Higher secondary polytopes and regular plabic graphs

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UCLA

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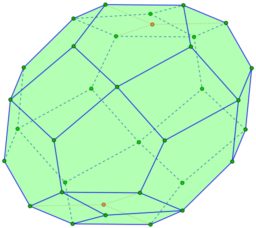

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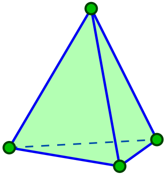

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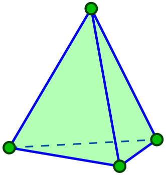

Simplex

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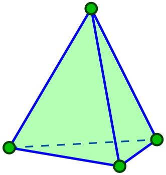

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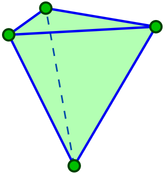

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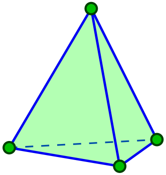

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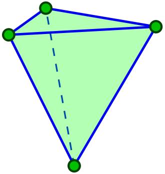

Upside-down simplex

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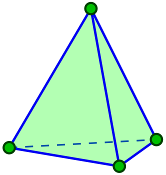

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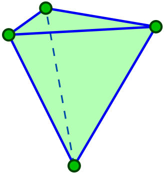

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What goes in the middle?

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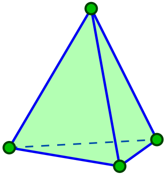

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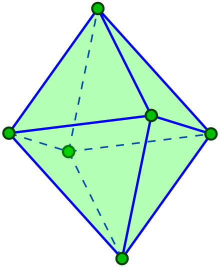
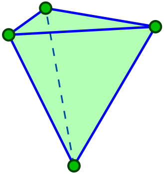


Hypersimplex



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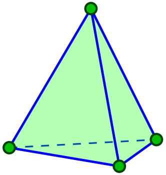

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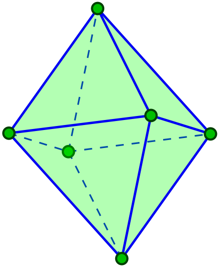

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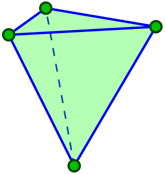

Upside-down simplex

$$\Delta_{k,n} := \operatorname{conv}\{\boldsymbol{e}_{i_1} + \cdots + \boldsymbol{e}_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\} \quad \subseteq \mathbb{R}^n.$$

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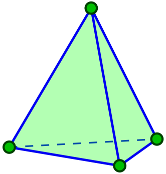
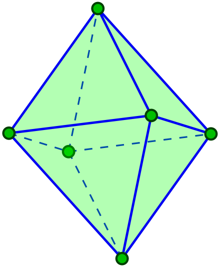


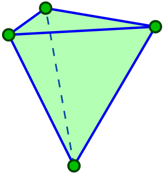
$\Delta_{1,4}$

Simplex



$\Delta_{2,4}$

Hypersimplex



$\Delta_{3,4}$

Upside-down simplex

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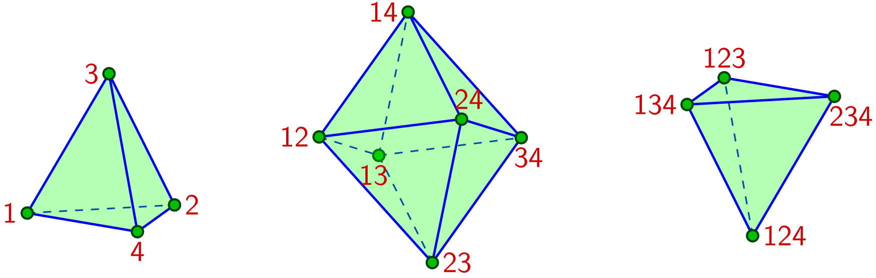

$\Delta_{1,4}$ Simplex

$\Delta_{2,4}$ Hypersimplex

$\Delta_{3,4}$ Upside-down simplex

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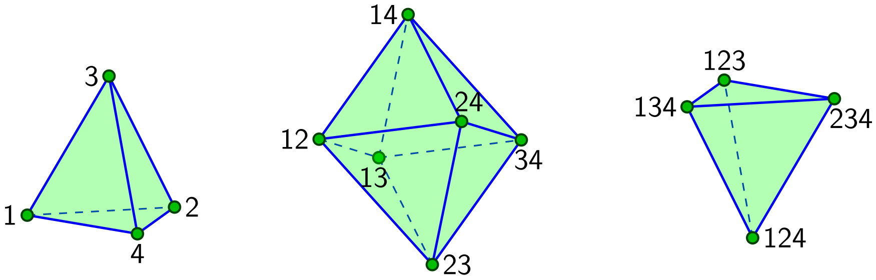

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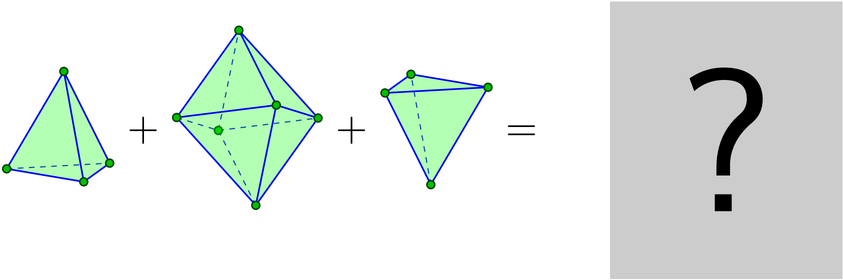
$$\Delta_{1,n} + \Delta_{2,n} + \cdots + \Delta_{n-1,n} = ?$$

Minkowski sum of hypersimplices



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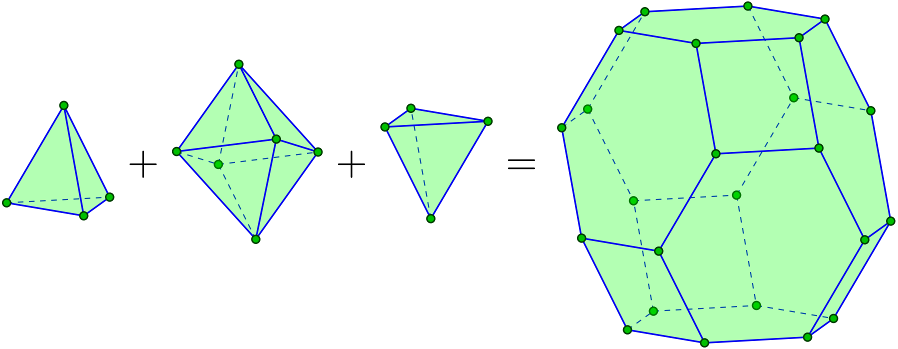

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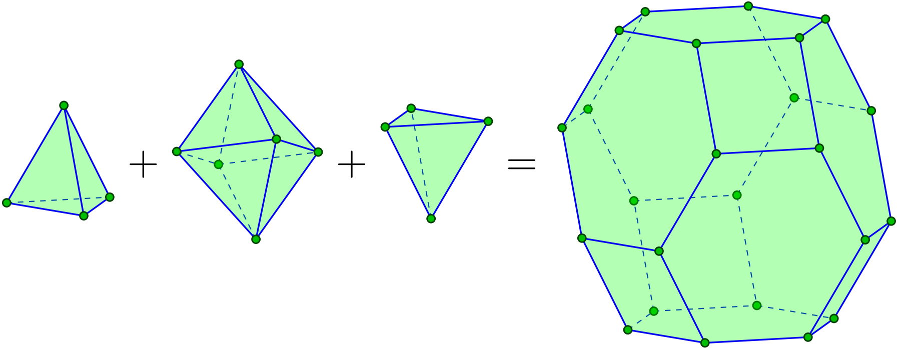
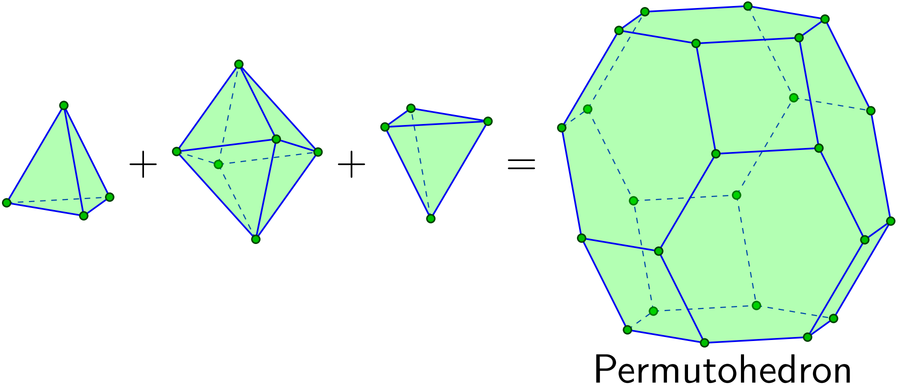


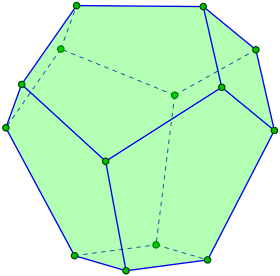
Permutohedron

Minkowski sum of hypersimplices



$$\Delta_{1,n} + \cdots + \Delta_{n-1,n} = \text{Perm}_n := \text{conv}\{(w_1, w_2, \ldots, w_n) \mid w \in S_n\}.$$

Name a polytope II



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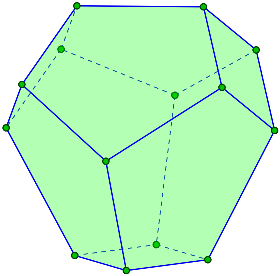

Associahedron

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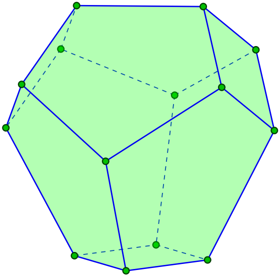

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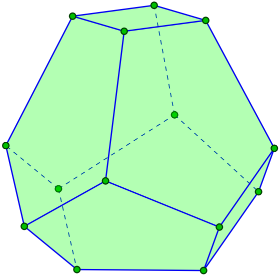

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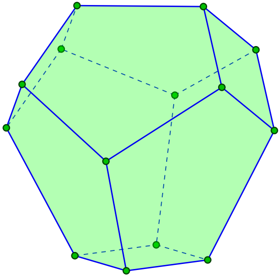

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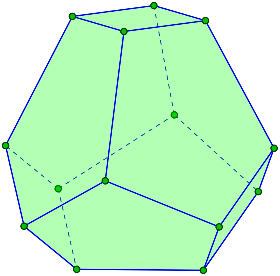

Upside-down
associahedron

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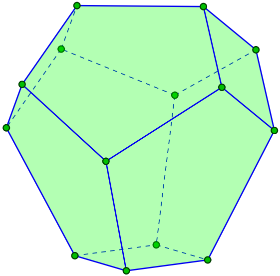

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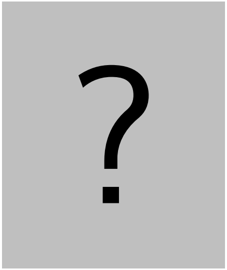

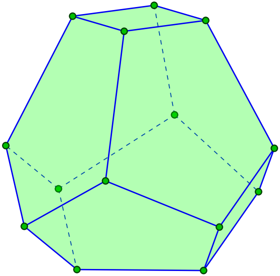

Upside-down associahedron

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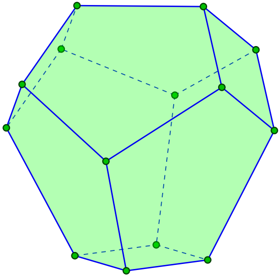

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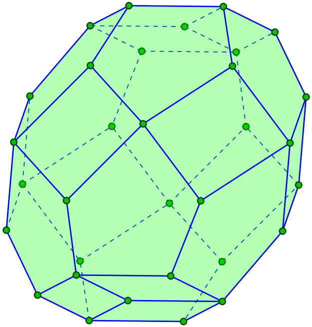
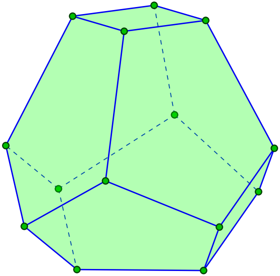


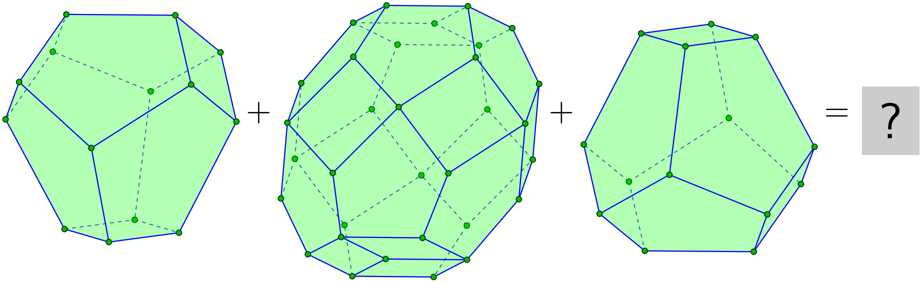
Higher associahedron



Upside-down associahedron

What goes in the middle?

Minkowski sum of higher associahedra



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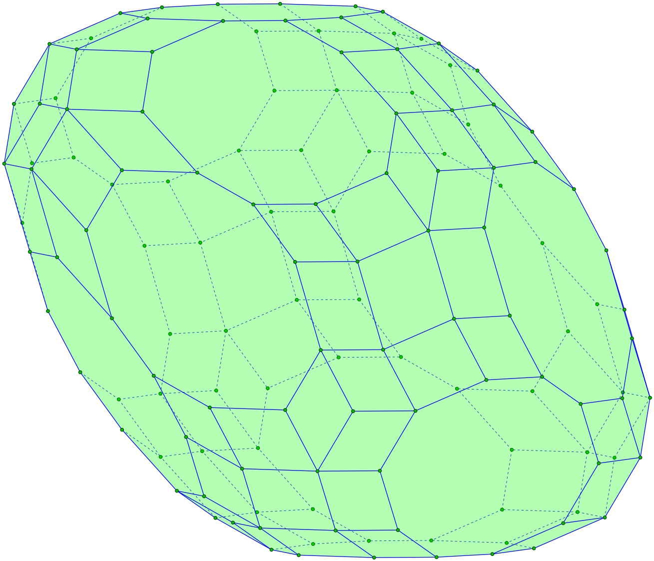

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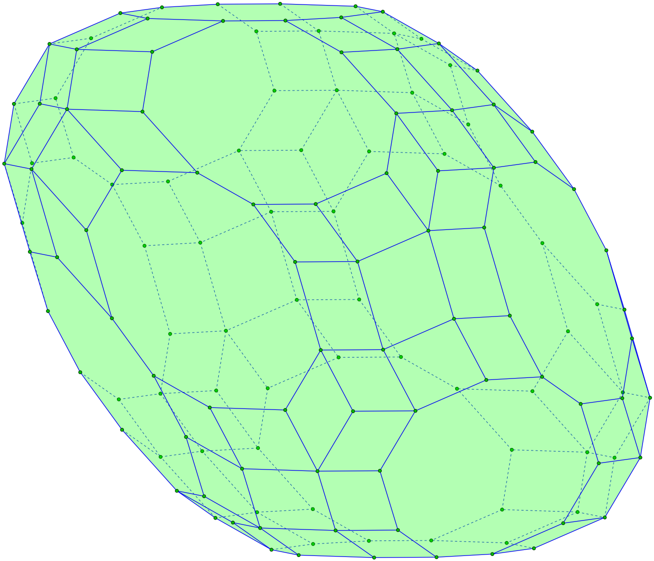

Fiber zonotope (Billera–Sturmfels (1992))

Let $\mathcal{A}$ be a configuration of n points in $\mathbb{R}^{d-1}$.

