \text{ matrices} \\ \text{with 1's on the diagonal} \end{array} \right\}$.

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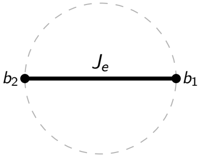

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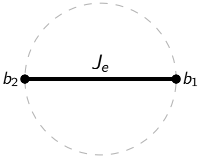

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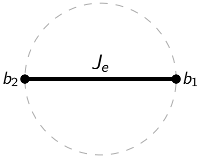

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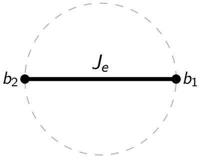

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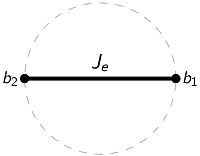

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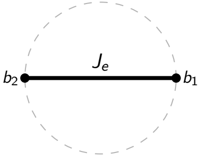

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Ising model: history

- Suggested by by W. Lenz to his student E. Ising in 1920
- Ising (1925): no phase transition in 1D $\Longrightarrow$ not a good model for ferromagnetism

Historically, we let $G := \mathbb{Z}^d \cap \Omega$ for some $\Omega \subset \mathbb{R}^d$
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- Kramers–Wannier (1941): critical temperature $\frac{1}{T_c} = \frac{1}{2}\log\left(\sqrt{2}+1\right)$ for $\mathbb{Z}^2$
- Onsager, Kaufman, Yang (1944–1952): exact expressions for the free energy and spontaneous magnetization
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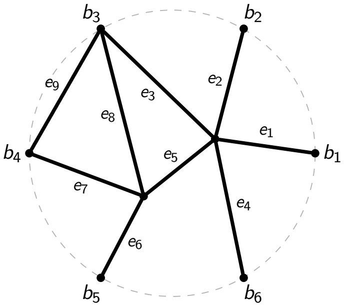

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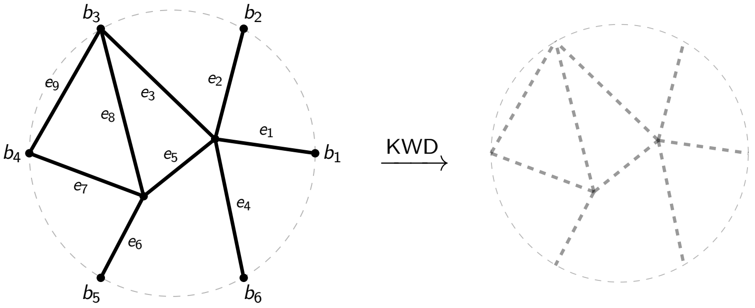

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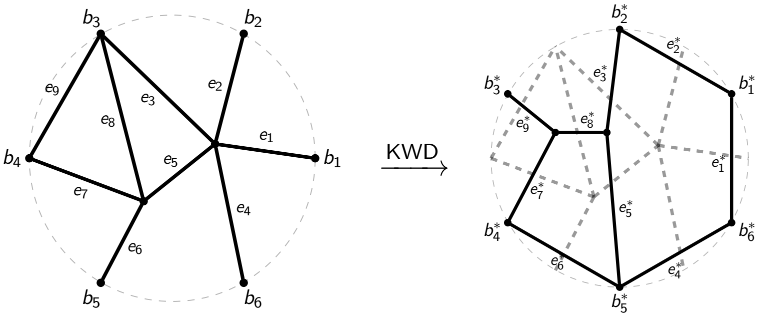

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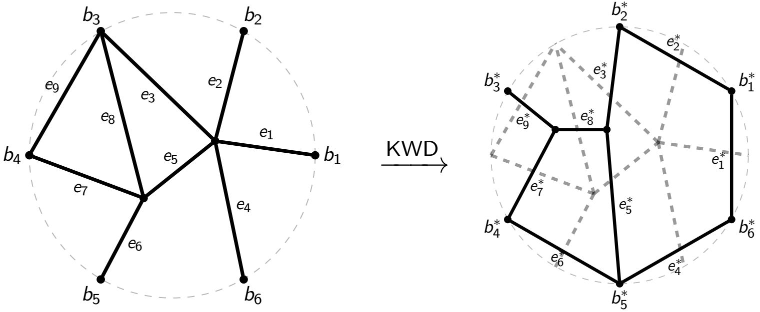

