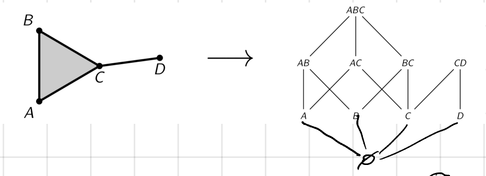
Топология и приложения
вполне неотрицательных пространств

k = Steven Karp [GKL17], [GKL18], [GKL19]
L = Thomas Lam [GL18]
P = Pavlo Pylyavskyy [GP18]

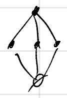
Опр. Рег. клет. комплекс — топ. пр-во разбитое на клетки = открытые шары
— замыкания = замкн. шары
— границы = сферы

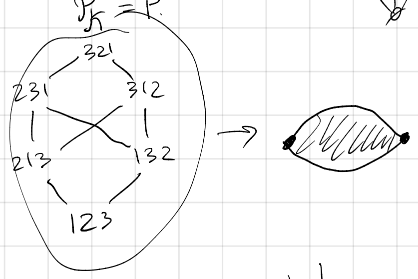


K — рег. клет. комплекс $\longrightarrow$ ч.у.м. P_K

Пример: многогранник

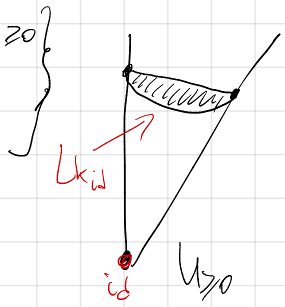
Теорема (Björner '84):
(1) Топология рег. клет. компл. K восстанавливается по P_K.
(2) ч.у.м. P — shellable + (...)
$\Rightarrow \exists\, K$ — рег. клет. комплекс: $P_K = P$.





$n \geq 1$

$$U_{\geq 0} = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \;\middle|\; \text{все миноры} \geq 0 \right\}$$



$n = 3$

$$U_{\geq 0} = \left\{ \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \;\middle|\; \begin{matrix} a, b, c \geq 0 \\ ac \geq b \end{matrix} \right\}$$

Вопрос: сколько клеток? ответ: $n!$

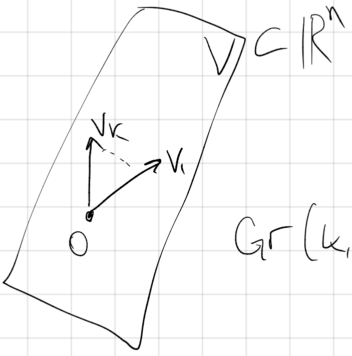
Клетки: $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & a & \\ & 1 & \\ & & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & & \\ & 1 & c \\ & & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & a & \\ & 1 & c \\ & & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & a & ac \\ & 1 & c \\ & & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}$

dim = 0, 1, 1, 2, 2, 3

Грассманиан.

$0 \leq k \leq n$

Опр. $Gr(k,n) = \{ V \subseteq \mathbb{R}^n \mid \dim V = k \}$



$$k \left\{ \underbrace{\begin{bmatrix} \text{———} \; v_1 \; \text{———} \\ \vdots \\ \text{———} \; v_k \; \text{———} \end{bmatrix}}_{n} \right. \qquad v_i \in \mathbb{R}^n$$

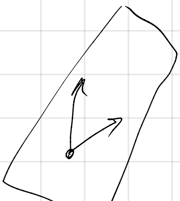
$$Gr(k,n) = \{ \text{матрицы } k \times n \text{ ранга } k \} \big/ GL_k$$

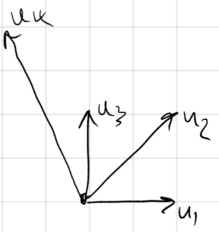
операц. над строками

$$\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} \qquad u_j \in \mathbb{R}^k$$