Higher secondary polytopes

Let $\mathcal{A}$ be a configuration of n points in $\mathbb{R}^{d-1}$.

We introduce higher secondary polytopes $\widehat{\Sigma}_{\mathcal{A},1}, \widehat{\Sigma}_{\mathcal{A},2}, \dots, \widehat{\Sigma}_{\mathcal{A},n-d} \subseteq \mathbb{R}^n$.

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Case $d = 1$

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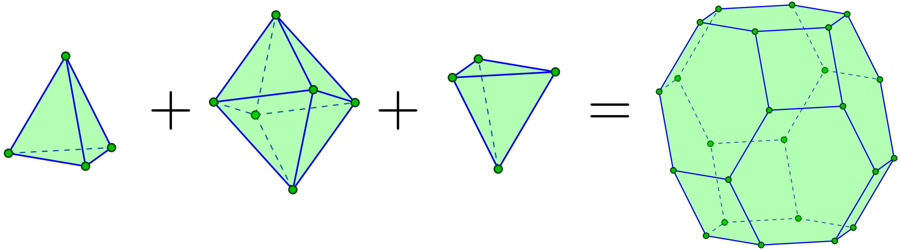

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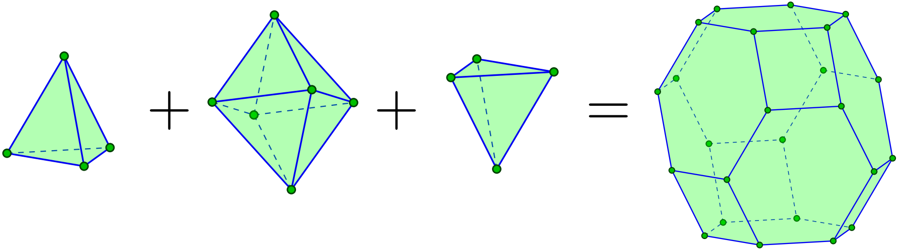

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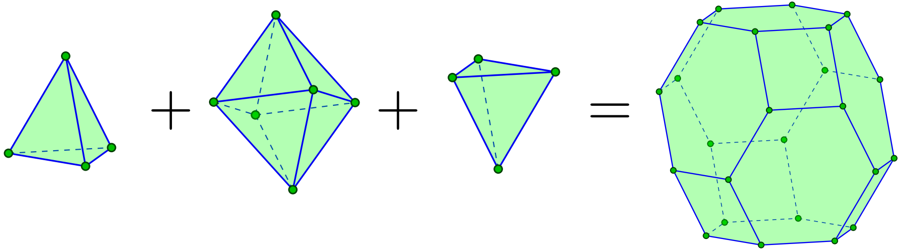

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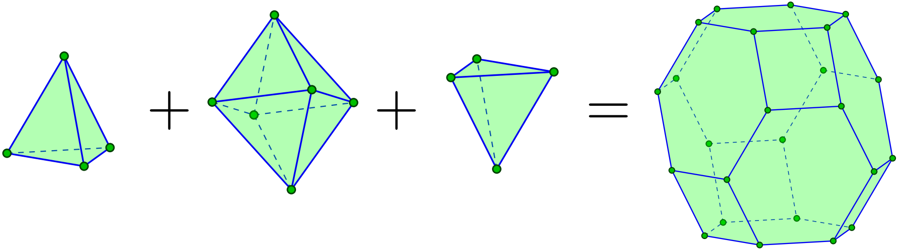

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Case $d = 3$: higher associahedra

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Case $d = 3$: higher associahedra

Assume that $\mathcal{A} \subseteq \mathbb{R}^{d-1} = \mathbb{R}^2$ consists of the vertices of a convex n-gon.

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Case $d = 3$: higher associahedra

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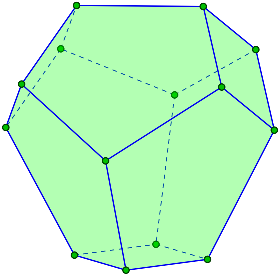

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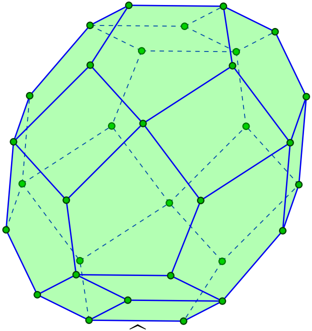
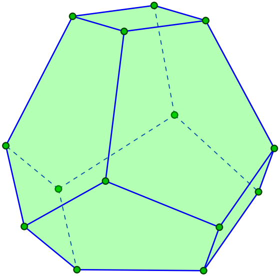


$\widehat{\Sigma}_{\mathcal{A},2}$



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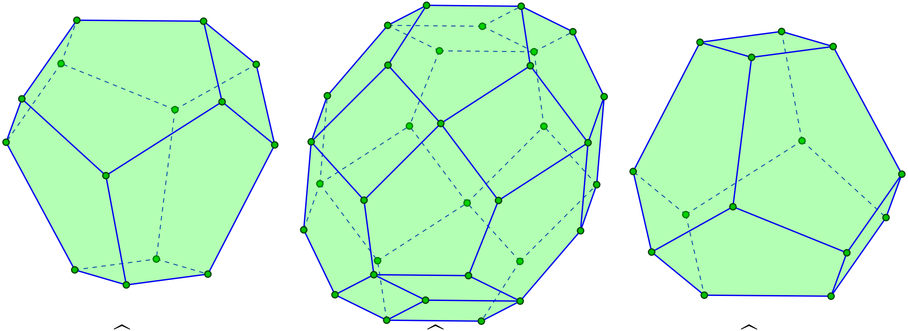

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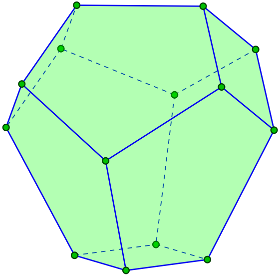

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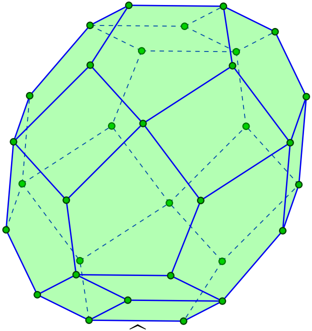

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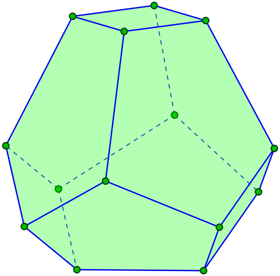

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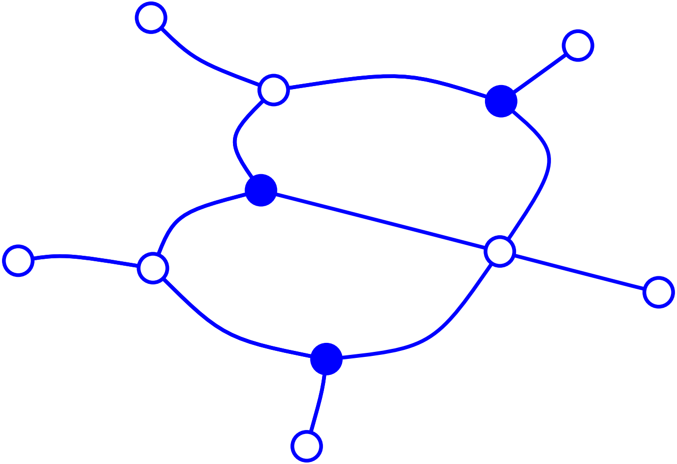
Plabic graphs		Higher associahedron $\widehat{\Sigma}_{\mathcal{A},k}$

Plabic graphs



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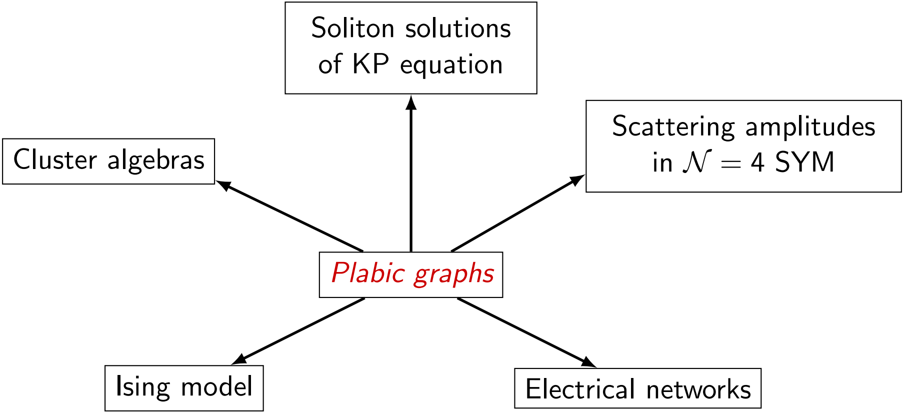

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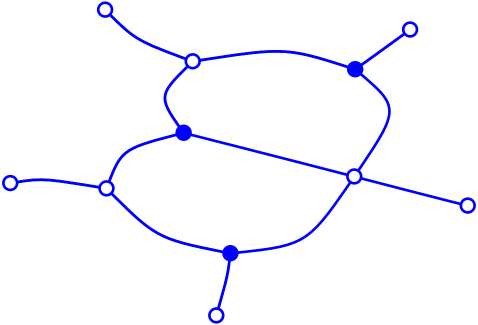
A *plabic graph* is a planar bipartite graph embedded in a disk, with n boundary vertices of degree 1.



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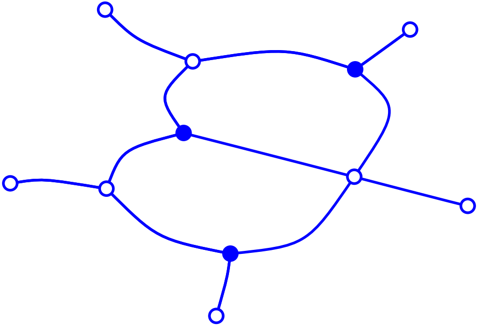

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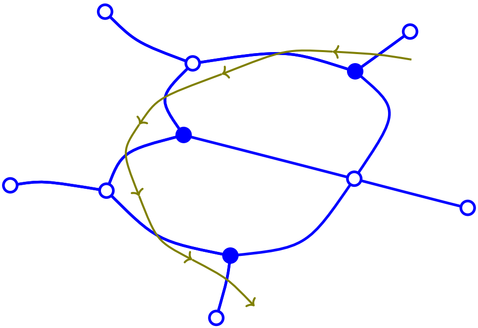

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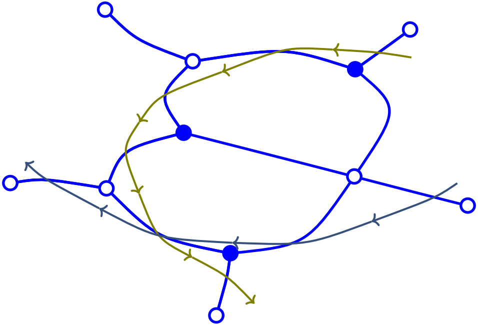

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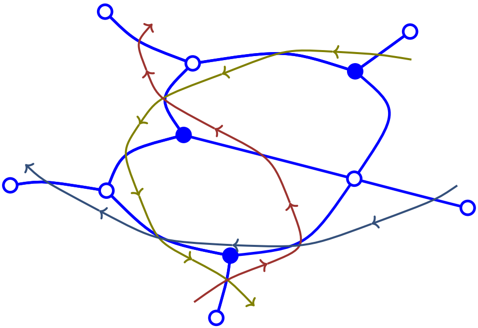

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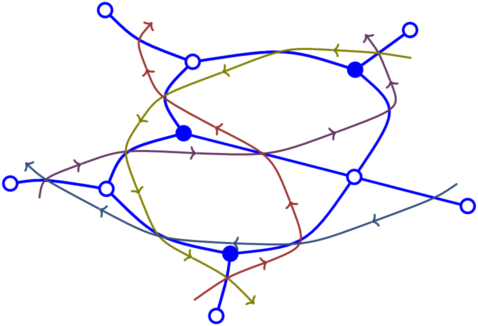

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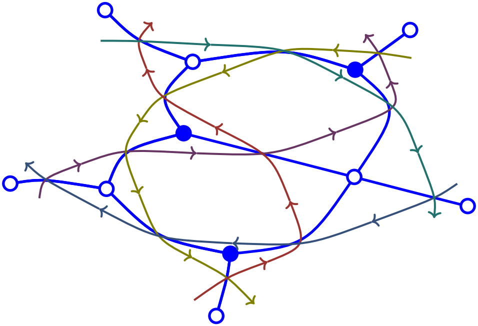

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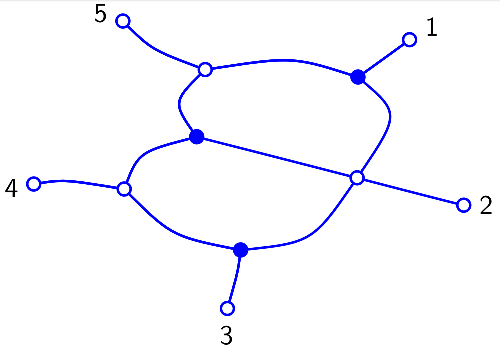

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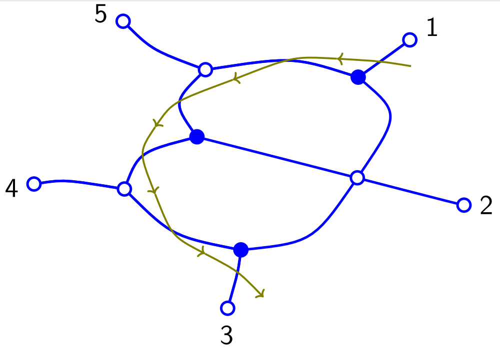

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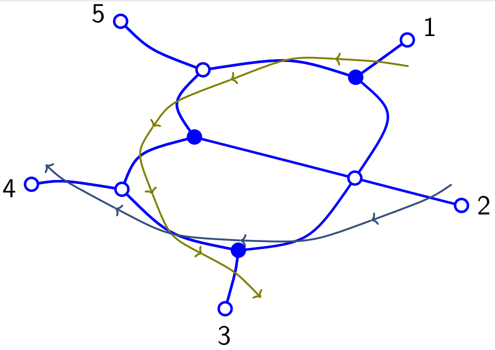