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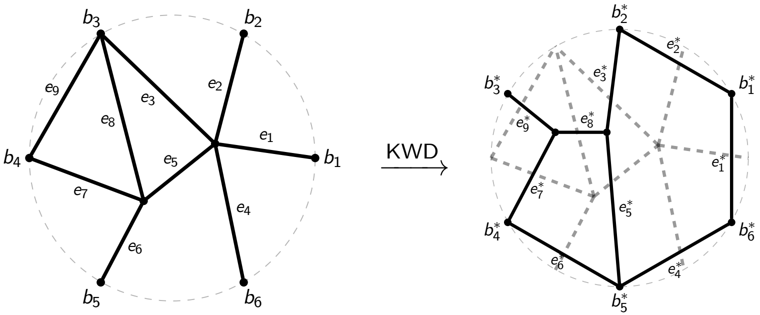

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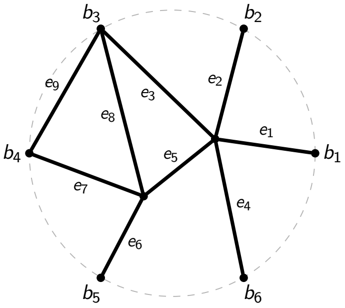

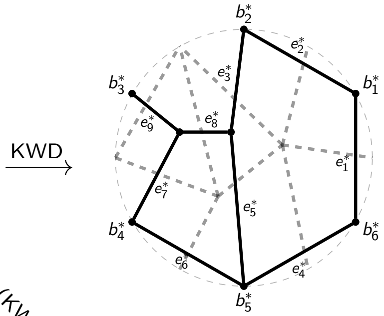

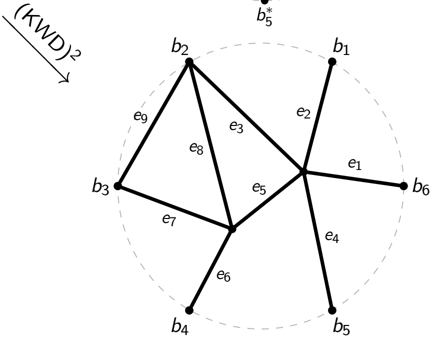

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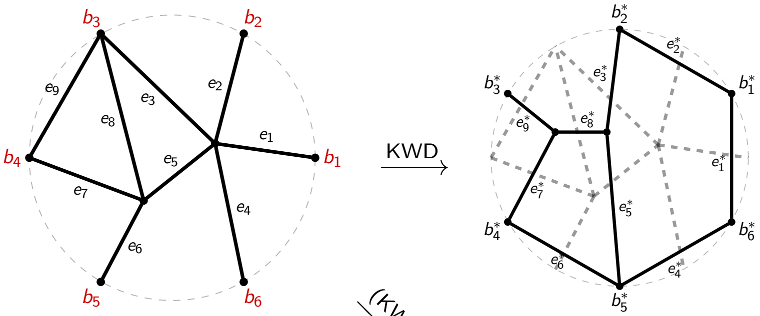

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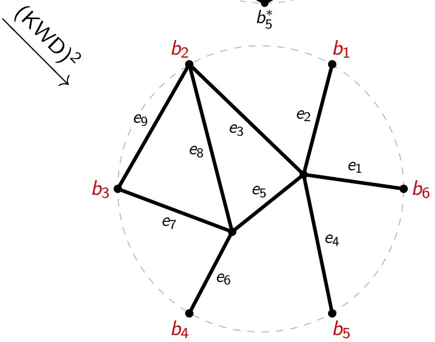

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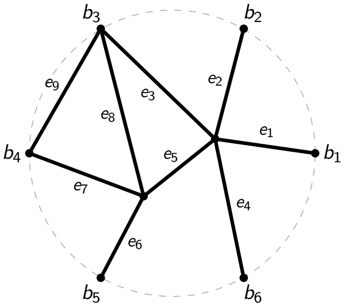

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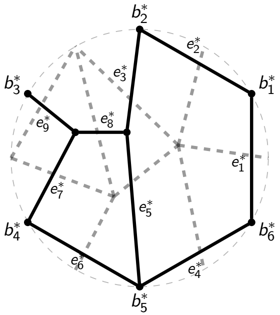

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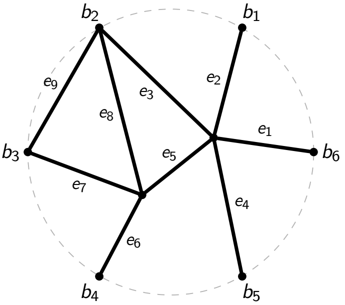

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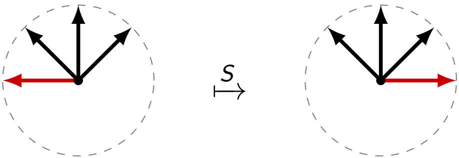

Cyclic shift on $\mathrm{Gr}_{\geq 0}(k,n)$

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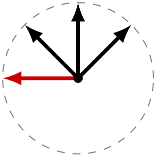