$Gr(2,4)$

$$\begin{array}{cccc c} u_1 & u_2 & u_3 & u_4 & \\ \end{array}$$
$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{matrix} v_1 \\ v_2 \end{matrix}$$





$$\dim Gr(k,n) = k(n-k).$$

Координаты Плюкера = макс. миноры матрицы

$I \subseteq \{1,2,\dots,n\}$, $|I|=k \Rightarrow \Delta_I$ = минор

$$\begin{array}{cc} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} & \\ \begin{vmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 \end{vmatrix} & \begin{array}{c} v_1 \\ v_2 \end{array} \end{array}$$

$$\boxed{\begin{array}{ccc} \Delta_{12}=1 & \Delta_{13}=1 & \Delta_{14}=2 \\ \Delta_{23}=1 & \Delta_{24}=3 & \Delta_{34}=1 \end{array}} \geq 0$$

__Опр.__ $Gr_{\geq 0}(k,n)=\{V\in Gr(k,n) \mid \Delta_I(V)\geq 0\ \forall I\}$

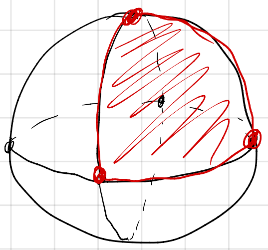
$Gr_{>0}(k,n)=$ —"— >0

замыкания клеток $\Leftrightarrow$ афф. порядок Брюа.

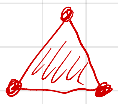
__Пример.__ $k=1$ $Gr(1,n)=\mathbb{RP}^{n-1}$

$n=3$ $Gr(1,3)=\mathbb{RP}^2$

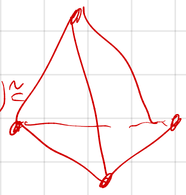
$Gr_{\geq 0}(1,n)\simeq\Delta^{n-1}$ симплекс разм. $n-1$



$Gr_{\geq 0}(1,3)\simeq$



$Gr_{\geq 0}(1,4)\simeq$



$Gr_{\geq 0}(k,n)$ — не многогранник

$\dim = k(n-k)$

кол-во граней $\dim=k(n-k)-1$ равно n

$n << k(n-k)$

$\geq d+1$ d

Частичные флаги: $\underline{k}=(k_1<k_2<\dots<k_r)$

$$Fl(\underline{k},n)=\{V_{k_1}\subset V_{k_2}\subset\dots\subset V_{k_r}\subset\mathbb{R}^n \mid \dim V_{k_i}=k_i\ \forall i\}$$

$\underline{k}=(k) \Rightarrow Fl(\underline{k},n)=Gr(k,n)$

__Lusztig__ (1998) определил $Fl_{\geq 0}(\underline{k},n)$

$\underline{k}=(k) \Rightarrow Fl_{\geq 0}(\underline{k},n)=Gr_{\geq 0}(k,n)$ (Rietsch, '09)

__Гипотеза__ (Постников '06, Williams '07)

— $Gr_{\geq 0}(k,n)$
— $Fl_{\geq 0}(\underline{k},n)$

— рег. клет. комплекс, гомео замкн. шару

$\boxed{\overline{B}^d = \text{замкн. шар } \dim = d}$

__Теорема__

— __GKL17__ $Gr_{\geq 0}(k,n)\simeq\overline{B}^{k(n-k)}$

— __GKL18__ $Fl_{\geq 0}(\underline{k},n)\simeq\overline{B}^{N(\underline{k},n)}$

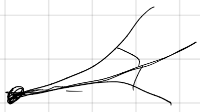
— __GKL19__ $Gr_{\geq 0}(k,n)$ и $Fl_{\geq 0}(\underline{k},n)$ — рег. клет. комплексы.