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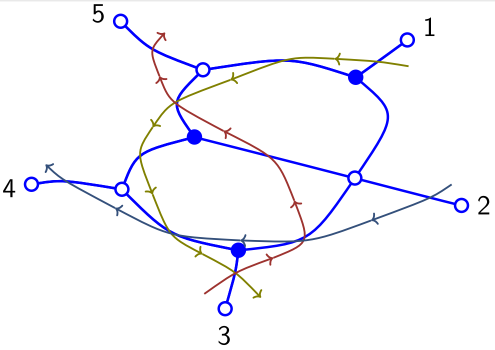

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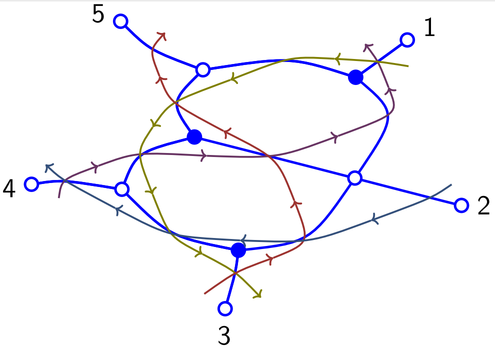

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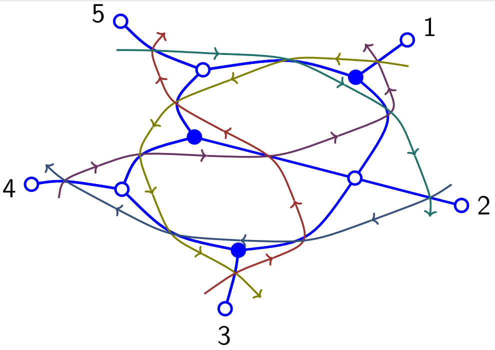

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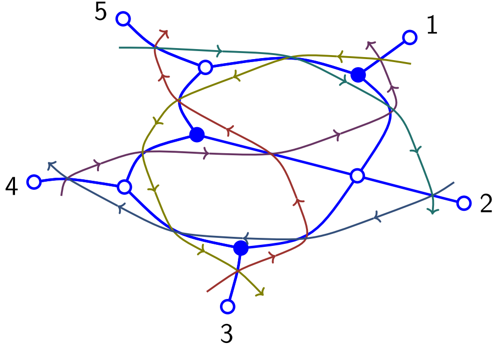
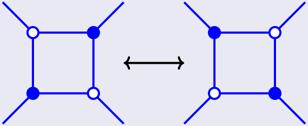


a $(2, 5)$-plabic graph

Square moves

Theorem (Postnikov (2006))

Any two (k, n)-plabic graphs are connected by a sequence of square moves:



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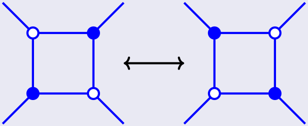

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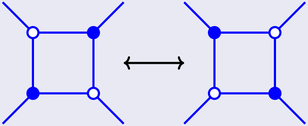

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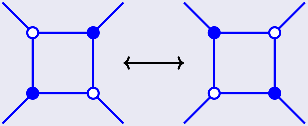

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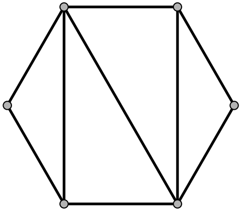

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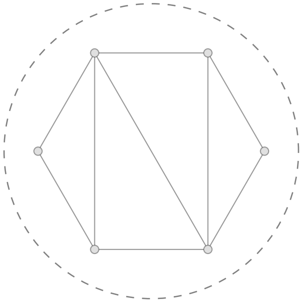

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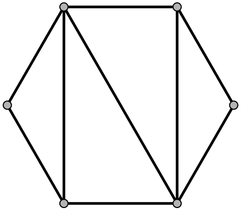

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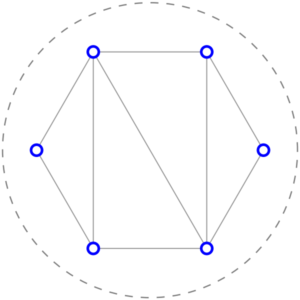

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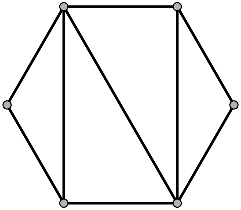

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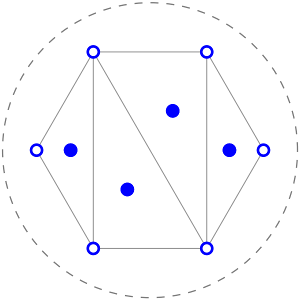

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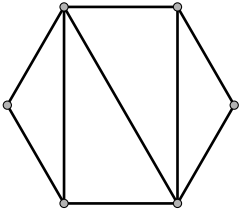

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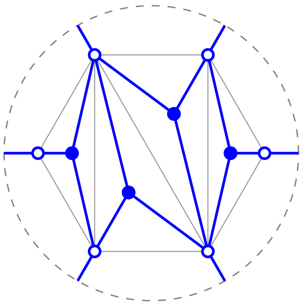

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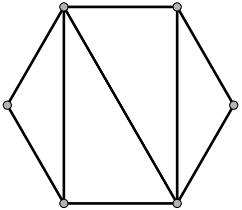

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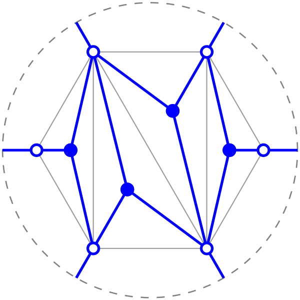

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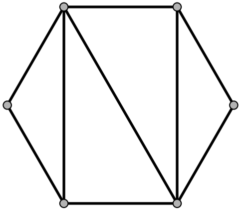

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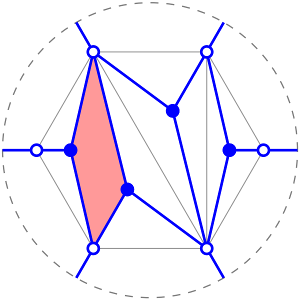

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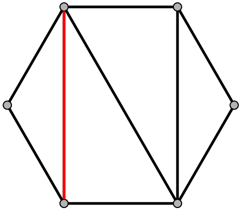

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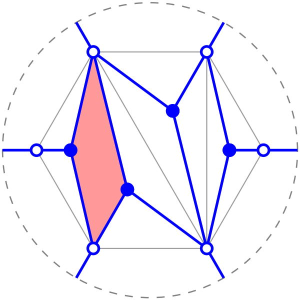

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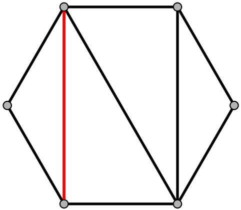

Thus $P_{2,n}$ is the usual associahedron.

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Connecting them by square moves, we get the following picture:

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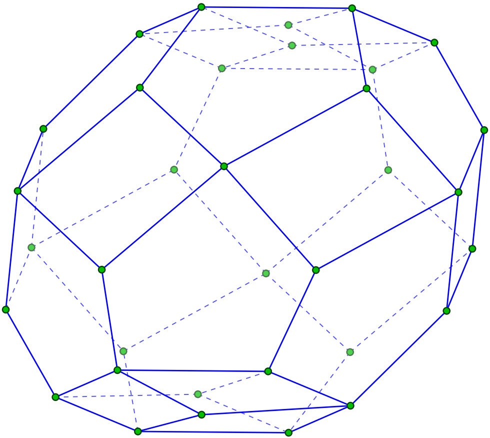

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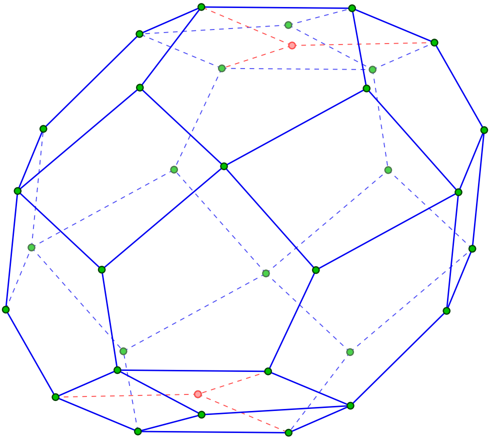

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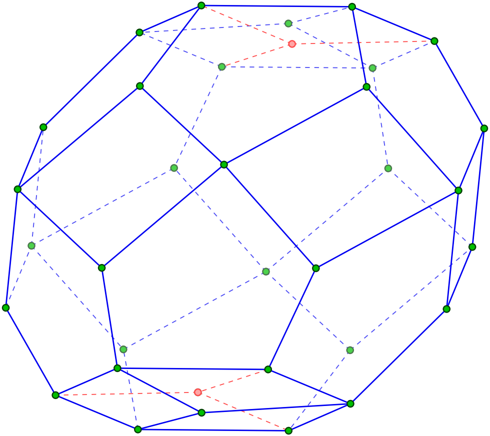

the polytope $P_{3,6}$ doesn't exist!

Motivation

Combinatorial objects	$\longrightarrow$	Polytope
k-element sets		Hypersimplex $\Delta_{k,n}$
Permutations in S_n		Permutohedron Perm_n
Triangulations of a convex n-gon		Associahedron
Triangulations of $\mathcal{A} \subseteq \mathbb{R}^{d-1}$		Secondary polytope
Zonotopal tilings		Fiber zonotope
Plabic graphs		Higher associahedron $\widehat{\Sigma}_{\mathcal{A},k}$

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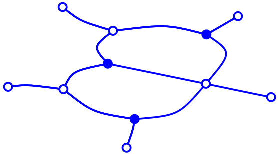
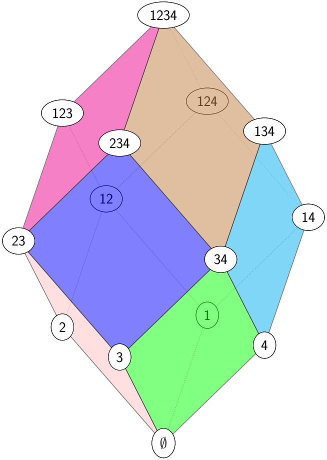
Regular(?) plabic graphs		Higher associahedron $\widehat{\Sigma}_{\mathcal{A},k}$

Plabic graphs and zonotopal tilings

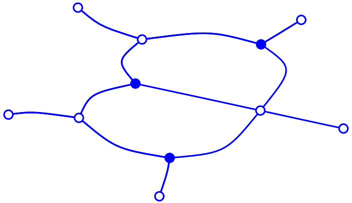


vs



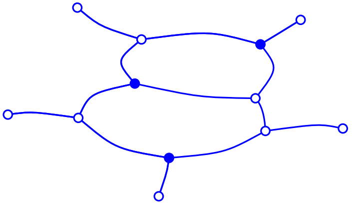
Trivalent plabic graphs

A *trivalent* (k, n)-plabic graph is obtained from a bipartite one by "uncontracting" vertices until each interior vertex has degree 3.



bipartite

$\longrightarrow$



trivalent

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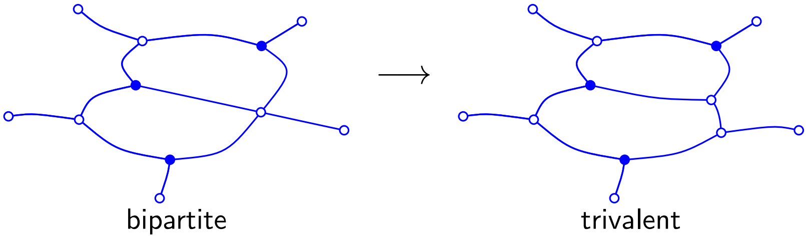
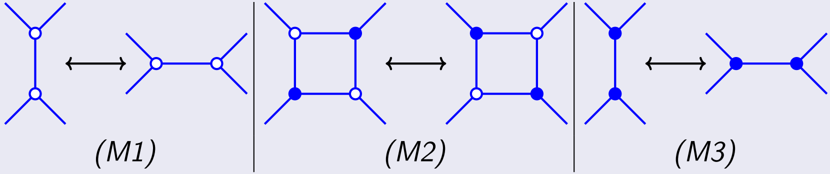


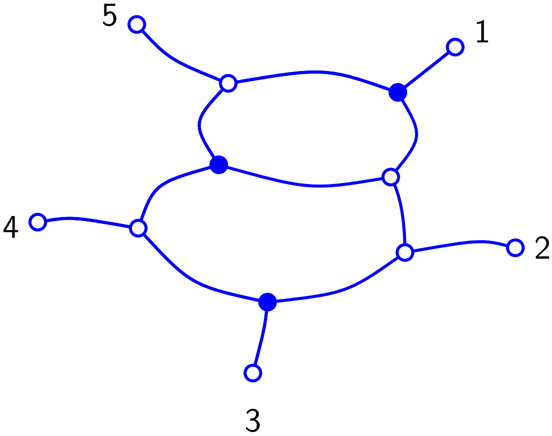
Theorem (Postnikov (2006))

Any two trivalent (k, n)-plabic graphs are connected by moves:



Planar duals of plabic graphs

Each (k, n)-plabic graph has $k(n-k)+1$ faces.



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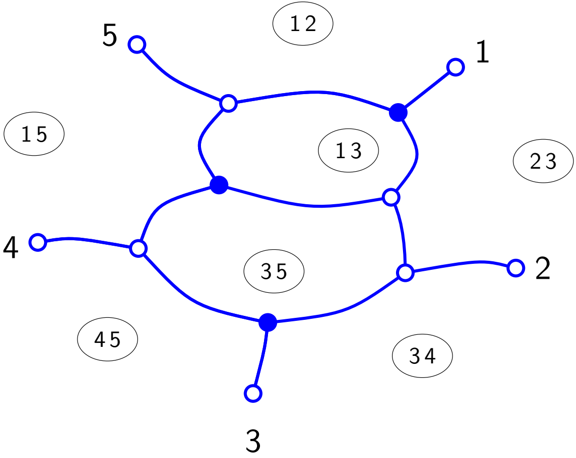
Label each face of a (k, n)-plabic graph by a k-element set:



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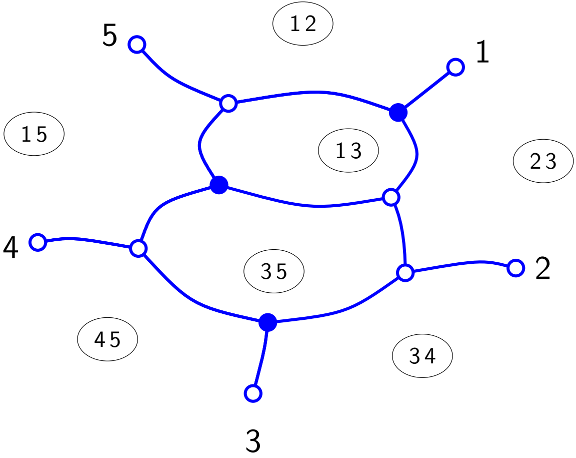

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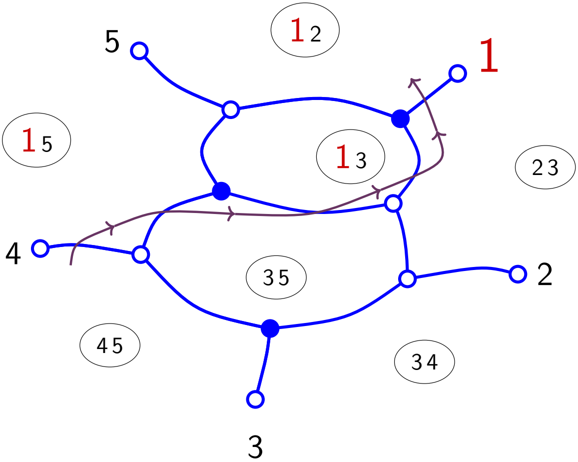

