Ising model and total positivity

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Joint work with Pavlo Pylyavskyy

arXiv:1807.03282

Part 1: Ising model

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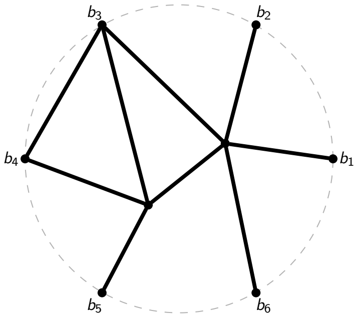

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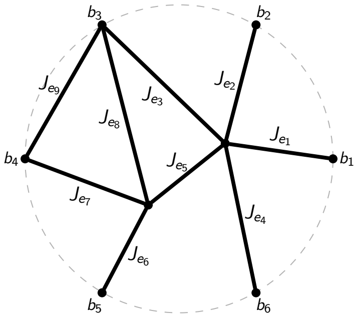

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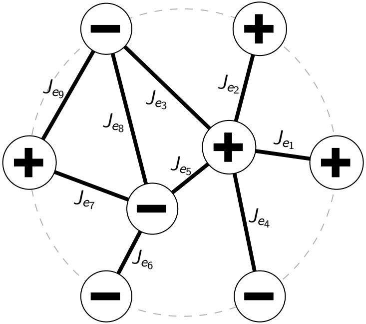

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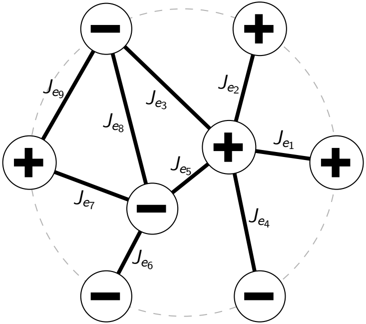

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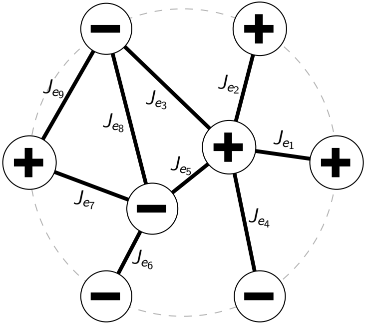

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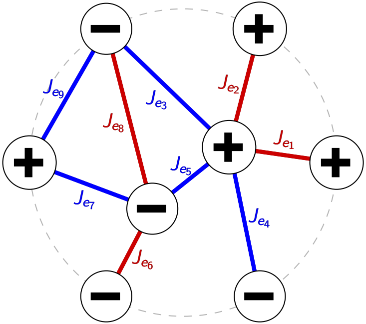

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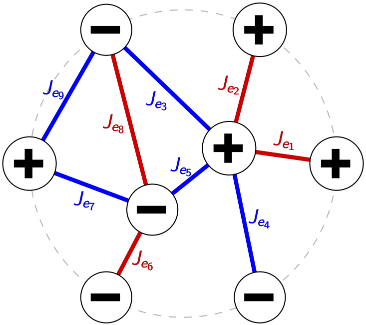

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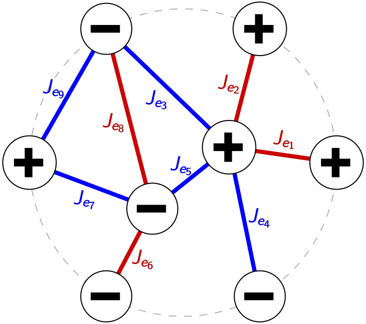

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$$\mathrm{Prob}(\sigma) := \frac{\mathrm{wt}(\sigma)}{Z}$$

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Theorem (G.–Pylyavskyy (2018))

Describe boundary correlations of the planar Ising model by inequalities.

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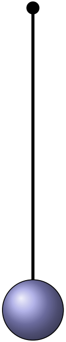

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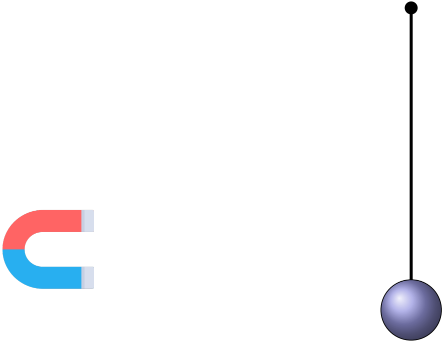

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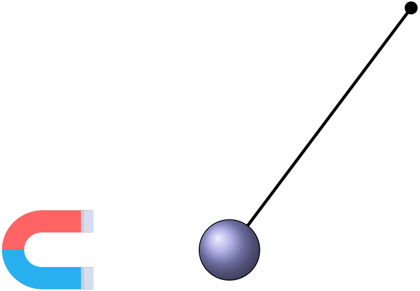

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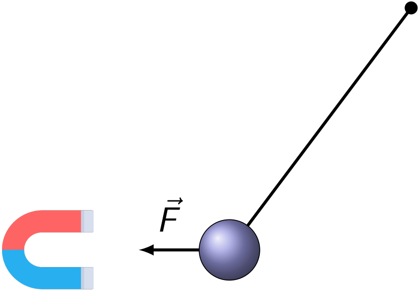