Следствие. $U_{\geq 0}$

Ингредиенты [GKL19]

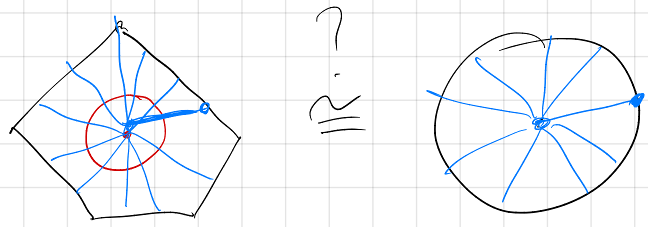
(i) Обобщенная гип. Пуанкаре

(ii) Вложение в афф. флаговое мн-е.



Идея GKL17

Вопрос.



Циклич. симм. $Gr_{\geq 0}(k,n)$:

$$S:\ \begin{pmatrix} | & & | \\ u_1 & \dots & u_n \\ | & & | \end{pmatrix} \longrightarrow \begin{pmatrix} | & | & & | \\ (-1)^{k-1}u_n & u_1 & \dots & u_{n-1} \\ | & | & & | \end{pmatrix}$$

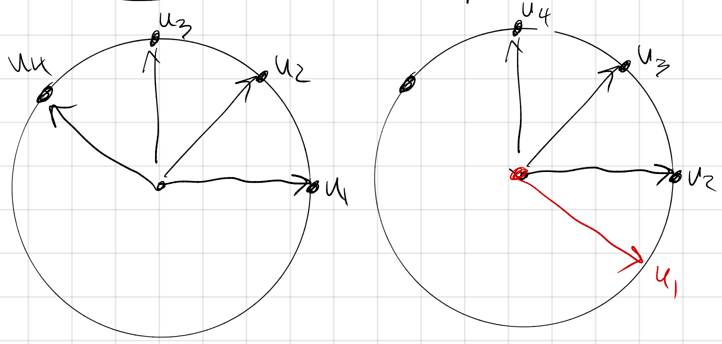
$$S:\ Gr_{\geq 0}(k,n) \longrightarrow Gr_{\geq 0}(k,n)$$

Вопрос: сколько точек $X \in Gr_{\geq 0}(k,n)$: $S(X)=X$?

Ответ: $\exists!\ X_0 \in Gr_{\geq 0}(k,n)$: $S(X_0)=X_0$

Пример: $Gr_{\geq 0}(2,4)$

$$\begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array}$$
$$\begin{pmatrix} 1 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1 & 1/\sqrt{2} \end{pmatrix} \longrightarrow \begin{pmatrix} 1/\sqrt{2} & 1 & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 1 \end{pmatrix}$$



$$\Delta_{12}=\Delta_{23}=\Delta_{34}=\Delta_{14}=1/\sqrt{2}$$

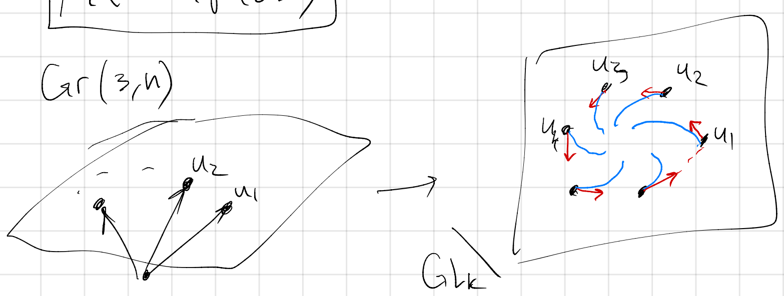
$$\Delta_{13}=\Delta_{24}=1.$$

$$S(X)=X\cdot \underbrace{\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ (-1)^{k-1} & & & & 0 \end{pmatrix}}_{S}$$