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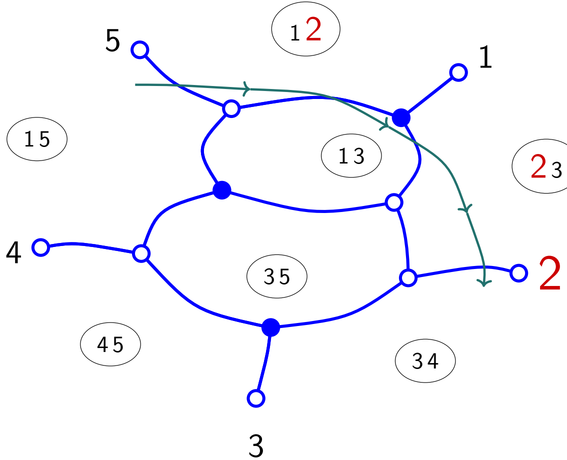

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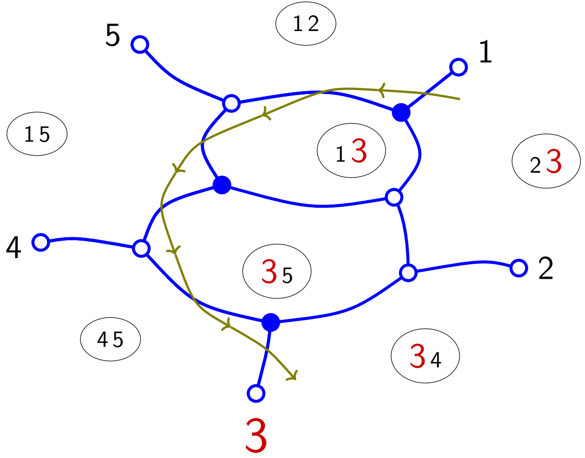

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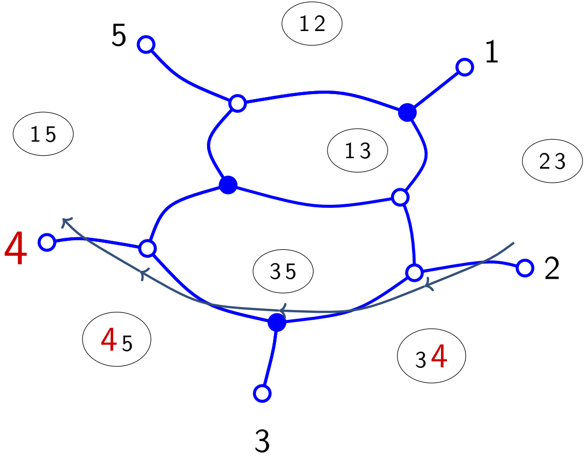

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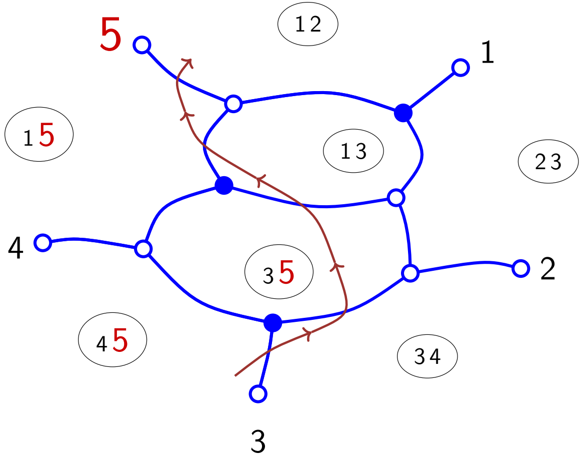

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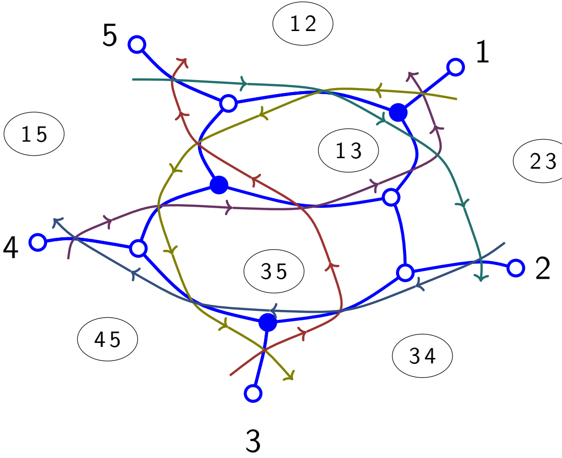

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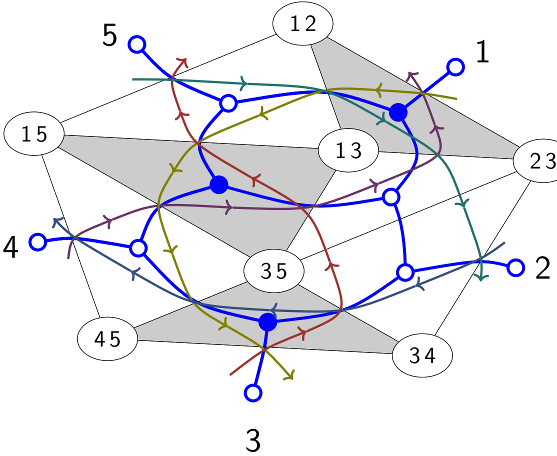
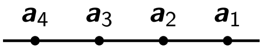


Zonotopal tilings

Zonotopal tilings

- Point configuration: $\mathcal{A} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n) \subseteq \mathbb{R}^{d-1}$;



$\mathcal{A}$

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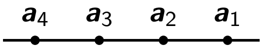

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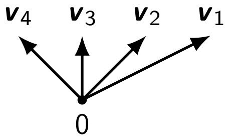

$\mathcal{V}$

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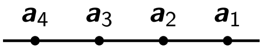

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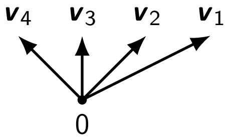
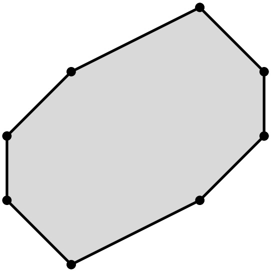


$\mathcal{V}$



$\mathcal{Z}_{\mathcal{V}}$

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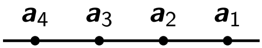

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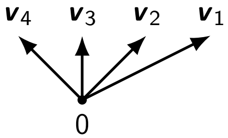

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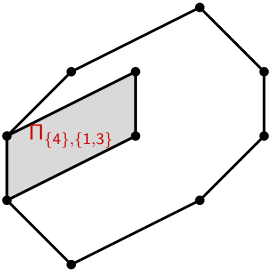

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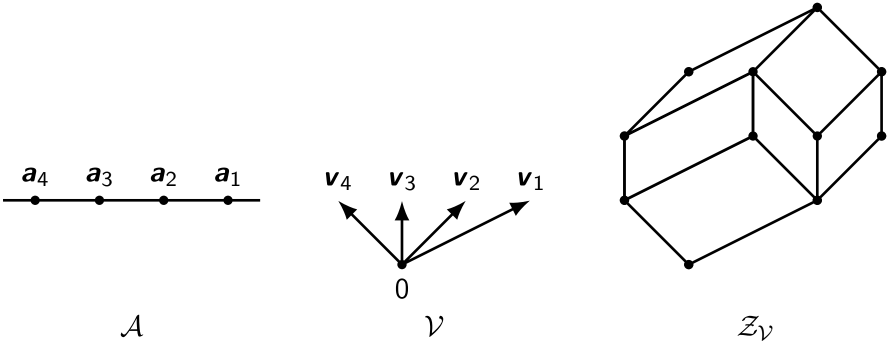

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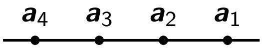
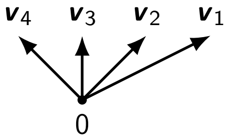


$\mathcal{A}$



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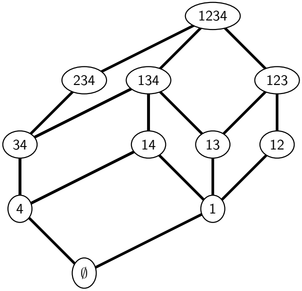
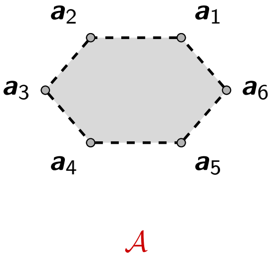


$\mathcal{Z}_\mathcal{V}$

3D cyclic zonotopes

3D cyclic zonotopes

From now on, assume that $\mathcal{A} \subseteq \mathbb{R}^2$ consists of vertices of a convex n-gon.



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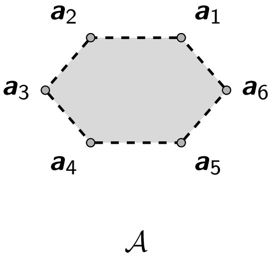

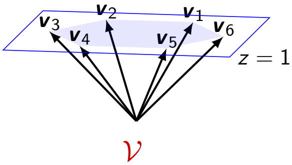

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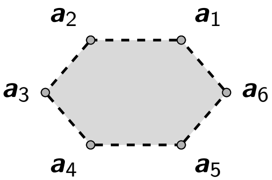

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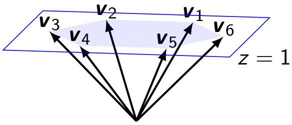
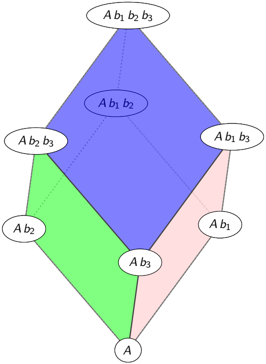
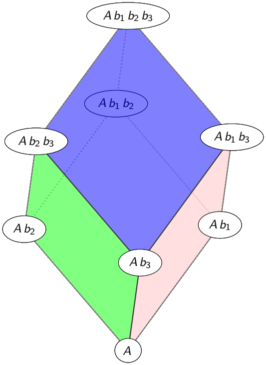


$\mathcal{V}$



$\Pi_{A,B}$

Sections of tiles



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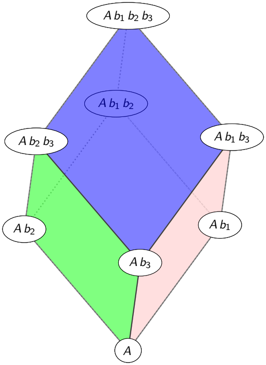

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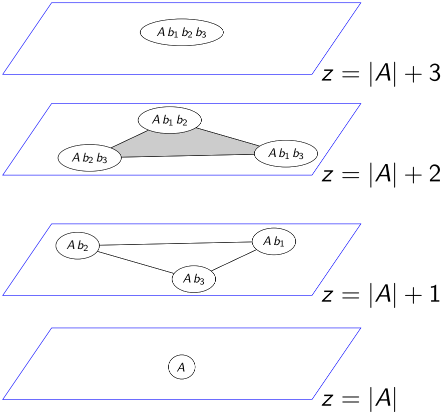

$\Pi_{A,B} \cap \{z = k\}$ for $k = |A|, \dots, |A| + 3$.

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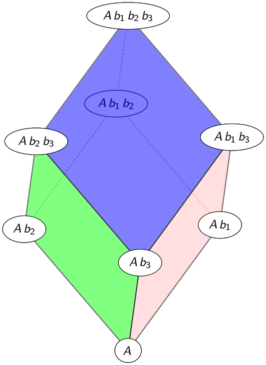

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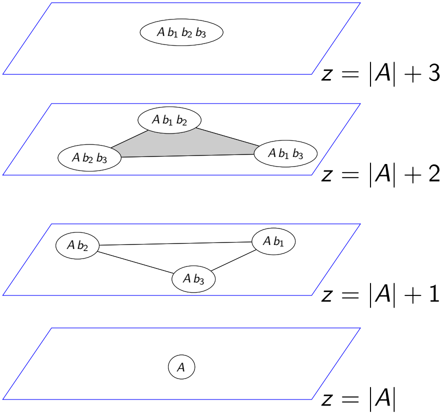

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Fine zonotopal tiling of $\mathcal{Z}_\mathcal{V}$ $\longrightarrow$ a subdivision of $\mathcal{Z}_\mathcal{V} \cap \{z = k\}$ into black and white triangles

Plabic graphs and zonotopal tilings

Theorem (G. (2017))

trivalent (k,n)*-plabic graphs* $\xleftrightarrow[\textit{dual}]{\textit{planar}}$ *horizontal sections at level* k *of fine zonotopal tilings of* $\mathcal{Z}_{\mathcal{V}}$

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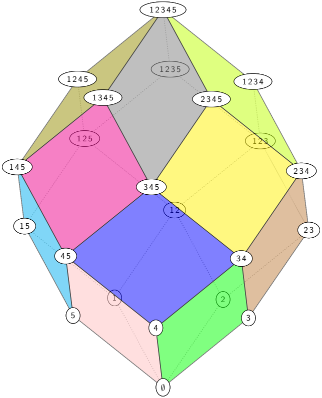

$\mathcal{Z}_{\mathcal{V}}$ for $d = 3$, $n = 5$

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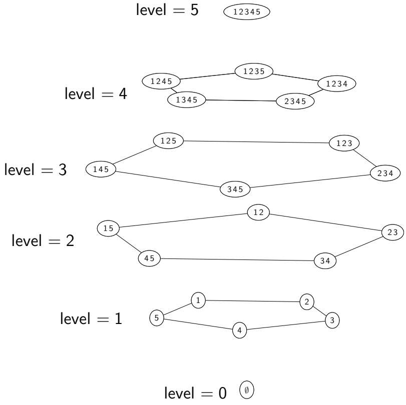

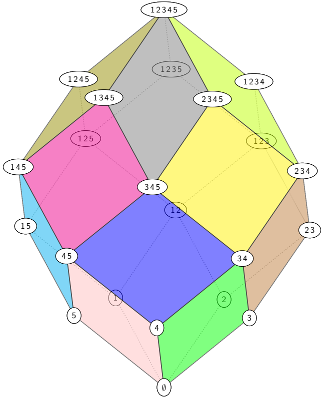

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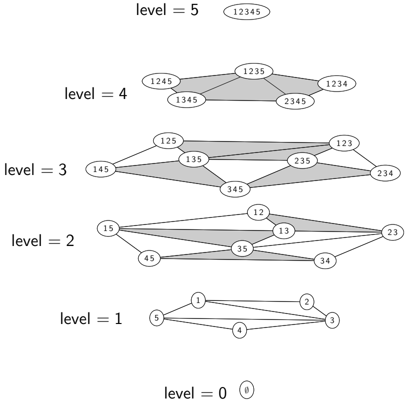