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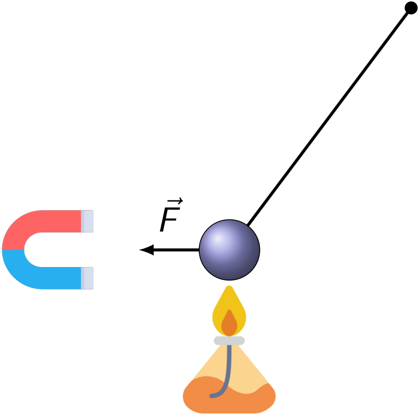

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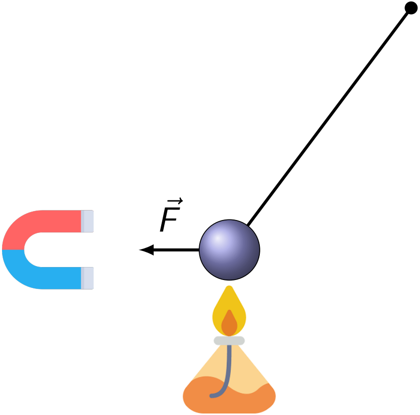

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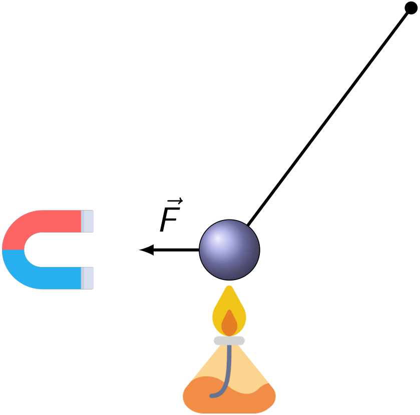

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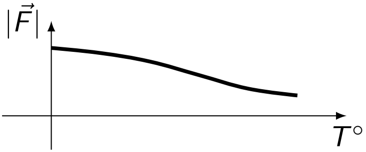

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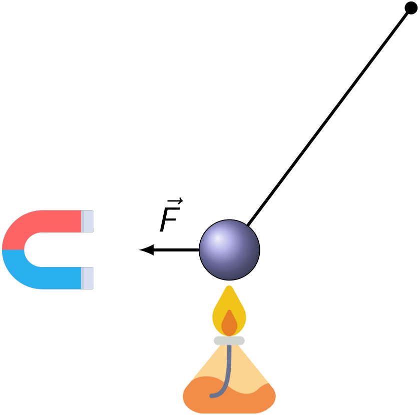

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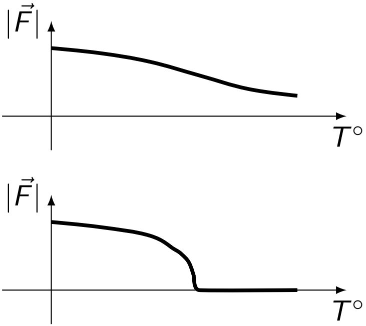

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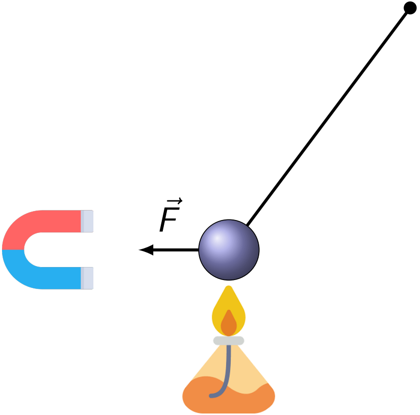

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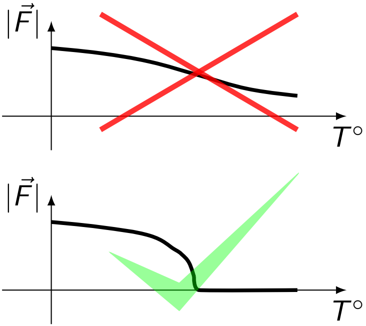

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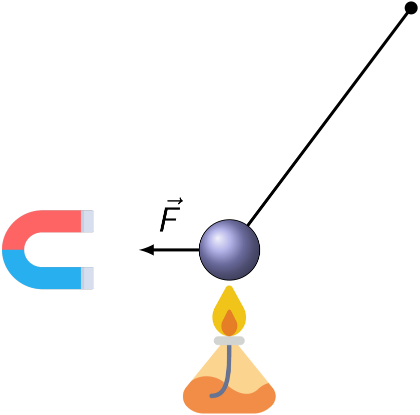

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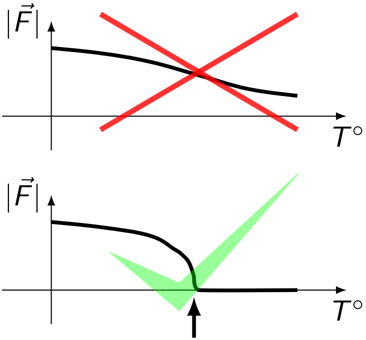

Curie point (P. Curie, 1895)

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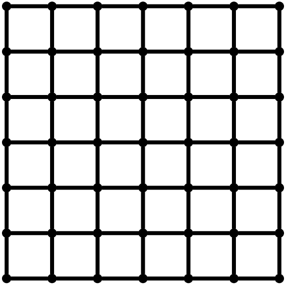