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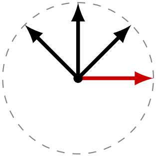

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Theorem (G.–Pylyavskyy (2018))

- *The map ϕ restricts to a homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n, 2n)$.*
- *Each of the spaces is homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*

$$\begin{array}{ccc} \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) & \xhookrightarrow[\phi]{} & \mathrm{OG}(n, 2n) \\ \uparrow & & \uparrow \\ \overline{\mathcal{X}}_n & \xrightarrow[\phi]{\sim} & \mathrm{OG}_{\geq 0}(n, 2n) \end{array}$$

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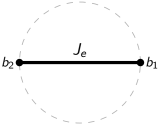
Example: 

$$M_0 \leftrightarrow J_e = \frac{1}{2} \log(\sqrt{2} + 1)$$

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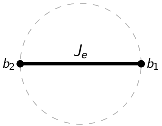

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Fixed point M_0 of KWD $\leftrightarrow$ Ising model at critical temperature $\leftrightarrow$ X_0?

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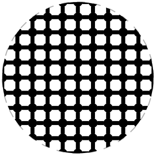
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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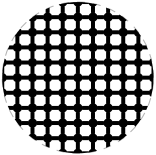
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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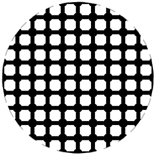
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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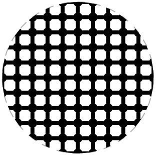
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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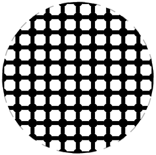
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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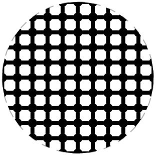
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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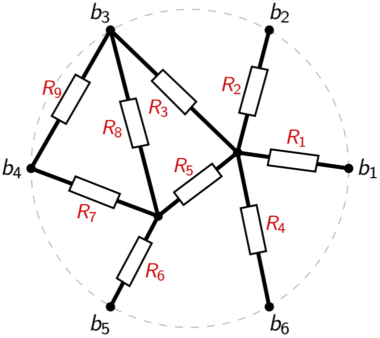

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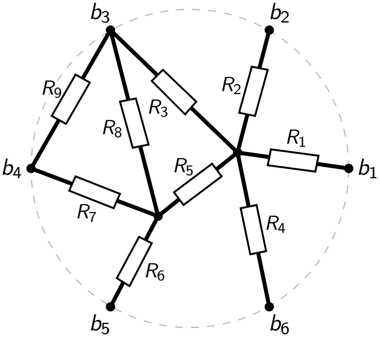
Electrical response matrix $\Lambda(G, R) : \mathbb{R}^n \to \mathbb{R}^n$, sending voltages to currents.



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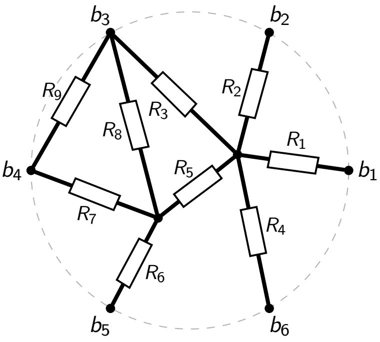

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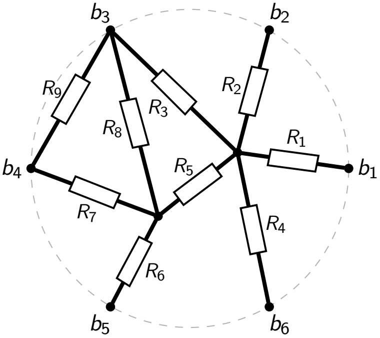

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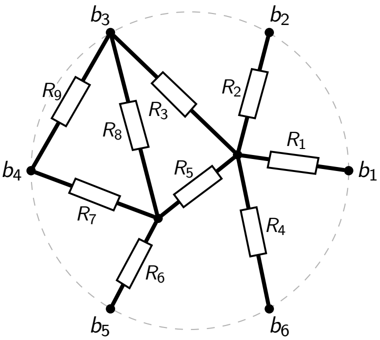

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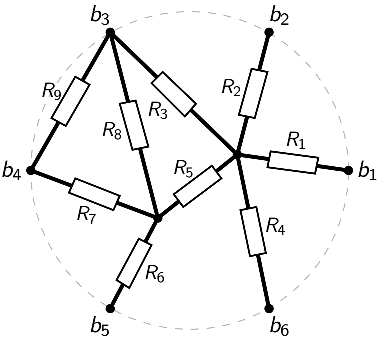