($n \times n$)

$$\boxed{X\cdot \exp(tS)}$$

$Gr(3,n)$



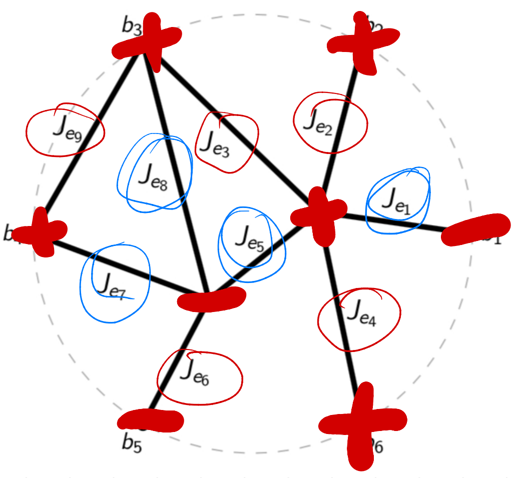
Лемма: $X \in Gr_{\geq 0}(k,n)$. Тогда:

(1) $\lim\limits_{t\to+\infty} X\cdot\exp(tS) = X_0$

(2) $X\cdot\exp(tS) \in Gr_{\geq 0}(k,n)$ для $t \geq 0$.

Приложения

Модель Изинга



G = план. граф в диске.

$J: E(G) \to \mathbb{R}_{>0}$

$\sigma \in \{\pm 1\}^V$

$$wt(\sigma) = \prod_{uv\in E} \exp(J_{uv}\cdot\sigma_u\sigma_v) = \frac{\prod_{\sigma_u=\sigma_v} \exp(J_{uv})}{\prod_{\sigma_u\neq\sigma_v} \exp(J_{uv})}$$

$$Z = \sum_{\sigma\in\{\pm1\}^V} wt(\sigma) \qquad Prob(\sigma) = \frac{wt(\sigma)}{Z}$$

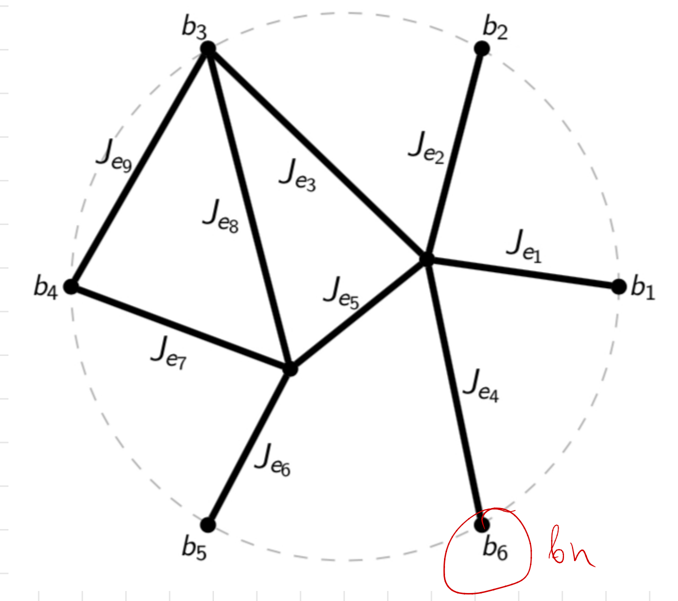
Корреляция: $\langle \sigma_u\sigma_v\rangle = Prob(\sigma_u=\sigma_v) - Prob(\sigma_u\neq\sigma_v)$

$u, v\in V(G)$

Теорема: $\langle\sigma_u\sigma_v\rangle \geq 0.$

$u,v,w\in V(G)$ $\quad \langle\sigma_u\sigma_w\rangle \geq \langle\sigma_u\sigma_v\rangle\cdot\langle\sigma_v\sigma_w\rangle$

Вопрос: как найти все неравенства?