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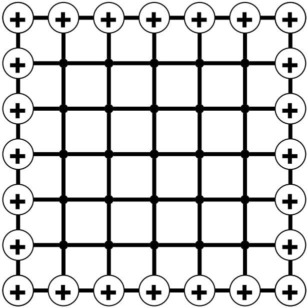

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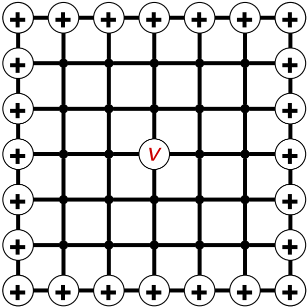

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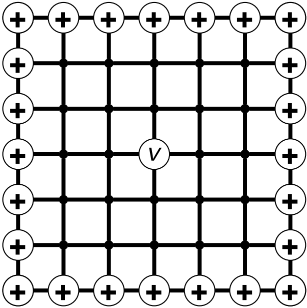
$$M(T) := \lim_{N \to \infty} (\text{Prob}(\sigma_v = +1) - \text{Prob}(\sigma_v = -1))$$



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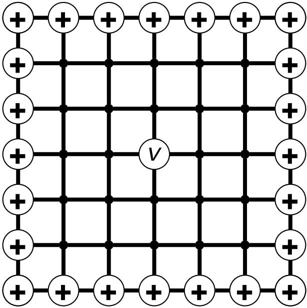

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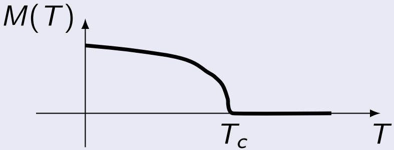

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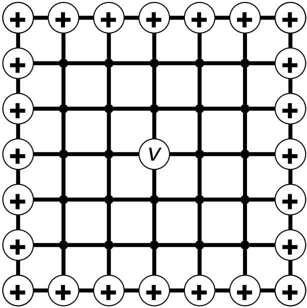

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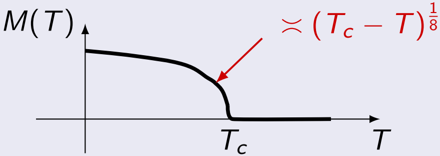

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- Belavin–Polyakov–Zamolodchikov (1984): conjectured conformal invariance of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$

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Historically, we let $G := \mathbb{Z}^d \cap \Omega$ for some $\Omega \subset \mathbb{R}^d$ and set all $J_e := \frac{1}{T}$ for some temperature $T \in \mathbb{R}_{>0}$.

- Peierls (1937): phase transition in $\mathbb{Z}^d$ for $d \geq 2$
- Kramers–Wannier (1941): critical temperature $\frac{1}{T_c} = \frac{1}{2}\log\left(\sqrt{2}+1\right)$ for $\mathbb{Z}^2$
- Onsager, Kaufman, Yang (1944–1952): exact expressions for the free energy and spontaneous magnetization
- Belavin–Polyakov–Zamolodchikov (1984): conjectured conformal invariance of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$
- Smirnov, Chelkak, Hongler, Izyurov, ... (2010–2015): proved conformal invariance and universality of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$

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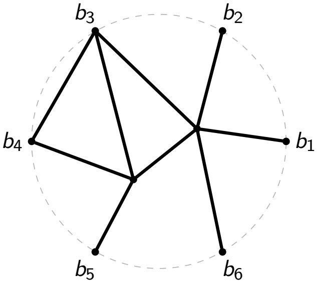

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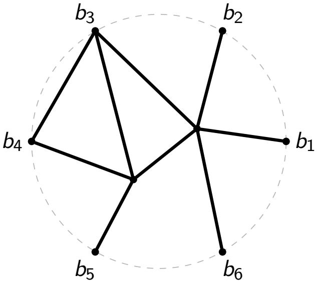

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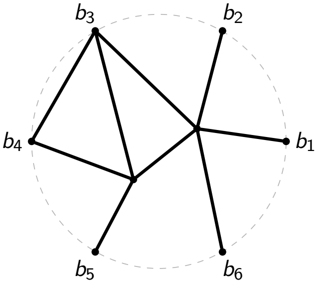

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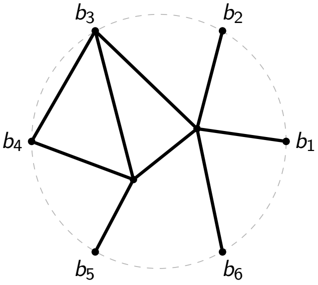

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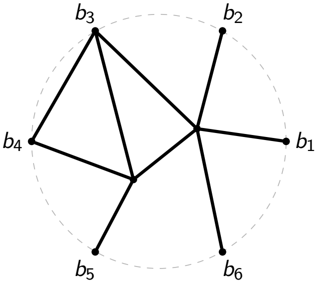

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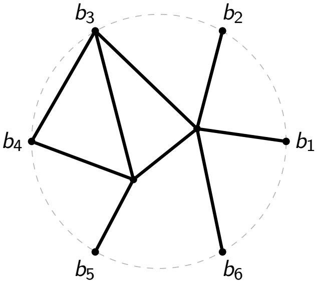

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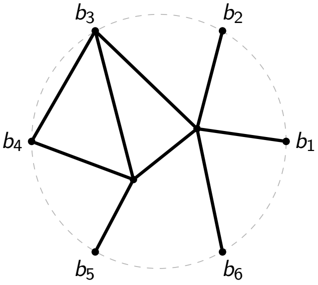