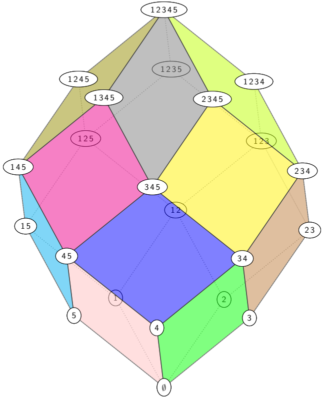

$\mathcal{Z}_{\mathcal{V}}$ for $d = 3$, $n = 5$

Plabic graphs and zonotopal tilings

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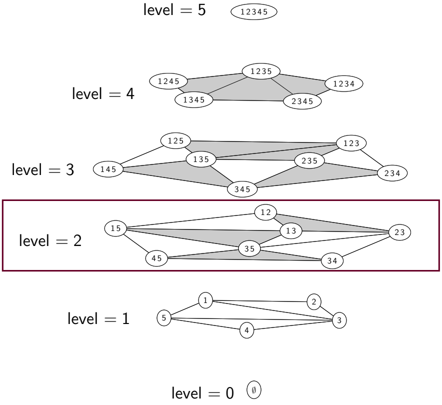
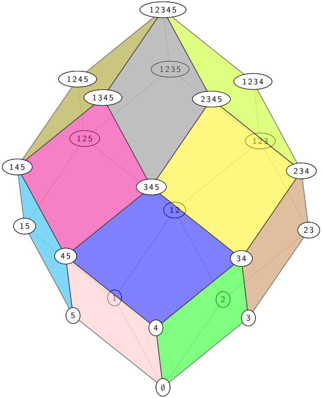
trivalent (k,n)-plabic graphs $\xleftrightarrow[\textit{dual}]{\textit{planar}}$ *horizontal sections at level k of fine zonotopal tilings of $\mathcal{Z}_\mathcal{V}$*



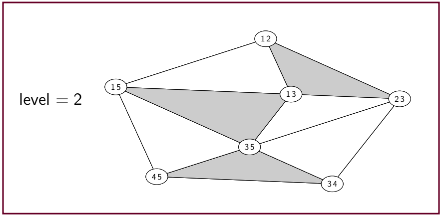


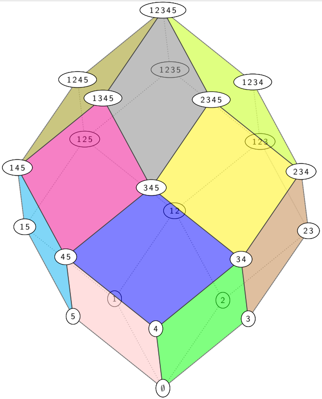
$\mathcal{Z}_\mathcal{V}$ for $d = 3$, $n = 5$

Plabic graphs and zonotopal tilings

Theorem (G. (2017))

trivalent (k,n)*-plabic graphs* $\xleftrightarrow[\textit{dual}]{\textit{planar}}$ *horizontal sections at level* k *of fine zonotopal tilings of* $\mathcal{Z}_{\mathcal{V}}$





$\mathcal{Z}_{\mathcal{V}}$ for $d = 3$, $n = 5$

Plabic graphs and zonotopal tilings

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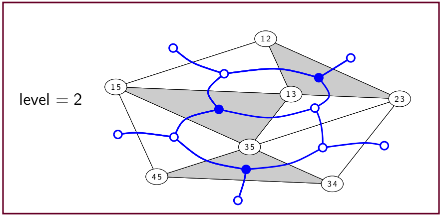
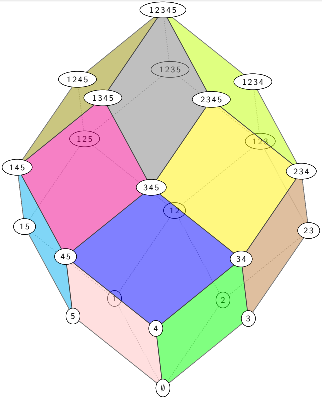


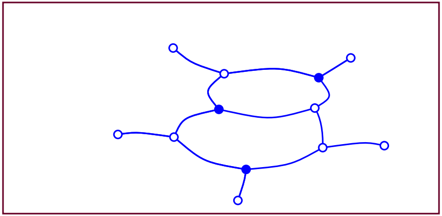


$\mathcal{Z}_{\mathcal{V}}$ for $d = 3$, $n = 5$

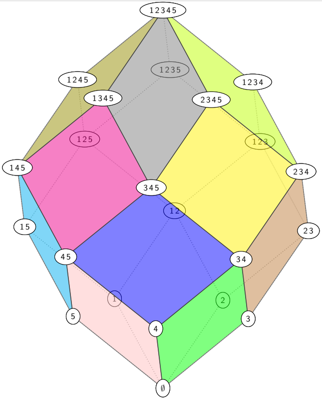
Plabic graphs and zonotopal tilings

Theorem (G. (2017))

trivalent (k, n)-plabic graphs $\overset{\textit{planar}}{\underset{\textit{dual}}{\longleftrightarrow}}$ *horizontal sections at level k of fine zonotopal tilings of $\mathcal{Z}_{\mathcal{V}}$*



a trivalent $(2, 5)$-plabic graph



$\mathcal{Z}_{\mathcal{V}}$ for $d = 3$, $n = 5$

Flips of fine zonotopal tilings

$n = d \quad \implies \quad \mathcal{Z}_{\mathcal{V}}$ admits one fine zonotopal tiling

Flips of fine zonotopal tilings

$$n = d \quad \Longrightarrow \quad \mathcal{Z}_{\mathcal{V}} \text{ admits one fine zonotopal tiling}$$
$$n = d + 1 \quad \Longrightarrow \quad \mathcal{Z}_{\mathcal{V}} \text{ admits two fine zonotopal tilings}$$

Flips of fine zonotopal tilings

$$\begin{aligned} n = d \quad &\implies \quad \mathcal{Z}_{\mathcal{V}} \text{ admits one fine zonotopal tiling} \\ n = d+1 \quad &\implies \quad \mathcal{Z}_{\mathcal{V}} \text{ admits two fine zonotopal tilings} \end{aligned}$$

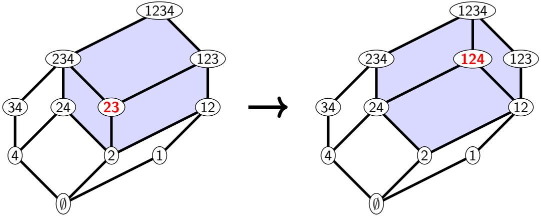
A flip consists of replacing a shifted copy of one tiling with the other one.

Flips of fine zonotopal tilings

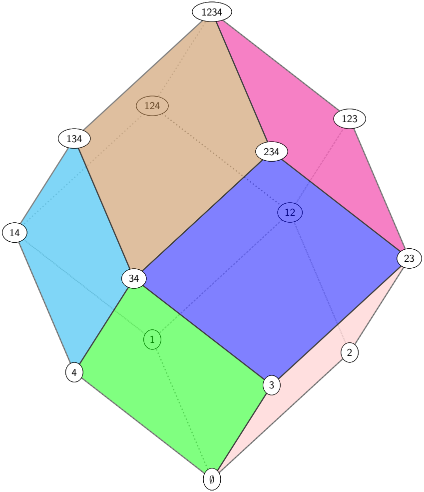
$$
\begin{array}{rcl}
n = d & \Longrightarrow & \mathcal{Z}_{\mathcal{V}} \text{ admits one fine zonotopal tiling} \\
n = d+1 & \Longrightarrow & \mathcal{Z}_{\mathcal{V}} \text{ admits two fine zonotopal tilings}
\end{array}
$$

A flip consists of replacing a shifted copy of one tiling with the other one.

Example for $d = 2$:



The case $d = 3$, $n = 4$



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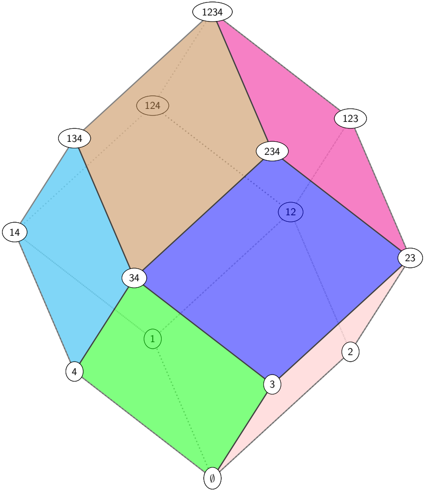
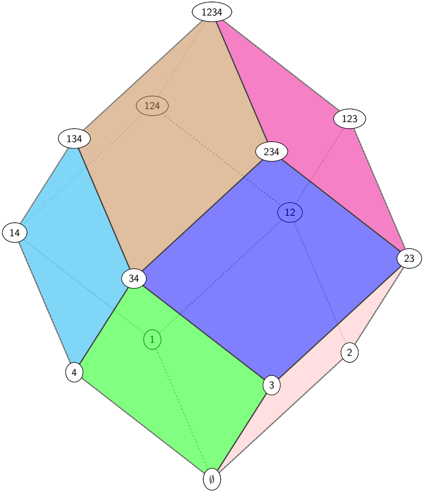


Q: How many fine zonotopal tilings?

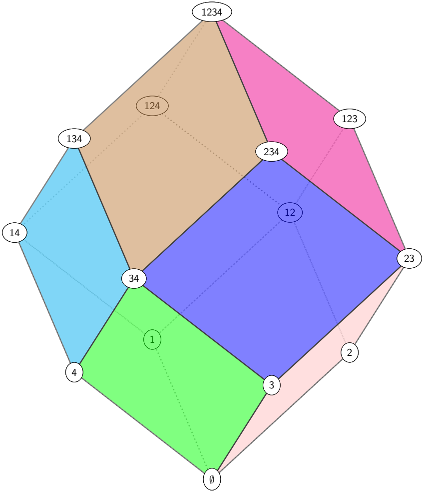
The case $d = 3$, $n = 4$



Q: How many fine zonotopal tilings?

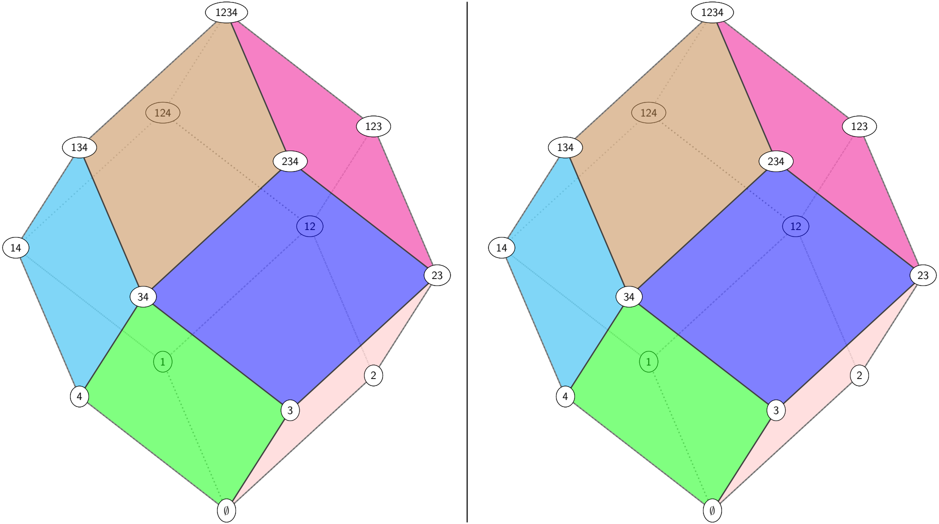
A: Two.

The case $d = 3$, $n = 4$



Q: How many fine zonotopal tilings?
A: Two. (because $n = d + 1$)

The case $d = 3$, $n = 4$



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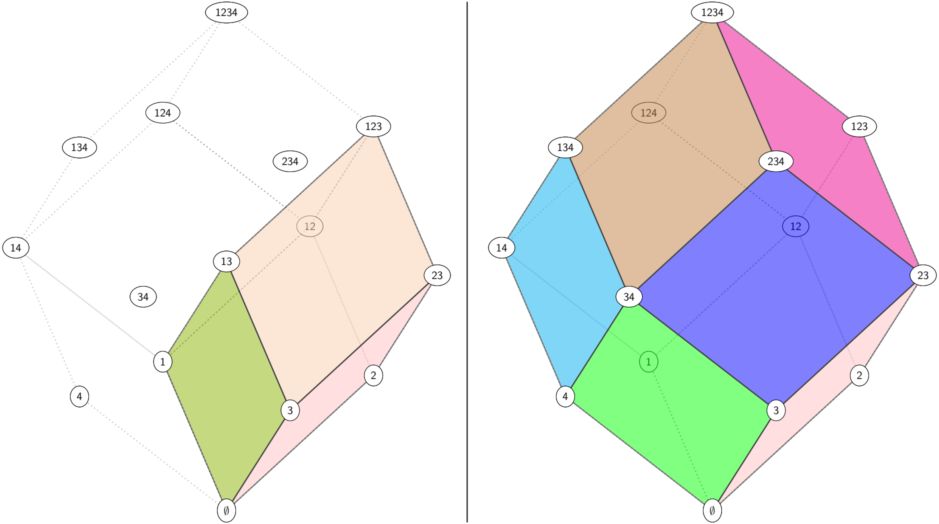

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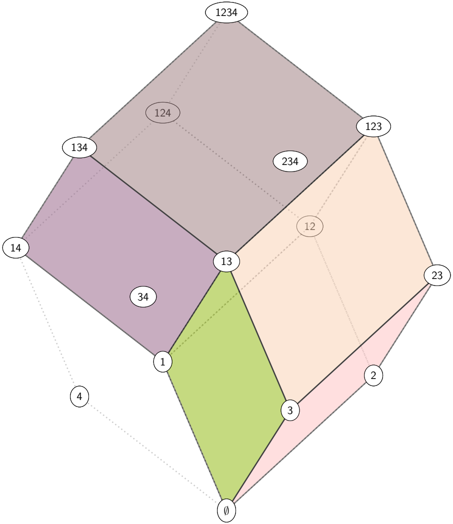

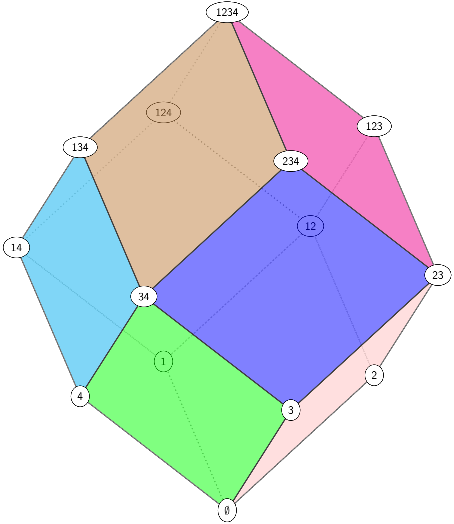

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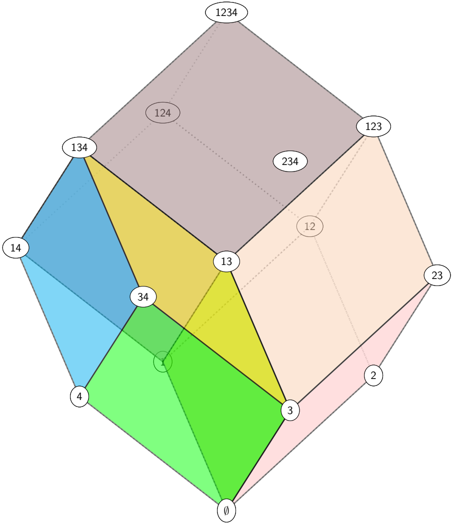