$\to M(G,J) = (m_{ij})_{i,j=1}^{n}$

$m_{ij} = \langle\sigma_{b_i}\sigma_{b_j}\rangle$

$m_{ij} = m_{ji}$

$m_{ii} = 1$

$0 \leq m_{ij} \leq 1$

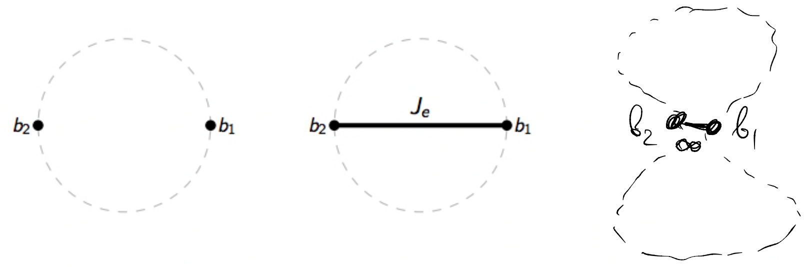
$(G, J) \qquad M \in [0,1]^{\binom{n}{2}}$

$$X_n = \{M(G,J)\} \subseteq [0,1]^{\binom{n}{2}}$$

$$\overline{X}_n = \text{замык. } X_n \text{ в } [0,1]^{\binom{n}{2}}$$

Пример

$n=2$



$$M(G,J) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}, \qquad m_{12} = \langle\sigma_1\sigma_2\rangle = \frac{2\exp(J_e) - 2\exp(-J_e)}{2\exp(J_e)+2\exp(-J_e)}$$

$J_e = 0$	$J_e \in (0,\infty)$	$J_e = \infty$
$m_{12} = 0$	$m_{12}\in(0,1)$	$m_{12} = 1$

$\left.\begin{matrix}+ & + \\ - & -\end{matrix}\right\}\exp(J)$ $\qquad Z = 2(\exp(J)+\exp(-J))$

$\left.\begin{matrix}+ & - \\ - & +\end{matrix}\right\}\exp(-J)$ $\qquad X_n = [0,1)$

$X_n \subseteq [0,1]^{\binom{n}{2}}$ $\qquad \overline{X_n} = [0,1]$.

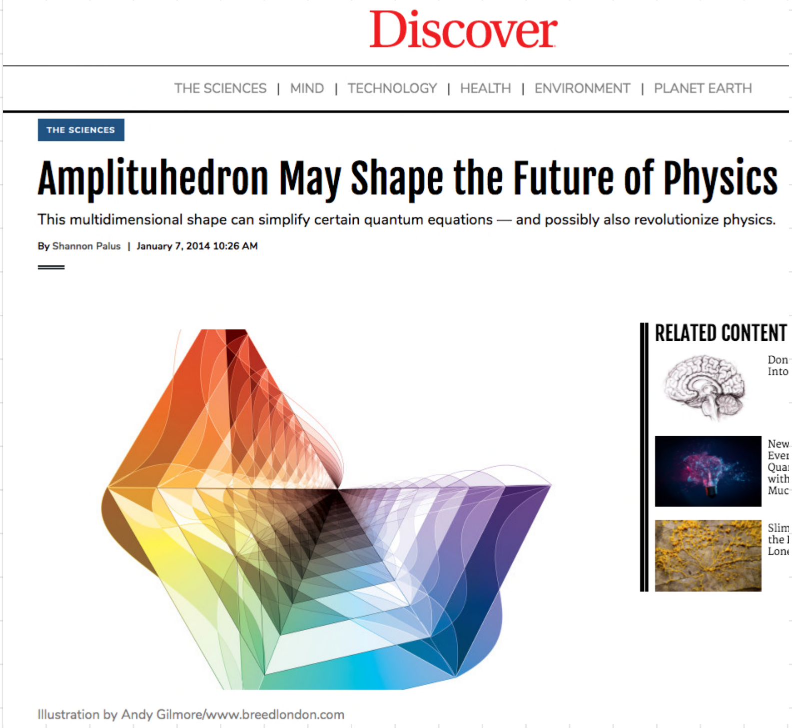
X_n — не открыто и не замкн

$\overline{X_n}$ — разрешаем стягивать ребра.

$$OG(n,2n) = \{W \in Gr(n,2n) \mid \Delta_I^{(W)} = \Delta_{[2n]\setminus I}^{(W)} \ \forall I\}$$

$$[2n] = \{1,2,\ldots,2n\}$$

$$OG_{\geq 0}(n,2n) = OG(n,2n) \cap Gr_{\geq 0}(n,2n).$$



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By Shannon Palus | January 7, 2014 10:26 AM