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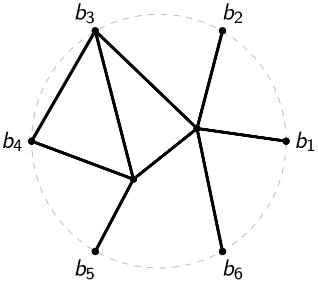

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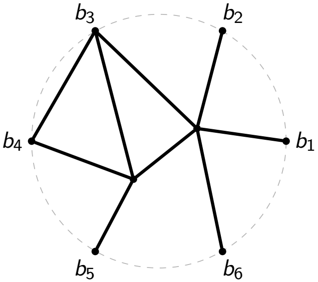

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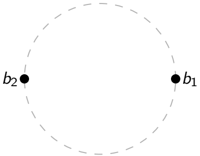

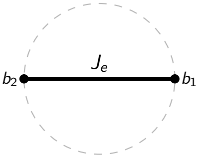

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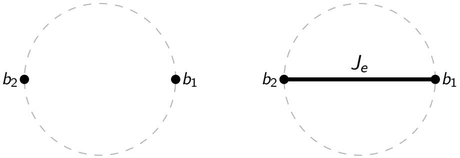

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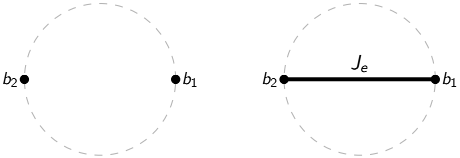

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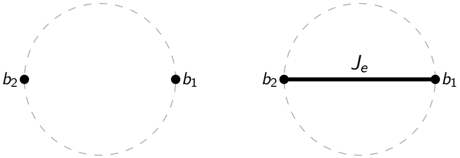

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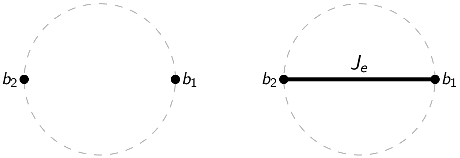

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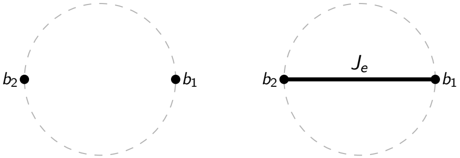

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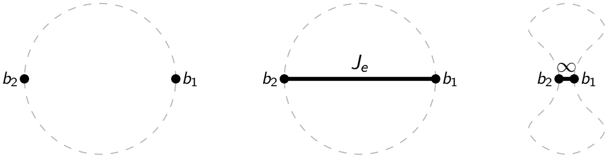

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Part 2: Total positivity

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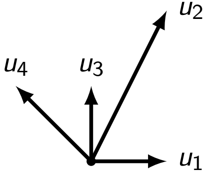

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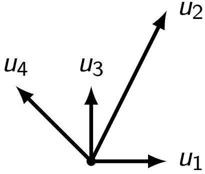

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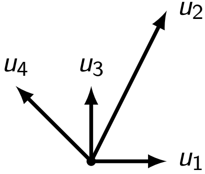

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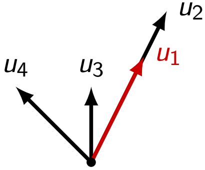

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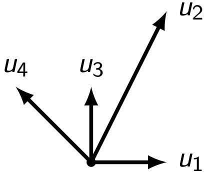

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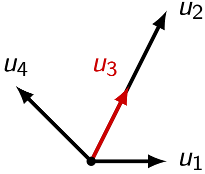

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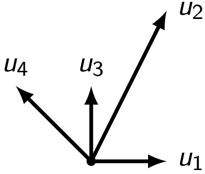

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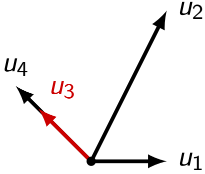

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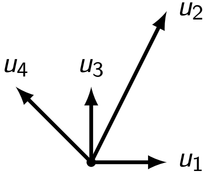

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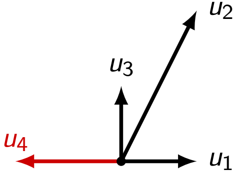

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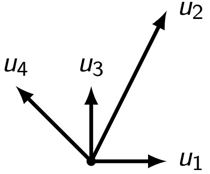

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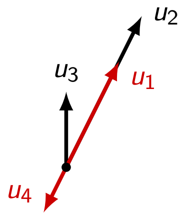

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Theorem (Postnikov (2006))

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Let C be a compact contractible topological manifold whose boundary is homeomorphic to a sphere. Then C is homeomorphic to a closed ball.