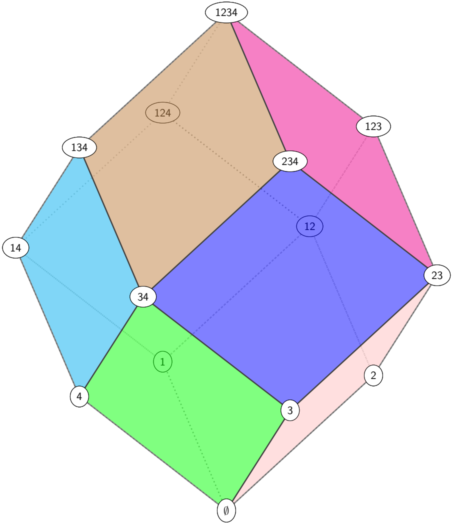

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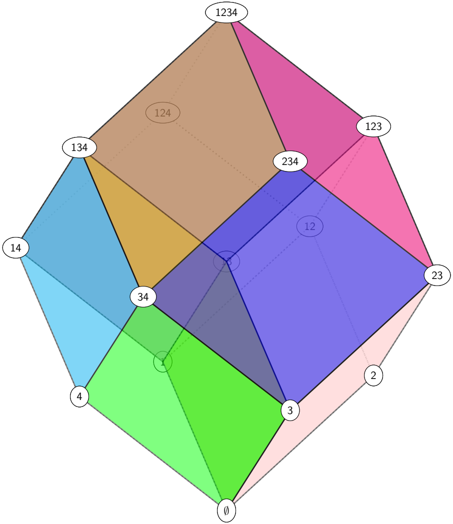

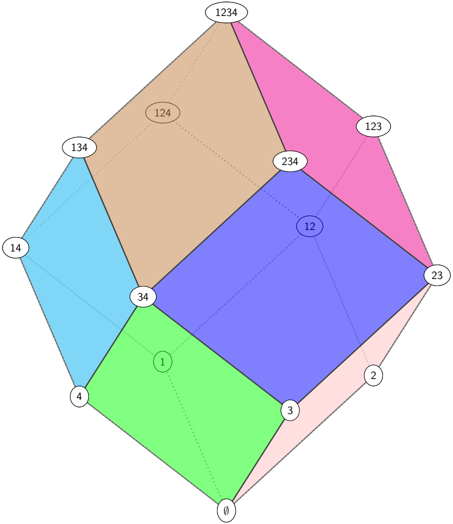

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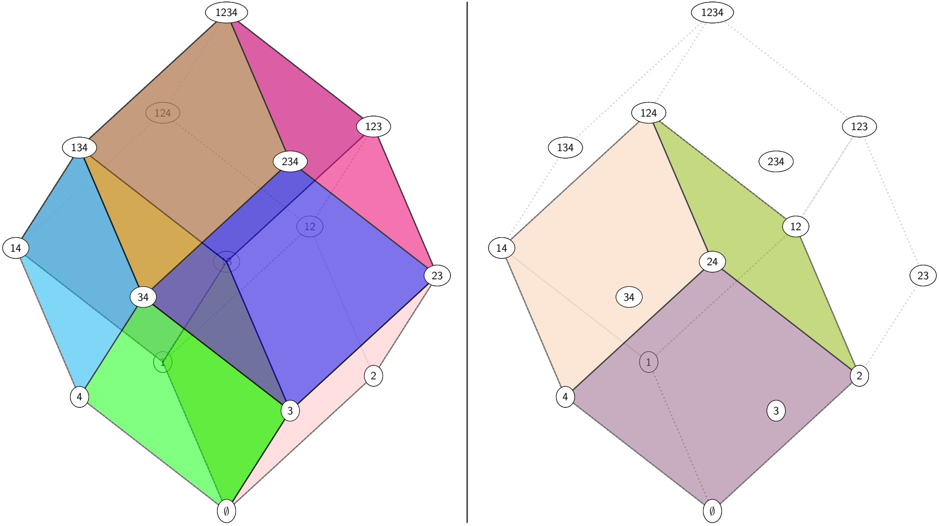

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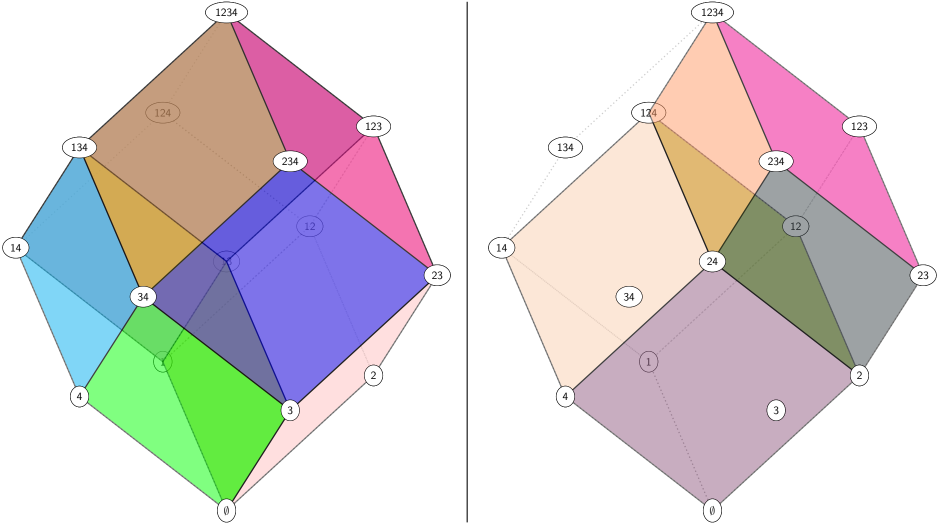

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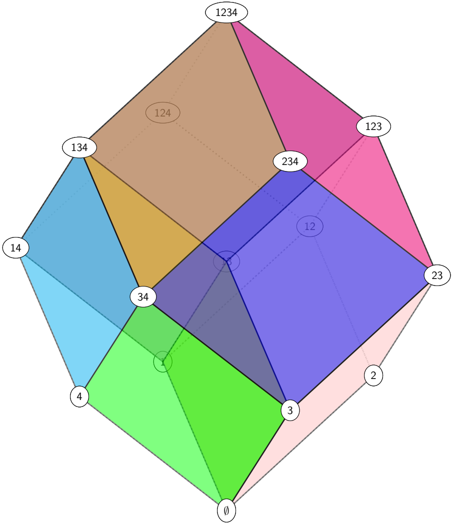

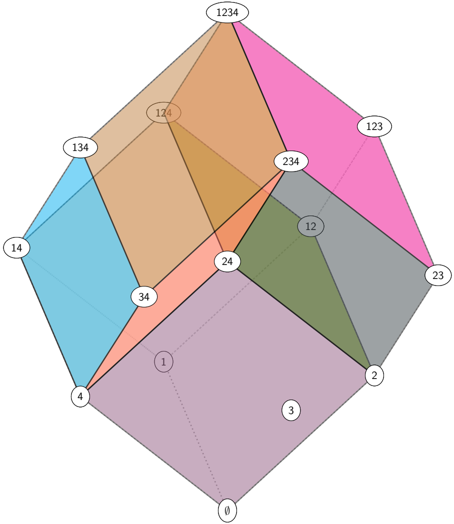

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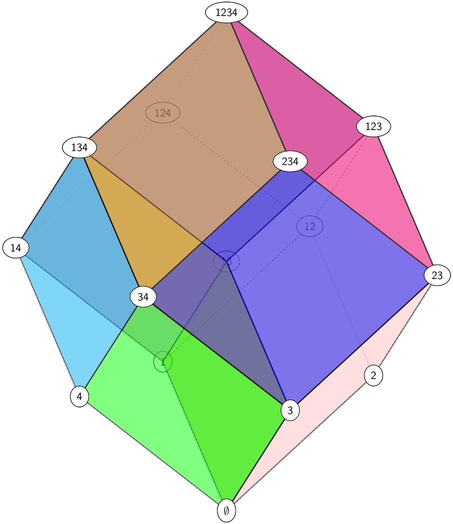
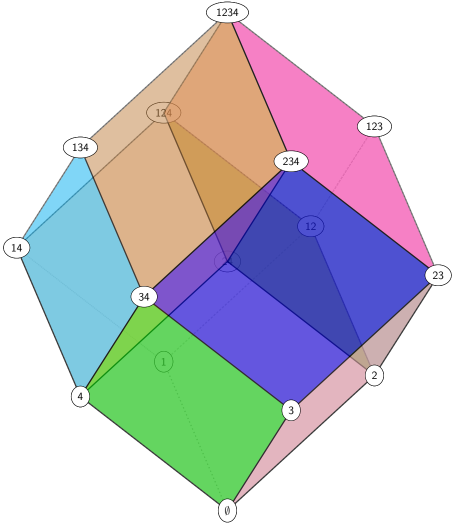
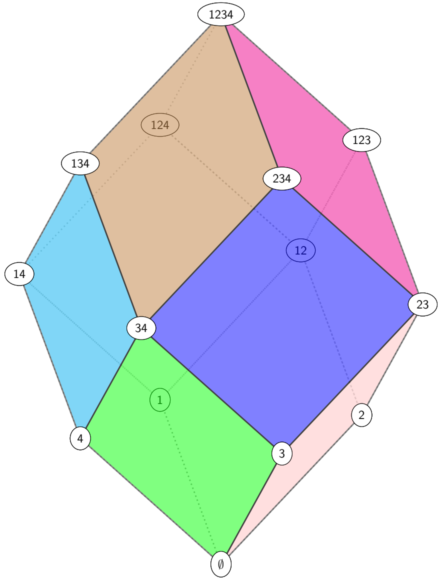
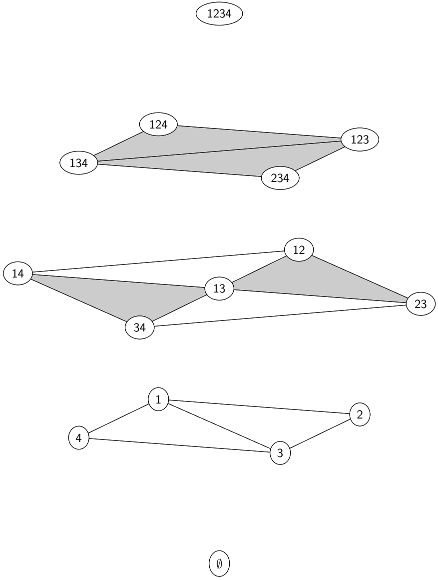




Sections of flips: $d = 3$, $n = 4$





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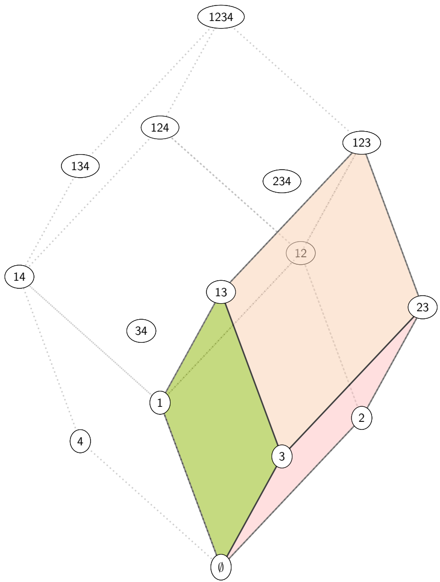

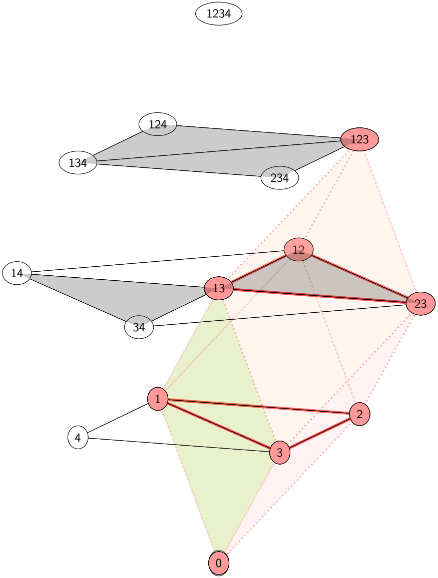

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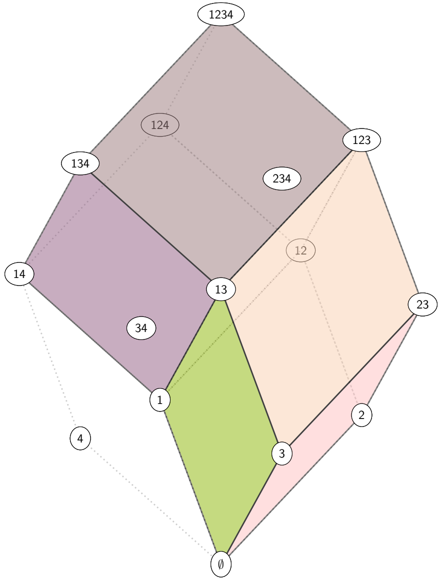

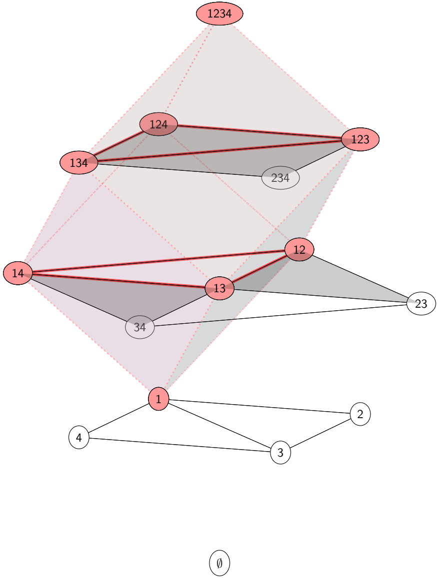

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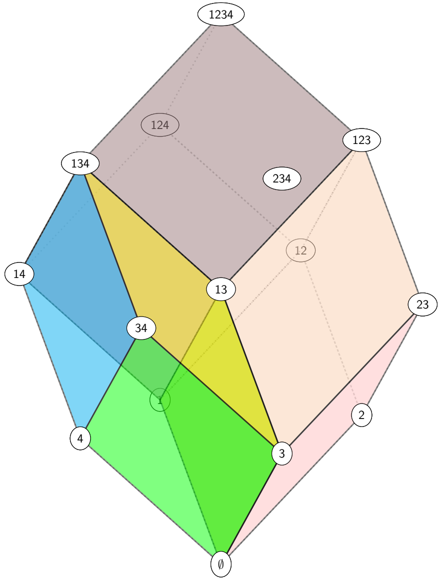
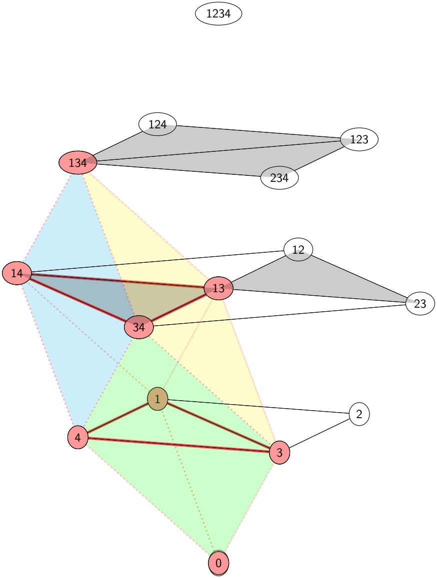
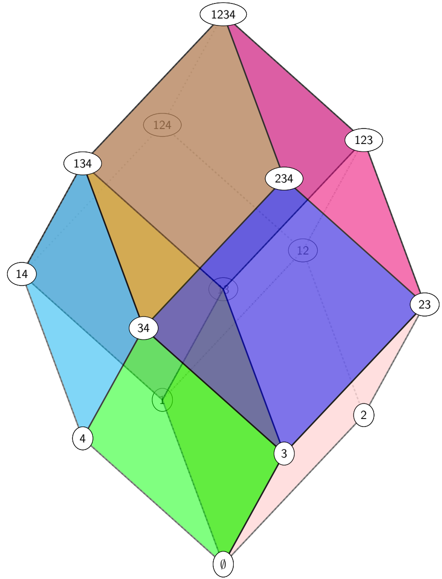
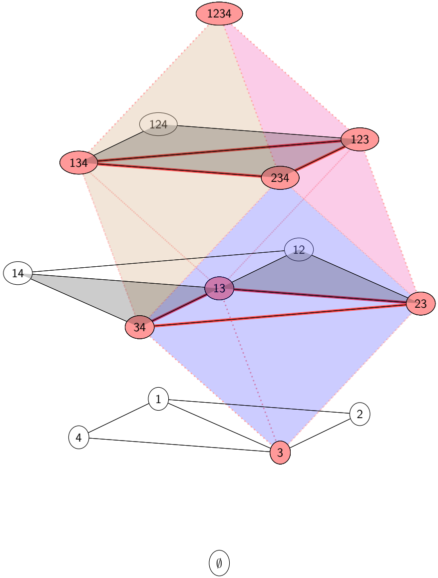