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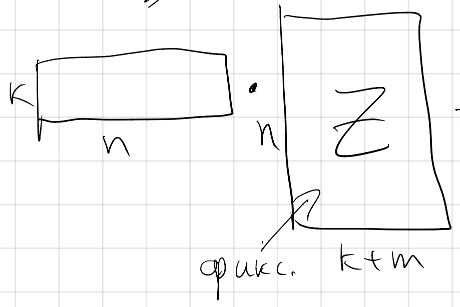
$Gr_{\geq 0}(k,n) \longrightarrow$ амплитуэдр $\rightarrow \mathcal{N}=4$ SYM



$= A \subseteq Gr(k,k+m)$

$OG_{\geq 0}(n,2n) \longrightarrow\ ???\ \longrightarrow$ $\mathcal{N}=6$ теория Черна-Саймонса

$$OG(n,2n) = \{W \in Gr(n,2n) \mid \Delta_I^{(W)} = \Delta_{[2n]\setminus I}^{(W)} \ \forall I\}$$

$$[2n] = \{1,2,\ldots,2n\}$$

$$OG_{\geq 0}(n,2n) = OG(n,2n) \cap Gr_{\geq 0}(n,2n).$$

$$\dim Gr_{\geq 0}(n,2n) = n^2$$

$$\dim OG_{\geq 0}(n,2n) = \binom{n}{2}$$

$$\begin{pmatrix}1 & m_{12} & m_{13} & m_{14}\\ m_{12} & 1 & m_{23} & m_{24}\\ m_{13} & m_{23} & 1 & m_{34}\\ m_{14} & m_{24} & m_{34} & 1\end{pmatrix} \mapsto n\,\underbrace{\begin{pmatrix}1 & 1 & m_{12} & -m_{12} & -m_{13} & m_{13} & m_{14} & -m_{14}\\ -m_{12} & m_{12} & 1 & 1 & m_{23} & -m_{23} & -m_{24} & m_{24}\\ m_{13} & -m_{13} & -m_{23} & m_{23} & 1 & 1 & m_{34} & -m_{34}\\ -m_{14} & m_{14} & m_{24} & -m_{24} & -m_{34} & m_{34} & 1 & 1\end{pmatrix}}_{2n}$$

$M = (m_{ij})$

$= M(G,J)$

$\tilde{M}: n \times 2n$

$\varphi(M) \in Gr(n,2n) \qquad \varphi(M) = GL_k \backslash \tilde{M}$

$$\varphi : [0,1]^{\binom{n}{2}} \longrightarrow Gr(n,2n)$$

Теорема

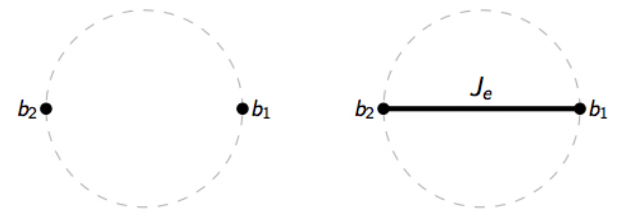
(1) φ – гомеоморфизм

$\overline{X_n} \xrightarrow{\sim} OG_{\geq 0}(n,2n)$

$$\begin{tikzcd}
{[0,1]^{\binom{n}{2}}} \arrow[r, "\varphi"] & OG(n,2n) \\
\overline{X_n} \arrow[u, hook] \arrow[r, "\sim", "\varphi"'] & OG_{\geq 0}(n,2n) \arrow[u, hook]
\end{tikzcd}$$