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Theorem (G.–Pylyavskyy (2018))

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$\mathcal{X}_n := \{M(G,J) \mid (G,J) \text{ is a planar Ising network with } n \text{ boundary vertices}\}$
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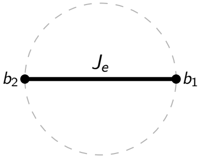

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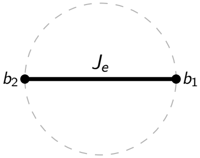

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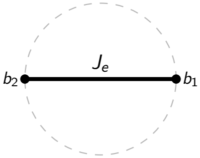

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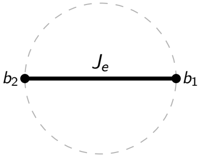

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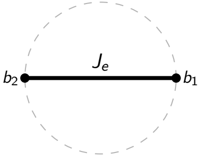

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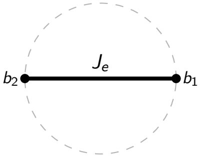

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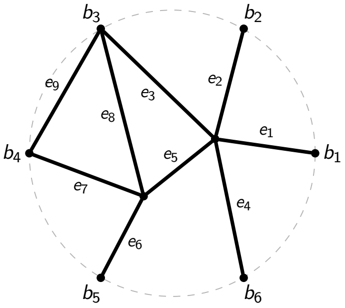

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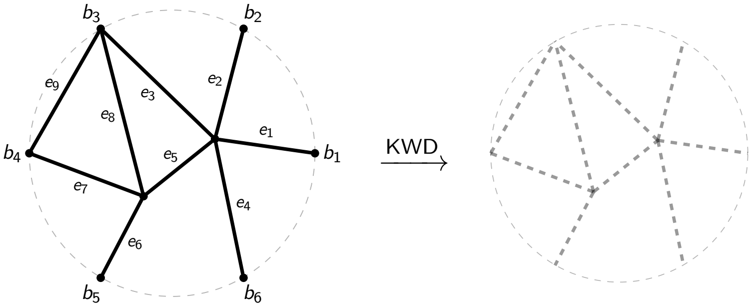

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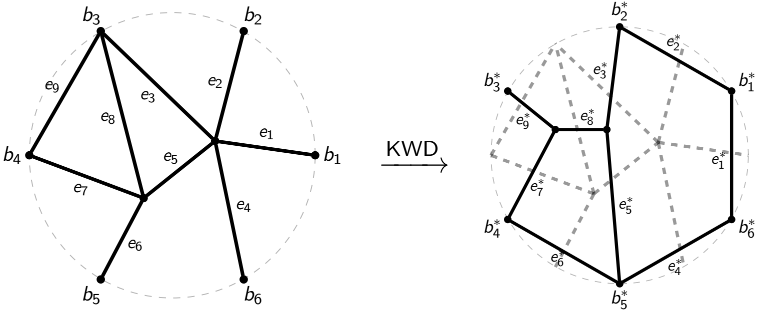

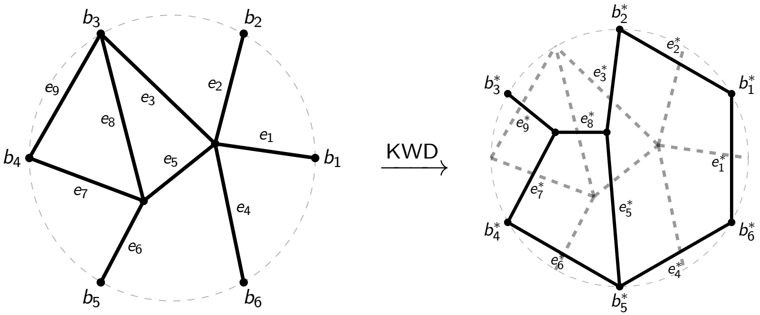

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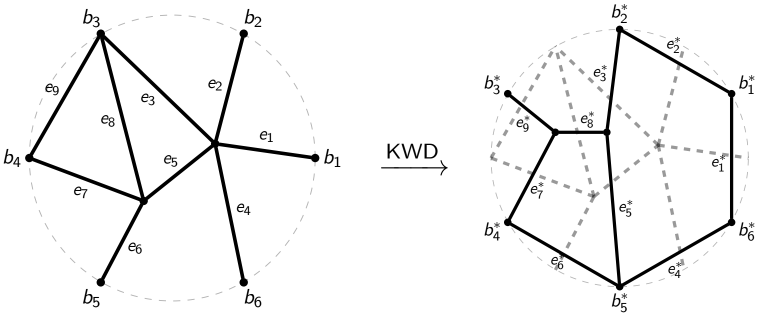

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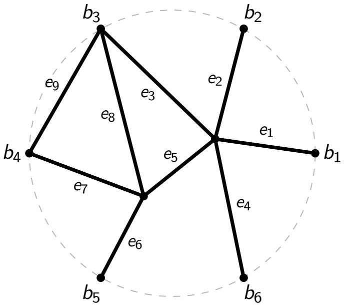

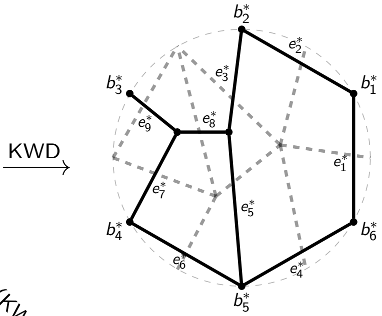

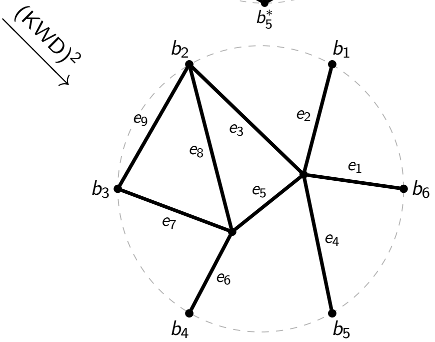

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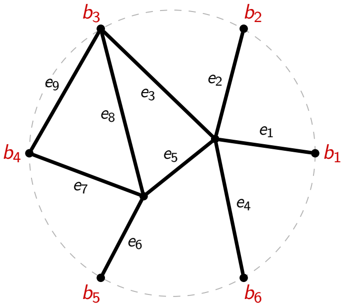