Sections of flips: $d = 3$, $n = 4$





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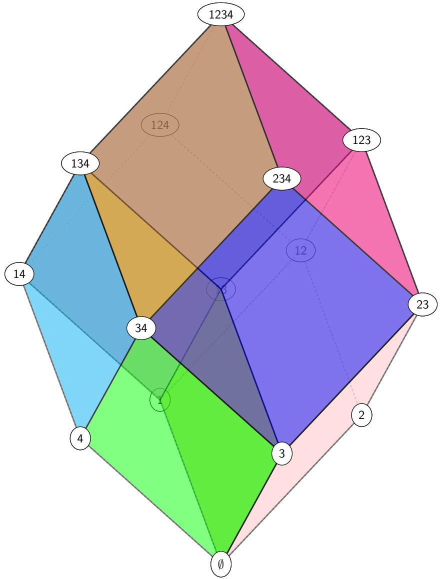

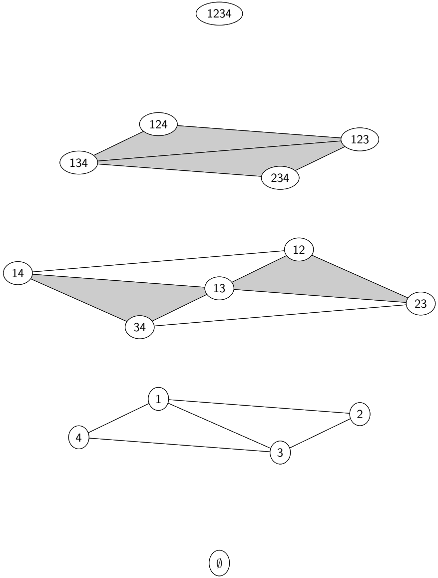

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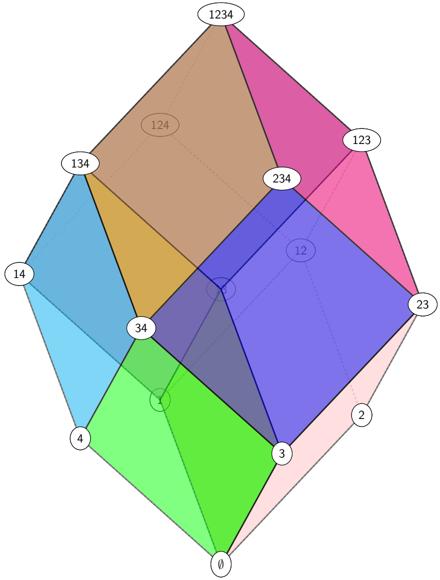
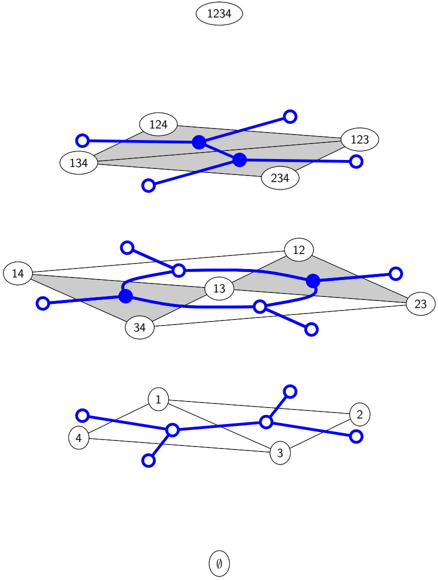
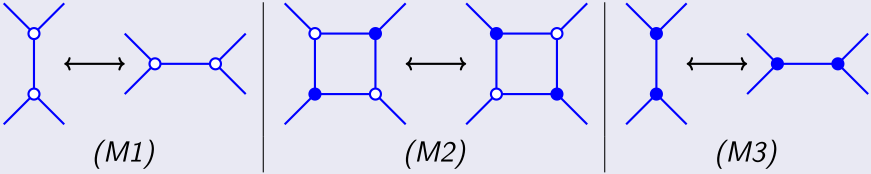




Moves and flips

Theorem (Postnikov (2006))

Any two trivalent (k, n)-plabic graphs are connected by a sequence of moves:



(M1) *(M2)* *(M3)*

Moves and flips

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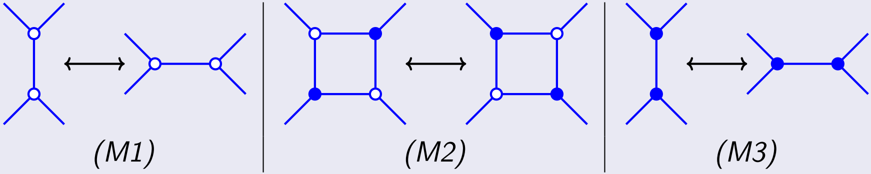

Recall: $\mathcal{A} \subseteq \mathbb{R}^2$ consists of vertices of a convex n-gon, and $\mathcal{V} \subseteq \mathbb{R}^3$ is the lift of $\mathcal{A}$.

Theorem (Ziegler (1993))

Any two fine zonotopal tilings of $\mathcal{Z}_{\mathcal{V}}$ are connected by a sequence of flips.

Moves and flips

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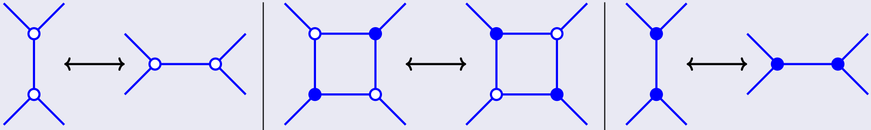

(M1) (M2) (M3)

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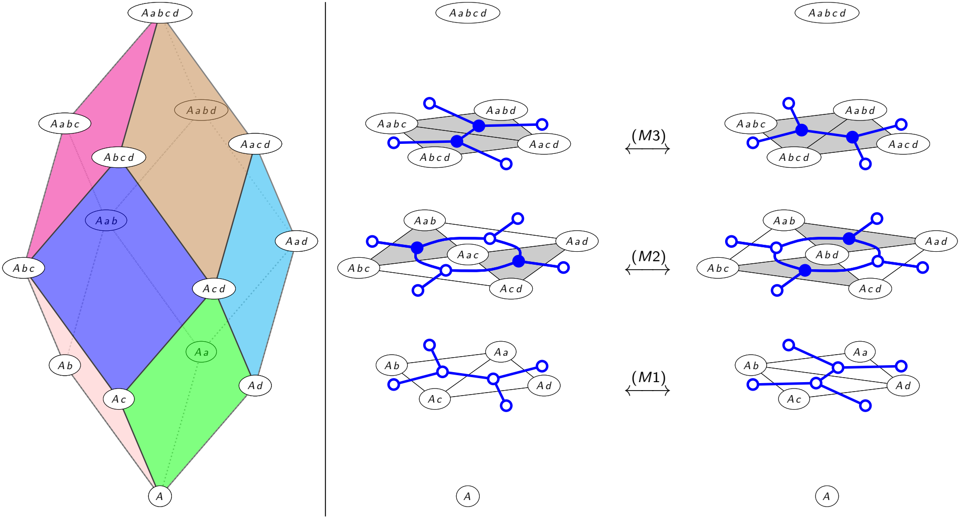
Theorem (Ziegler (1993))

Any two fine zonotopal tilings of $\mathcal{Z}_{\mathcal{V}}$ are connected by a sequence of flips.

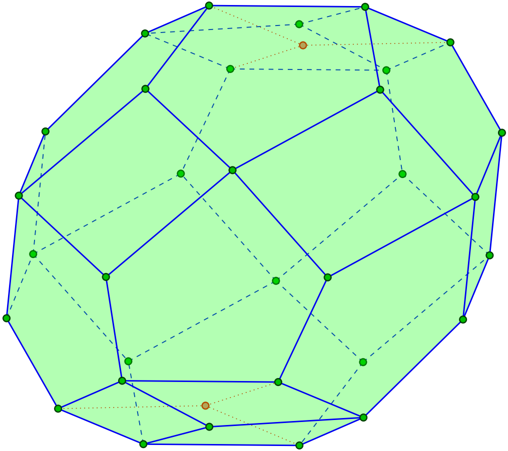
Theorem (G. (2017))

Moves (M1)–(M3) of (k,n)-plabic graphs $\xleftrightarrow[\text{dual}]{\text{planar}}$ *horizontal sections of flips of fine zonotopal tilings of $\mathcal{Z}_{\mathcal{V}}$*

Moves = sections of flips



Higher secondary polytopes



Regular triangulations and zonotopal tilings

Let $\mathcal{A} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ be a point configuration in $\mathbb{R}^{d-1}$.
Choose a height vector $\boldsymbol{h} = (h_1, \dots, h_n) \in \mathbb{R}^n$.

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Definition

A regular $\mathcal{A}$-triangulation is obtained by projecting the upper boundary of $\mathrm{conv}\{(\boldsymbol{a}_i, h_i) \mid i = 1, \dots, n\} \subseteq \mathbb{R}^d$ onto $\mathrm{conv}\mathcal{A}$.

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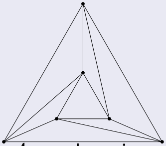

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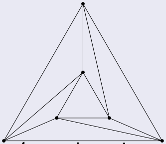

not an $\mathcal{A}$-regular triangulation

Definition

A regular fine zonotopal tiling of $\mathcal{Z}_\mathcal{V}$ is obtained by projecting the upper boundary of $\mathcal{Z}_{\widetilde{\mathcal{V}}} := [0, \tilde{\boldsymbol{v}}_1] + \dots + [0, \tilde{\boldsymbol{v}}_n]$ (where $\tilde{\boldsymbol{v}}_i = (\boldsymbol{v}_i, h_i) \in \mathbb{R}^{d+1}$) onto $\mathcal{Z}_\mathcal{V}$.

Regular triangulations and zonotopal tilings

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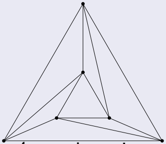

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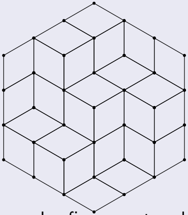

not a regular fine zonotopal tiling

Secondary polytopes

Given an $\mathcal{A}$-triangulation τ, define a vector

$$\mathrm{vert}^{\mathsf{GKZ}}(\tau) := \sum_{\Delta_B \in \tau} \mathrm{Vol}^{d-1}(\Delta_B) \cdot \sum_{b \in B} \boldsymbol{e}_b \quad \in \mathbb{R}^n.$$

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Example

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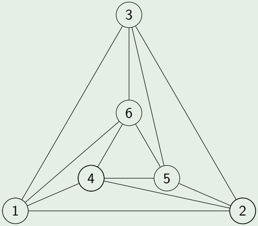

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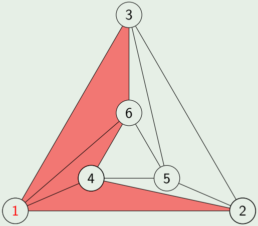

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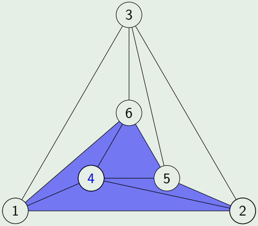

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The secondary polytope $\Sigma_{\mathcal{A}}^{\mathsf{GKZ}}$ of $\mathcal{A}$ is defined as the convex hull

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The vertices of $\Sigma_{\mathcal{A}}^{\mathsf{GKZ}}$ correspond precisely to regular $\mathcal{A}$-triangulations.

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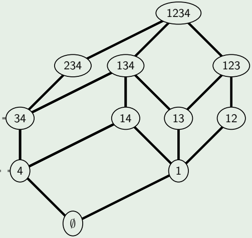

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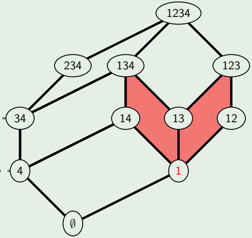

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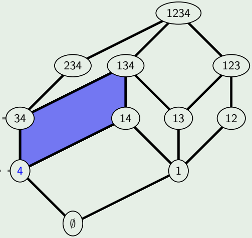

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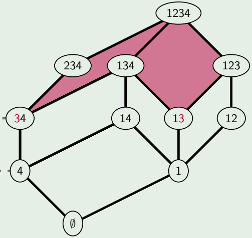

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Conjecture

The word regular can be omitted from the above definition.

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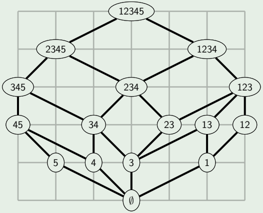

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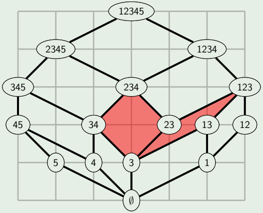

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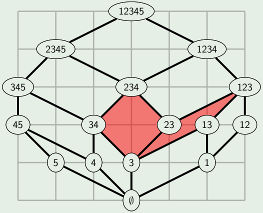

Higher secondary polytopes vs secondary polytopes

- $\widehat{\Sigma}_{\mathcal{A},1} = \Sigma^{\mathrm{GKZ}}_{\mathcal{A}}$ is the secondary polytope.