(2) $\overline{X_n} \cong OG_{\geq 0}(n,2n) \cong \overline{B}^{\binom{n}{2}}$

Пример: $n=2$



$$M(G,J) = \begin{pmatrix}1 & m_{12}\\ m_{12} & 1\end{pmatrix}, \qquad m_{12} = \langle \sigma_1\sigma_2 \rangle = \frac{2\exp(J_e) - 2\exp(-J_e)}{2\exp(J_e) + 2\exp(-J_e)}$$

$J_e = 0$	$J_e \in (0,\infty)$	$J_e = \infty$
$m_{12} = 0$	$m_{12} \in (0,1)$	$m_{12} = 1$

$$M = \begin{pmatrix}1 & m\\ m & 1\end{pmatrix} \longmapsto \tilde{M} = \begin{pmatrix}1 & 1 & m & -m\\ -m & m & 1 & 1\end{pmatrix}$$

$M \in \overline{X_n} \iff 0 \leq m \leq 1$

$\Updownarrow$

$\tilde{M} \in OG_{\geq 0}(n,2n)$

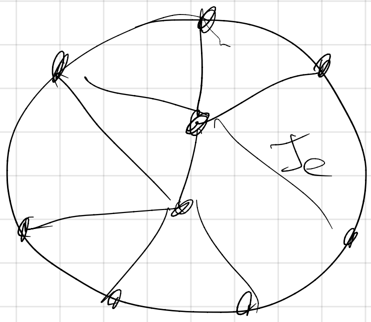
$$\left[\begin{array}{l}\Delta_{12} = \Delta_{34} = 2m\\ \Delta_{13} = \Delta_{24} = 1+m^2\\ \Delta_{14} = \Delta_{23} = 1-m^2\end{array}\right]$$

$\in OG(n,2n)$

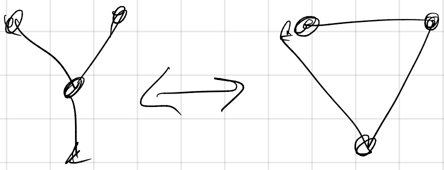
— $\Delta_I(\tilde{M})$ = полин. ф-ция от m_{ij}

$\Delta_I(\tilde{M}) \geq 0$ — описывают $\overline{X_n}$

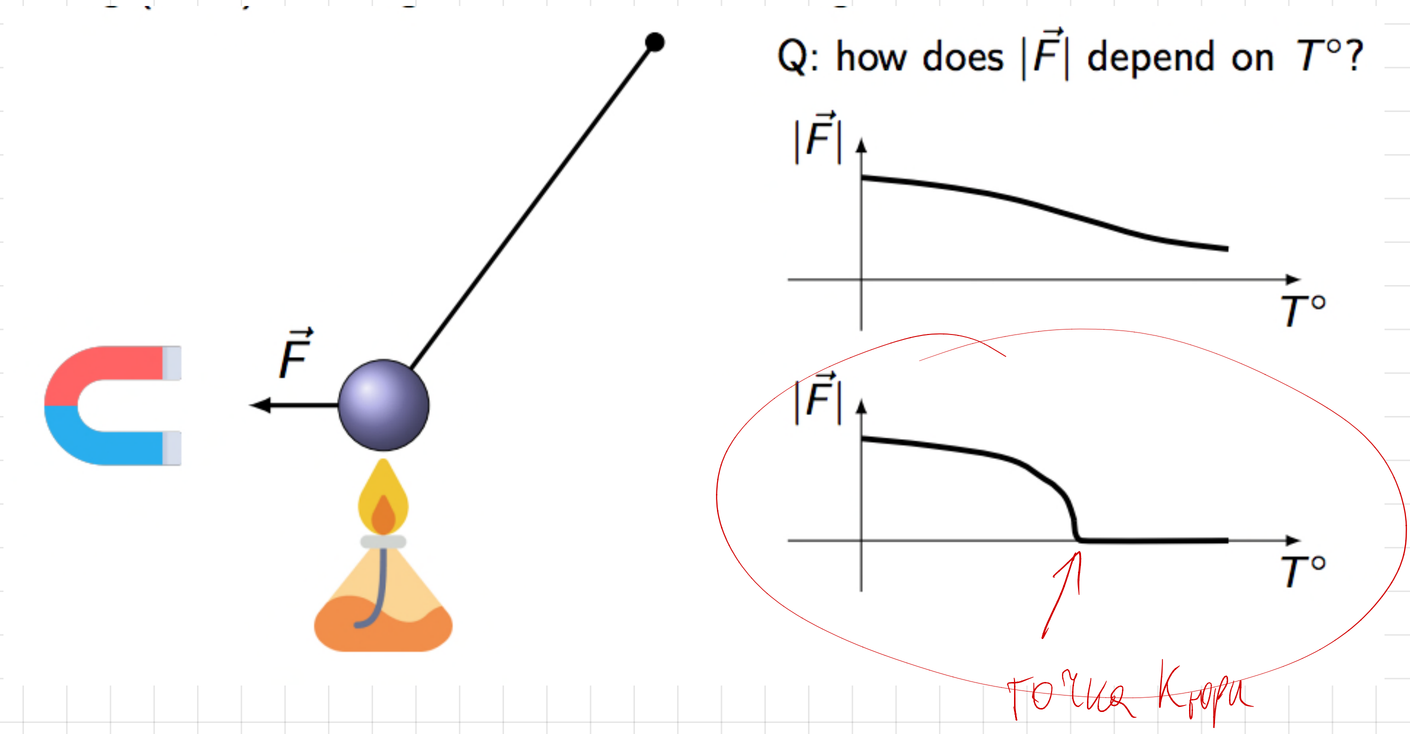
— решает обратную задачу!