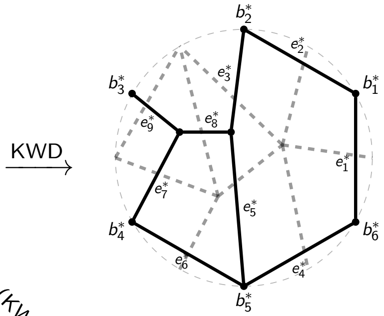

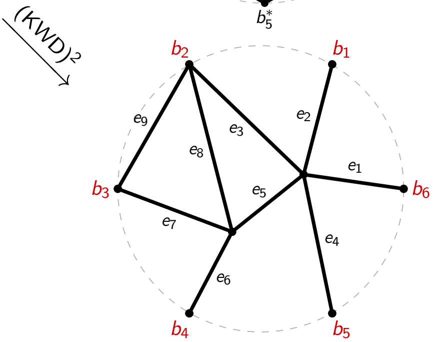

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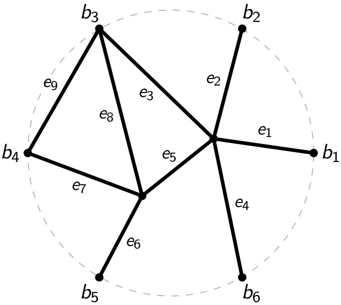

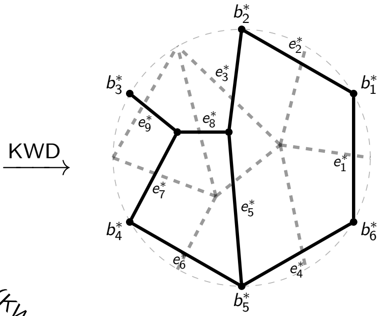

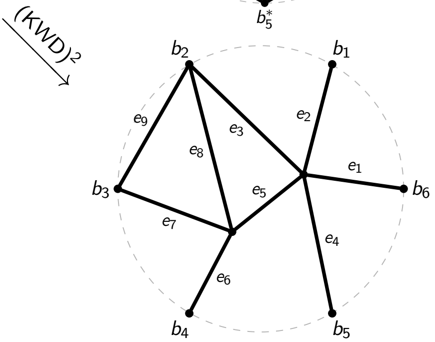

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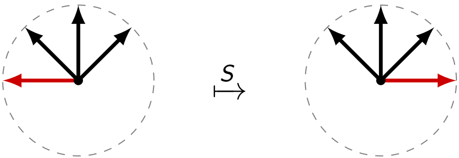

Cyclic shift on $\mathrm{Gr}_{\geq 0}(k, n)$

Theorem (G.–Karp–Lam (2017))

$\mathrm{Gr}_{\geq 0}(k, n)$ *is homeomorphic to a* $k(n-k)$*-dimensional closed ball.*

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This map preserves $\mathrm{Gr}_{\geq 0}(k, n)$.

Example: For $\mathrm{Gr}_{\geq 0}(2, 4)$, we have

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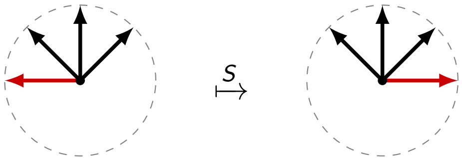

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Theorem (G.–Pylyavskyy (2018))

- *The map ϕ restricts to a homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n, 2n)$.*
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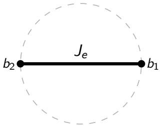

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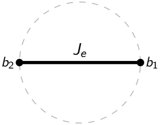

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Fixed point M_0 of KWD $\leftrightarrow$ Ising model at critical temperature $\leftrightarrow$ X_0?

Kramers–Wannier's duality vs. cyclic shift

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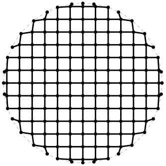
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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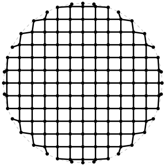
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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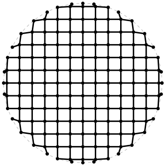
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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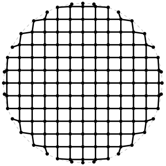
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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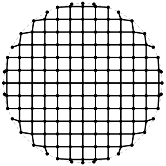
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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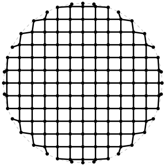
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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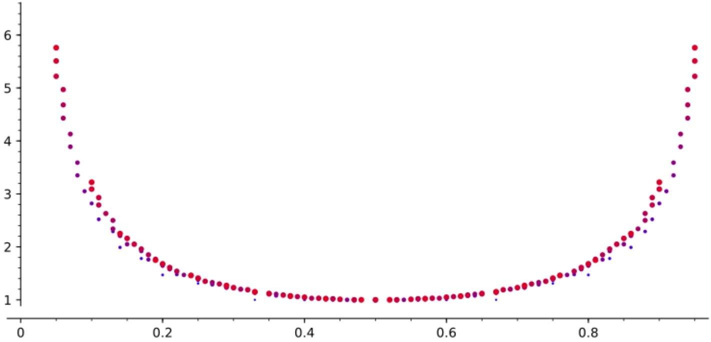