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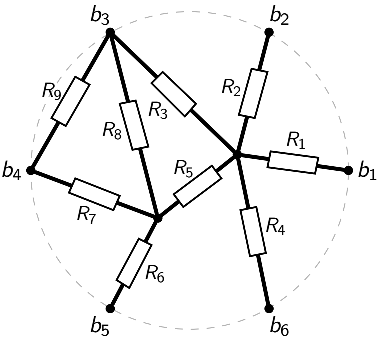

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$\overline{E}_n$: compactification of the space of $n \times n$ electrical response matrices [Lam (2014)]

Ising networks vs. Electrical networks

$\overline{\mathcal{X}}_n$: space of $n \times n$ boundary correlation matrices of planar Ising networks
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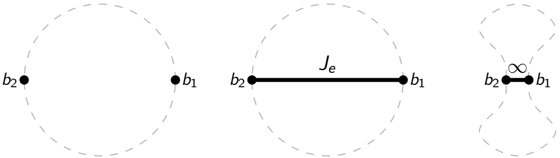

$$M(G,J) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}$$

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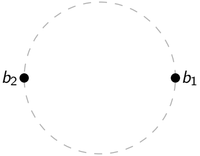

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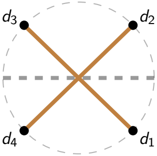

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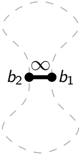

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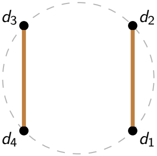

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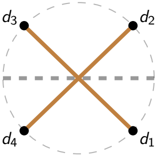

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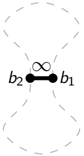

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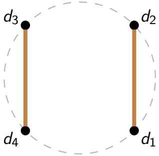
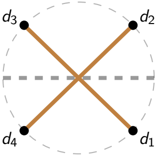
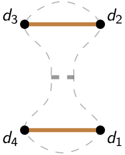
$$M(G,J) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}$$



$J_e = 0$
$m_{12} = 0$

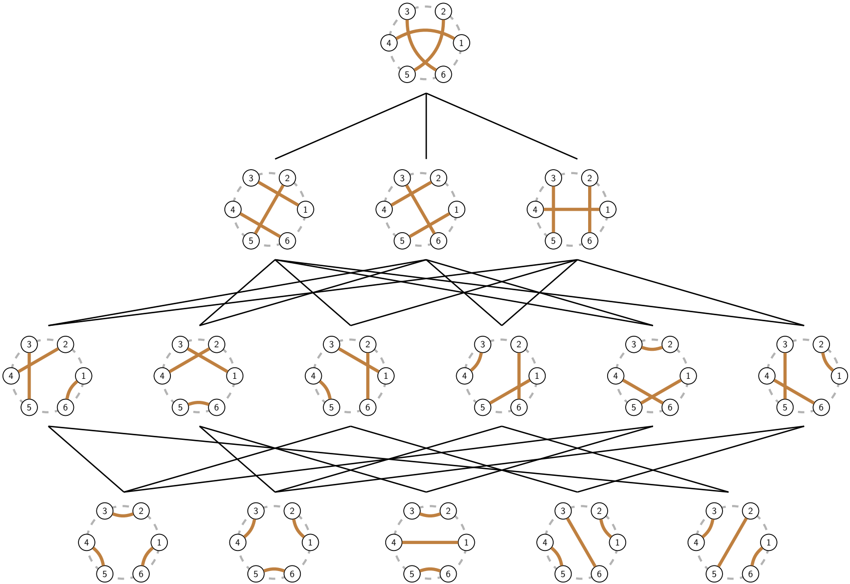


$J_e \in (0, \infty)$
$m_{12} \in (0, 1)$



$J_e = \infty$
$m_{12} = 1$

Stratification: $n = 3$



Ising networks vs. Electrical networks

$\overline{\mathcal{X}}_n$: space of $n \times n$ boundary correlation matrices of planar Ising networks
$\overline{E}_n$: compactification of the space of $n \times n$ electrical response matrices

- Stratification: $\overline{\mathcal{X}}_n = \bigsqcup_{\tau \in \mathrm{Match}(2n)} \mathcal{X}_\tau \qquad \overline{E}_n = \bigsqcup_{\tau \in \mathrm{Match}(2n)} E_\tau$

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- Two planar electrical networks give the same matrix $\Lambda(G, R)$

  $\iff$ they are related by Y-Δ moves.

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Problem

Construct a stratification-preserving homeomorphism between $\overline{\mathcal{X}}_n$ and $\overline{E}_n$.

Thank you!

Slides: `http://math.mit.edu/~galashin/slides/ucla_ising.pdf`

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