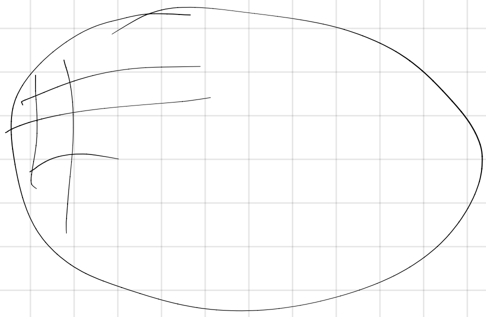
Зная M, восстановить (G, J)

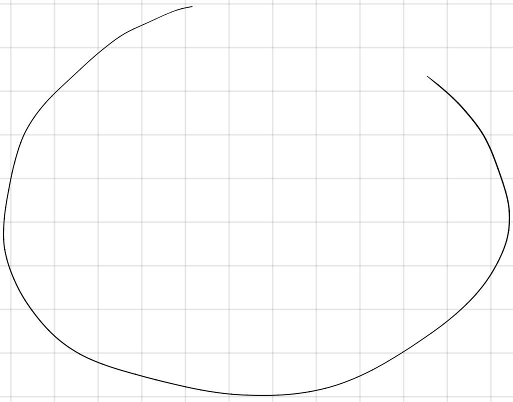


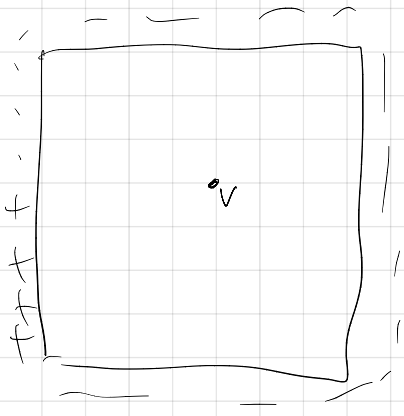
Приложения к крит. температуре



Анонс: $X_0 \in Gr_{\geq 0}(n, 2n) \to$ крит. модель Изинга в диске







$Prob(\sigma_v = +) = ?$

Lusztig $(G/P)_{\geq 0}$

$G/P \simeq OG(n, 2n)$

$(G/P)_{\geq 0} \neq OG_{\geq 0}(n, 2n)$