M_0 for $n = 2 \dots 20$

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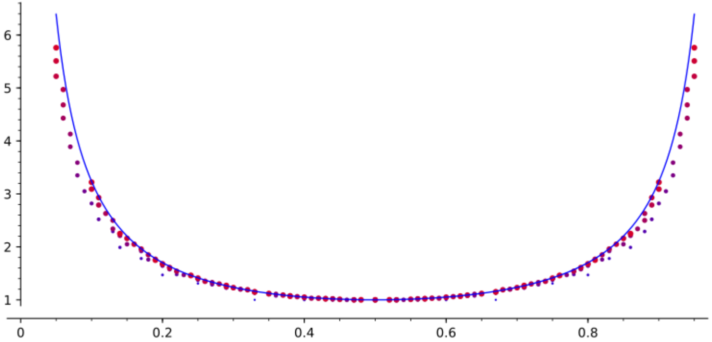

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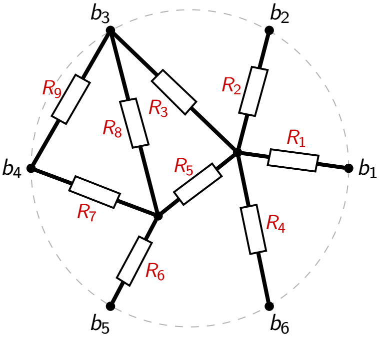

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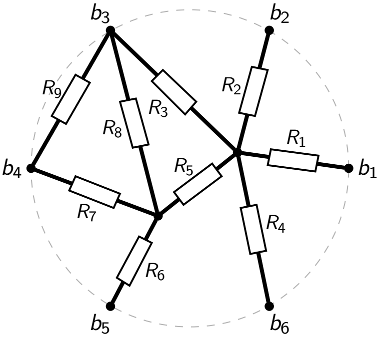
Definition

Electrical response matrix $\Lambda(G, R) : \mathbb{R}^n \to \mathbb{R}^n$, sending voltages to currents.



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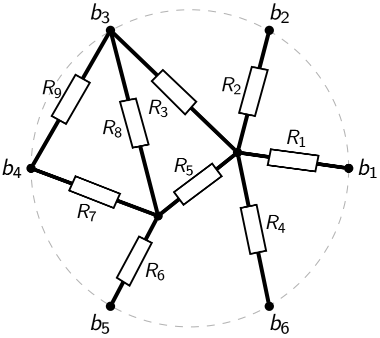

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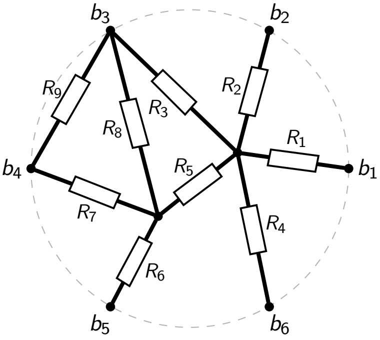

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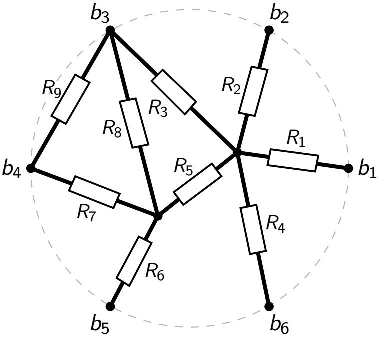

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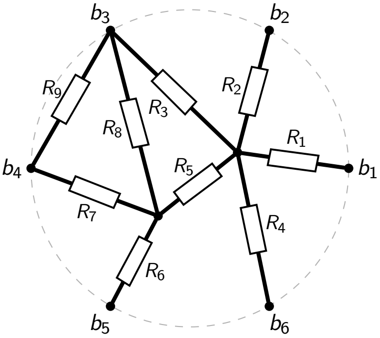

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$\overline{E}_n$: compactification of the space of $n \times n$ electrical response matrices [Lam (2014)]

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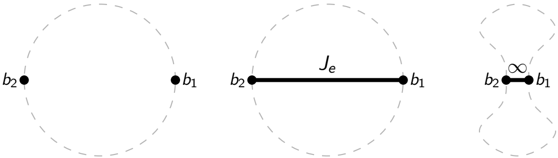

$$M(G, J) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}$$

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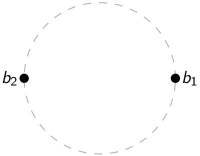

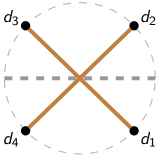

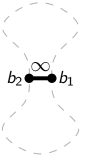

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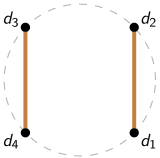

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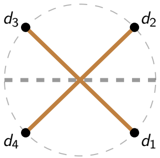

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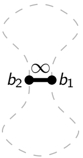

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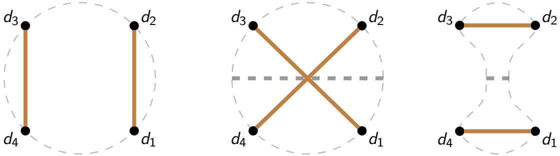
$m_{12} = 1$

Stratification: $n = 2$

Stratification: $\overline{\mathcal{X}}_n = \bigsqcup_{\tau \in \mathrm{Match}(2n)} \mathcal{X}_\tau \qquad \overline{E}_n = \bigsqcup_{\tau \in \mathrm{Match}(2n)} E_\tau$



$M(G, J) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}$

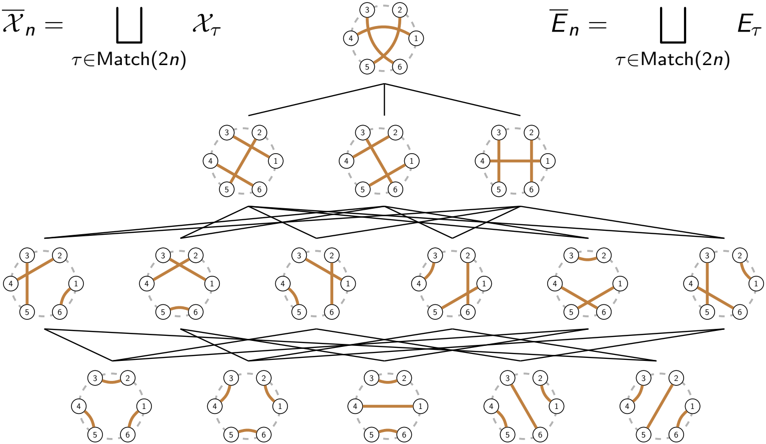
$J_e = 0$, $m_{12} = 0$

$J_e \in (0, \infty)$, $m_{12} \in (0, 1)$

$J_e = \infty$, $m_{12} = 1$

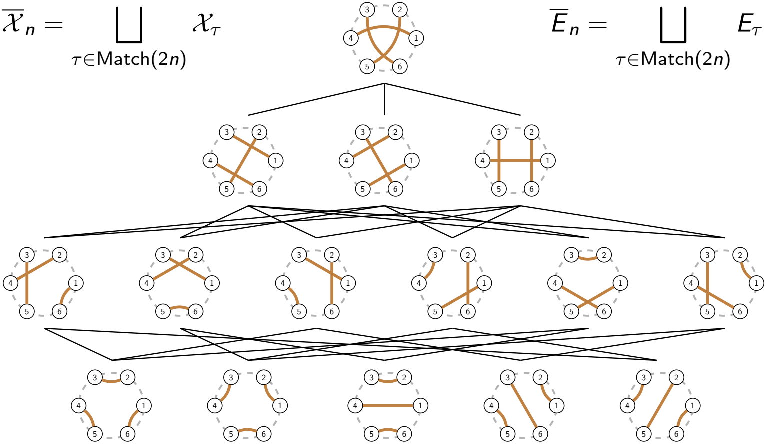
Ising model vs. Electrical networks

$$\overline{\mathcal{X}}_n = \bigsqcup_{\tau \in \mathrm{Match}(2n)} \mathcal{X}_\tau \qquad \overline{E}_n = \bigsqcup_{\tau \in \mathrm{Match}(2n)} E_\tau$$



Ising model vs. Electrical networks

$$\overline{\mathcal{X}}_n = \bigsqcup_{\tau \in \mathrm{Match}(2n)} \mathcal{X}_\tau \qquad \overline{E}_n = \bigsqcup_{\tau \in \mathrm{Match}(2n)} E_\tau$$

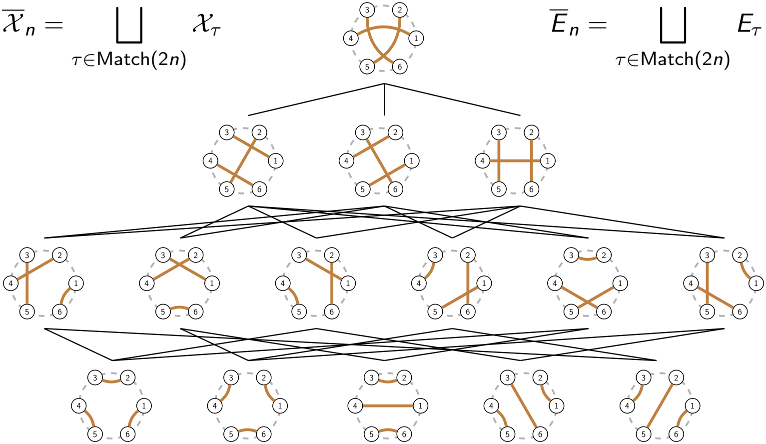


Problem

- *Construct a stratification-preserving homeomorphism between $\overline{\mathcal{X}}_n$ and $\overline{E}_n$.*

Ising model vs. Electrical networks

$$\overline{\mathcal{X}}_n = \bigsqcup_{\tau \in \mathrm{Match}(2n)} \mathcal{X}_\tau \qquad \overline{E}_n = \bigsqcup_{\tau \in \mathrm{Match}(2n)} E_\tau$$



Problem

- *Construct a stratification-preserving homeomorphism between $\overline{\mathcal{X}}_n$ and $\overline{E}_n$.*
- *Show that the closure of $\mathcal{X}_\tau$ and of E_τ is a ball.*

Thank you!

Slides: http://math.mit.edu/~galashin/slides/toronto_ising.pdf

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