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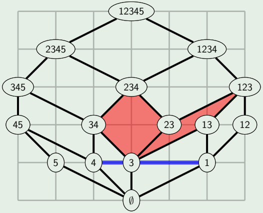

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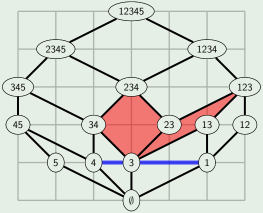

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Claim: up to shift and dilation,

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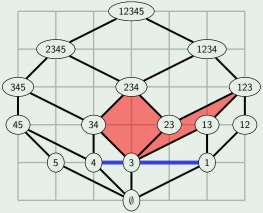

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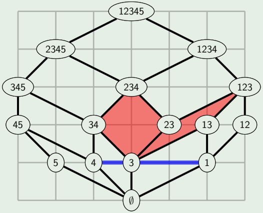

Back to plabic graphs

Theorem (G. (2017))

trivalent (k,n)*-plabic graphs* $\xleftrightarrow[\textit{dual}]{\textit{planar}}$ *horizontal sections of fine zonotopal tilings of* $\mathcal{Z}_{\mathcal{V}}$

Back to plabic graphs

Theorem (G. (2017))

trivalent (k, n)-plabic graphs	$\xleftrightarrow[\text{dual}]{\text{planar}}$	*horizontal sections of fine zonotopal tilings of $\mathcal{Z}_{\mathcal{V}}$*
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We say that a bipartite/trivalent (k,n)-plabic graph is $\mathcal{A}$-regular if it arises from a regular fine zonotopal tiling of $\mathcal{Z}_{\mathcal{V}}$.

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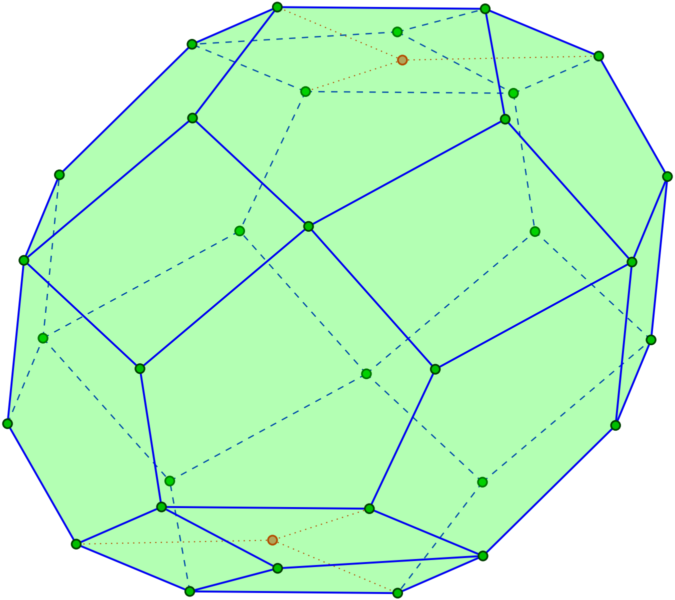
Non-regular plabic graphs



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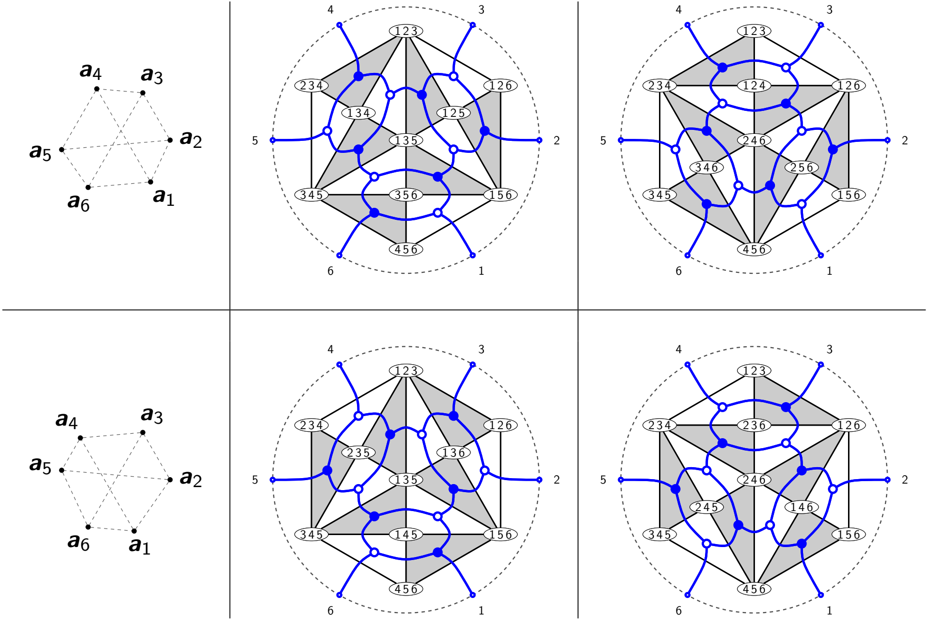

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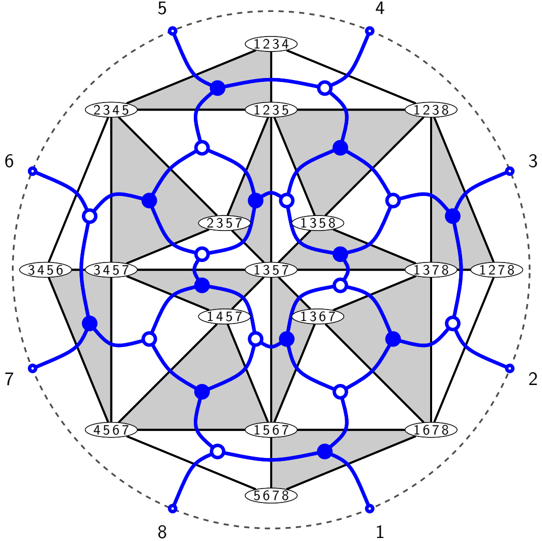

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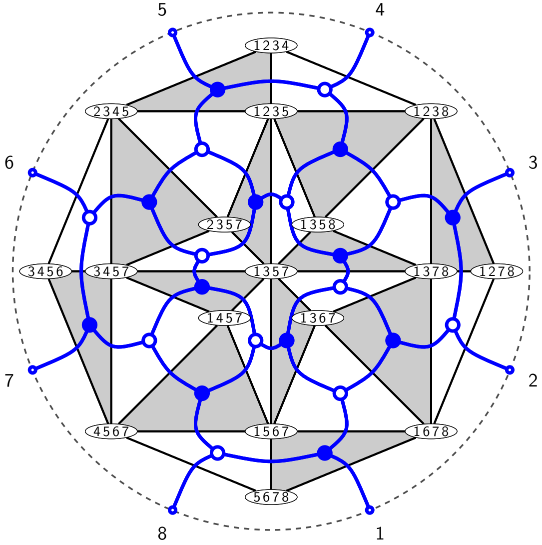

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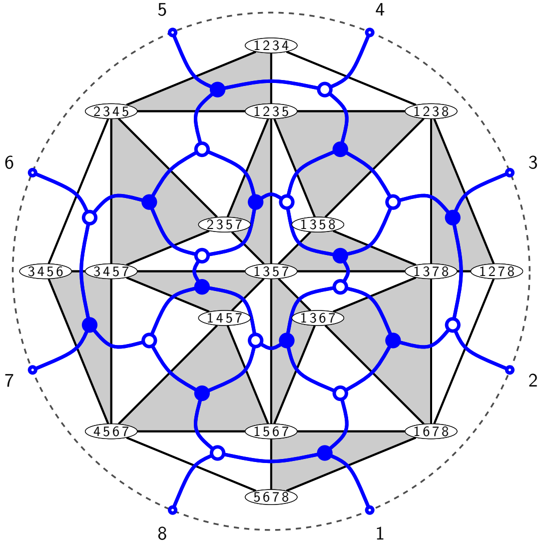

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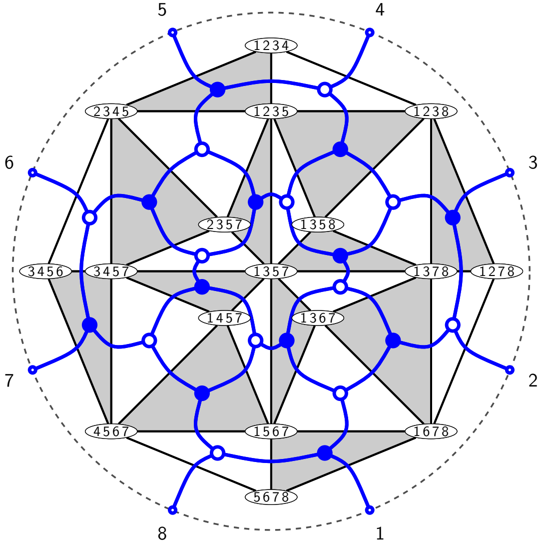

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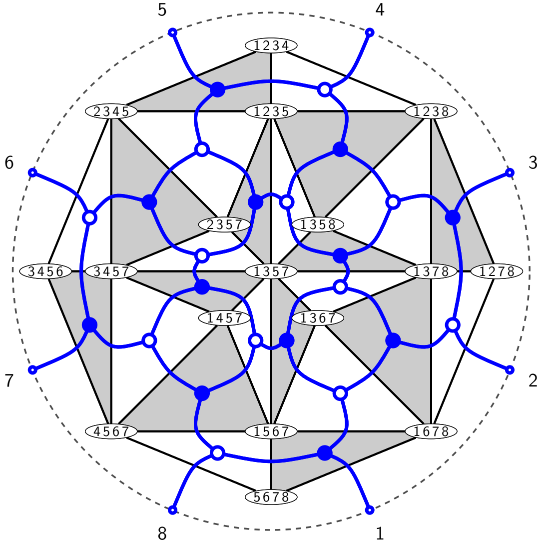

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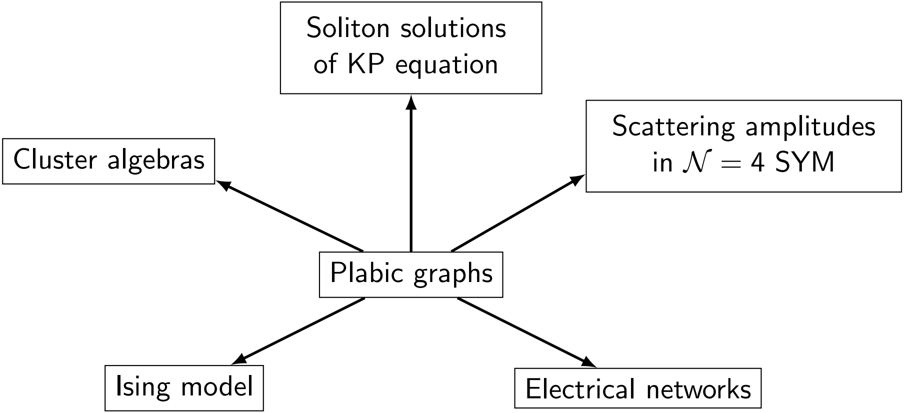
This plabic graph admits only 4 square moves. □

Soliton solutions of Kadomtsev-Petviashvili (KP) equation



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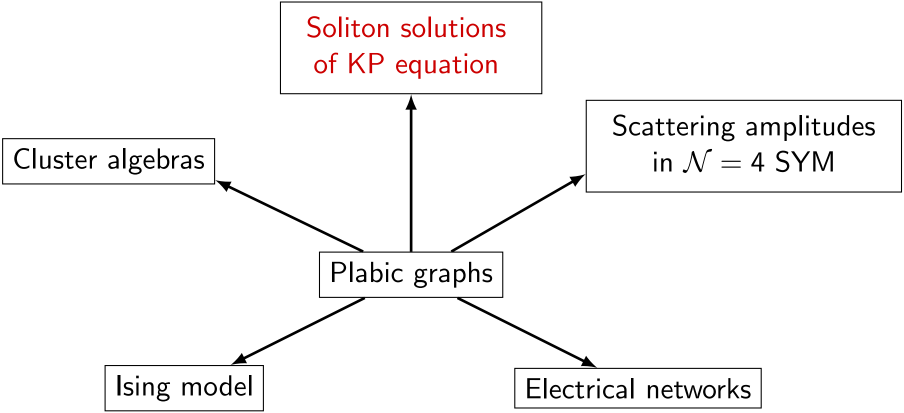

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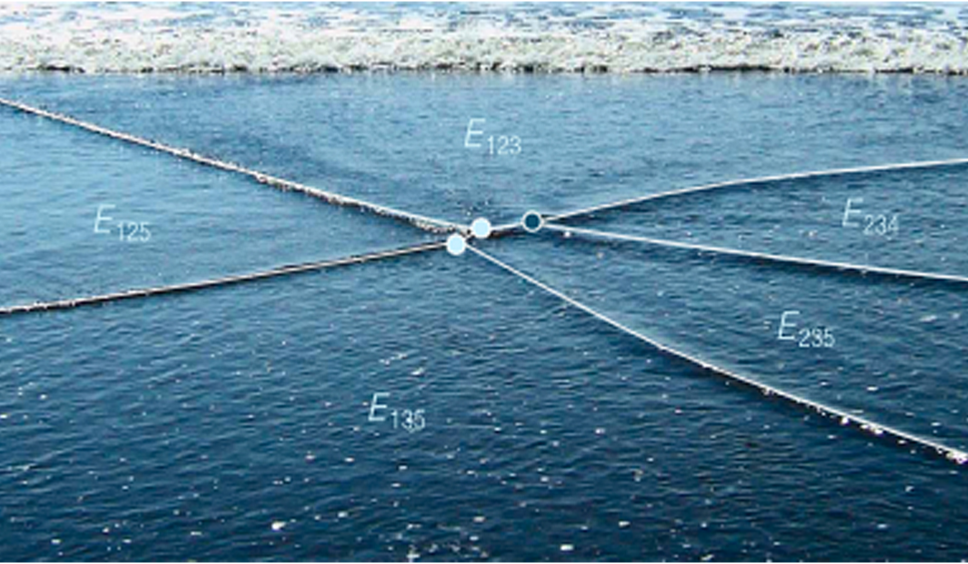

Photo Credit: Mark Ablowitz, Colorado.

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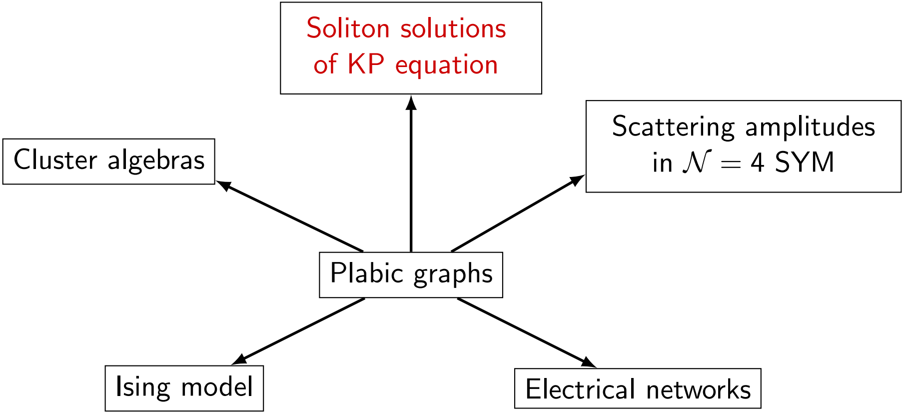

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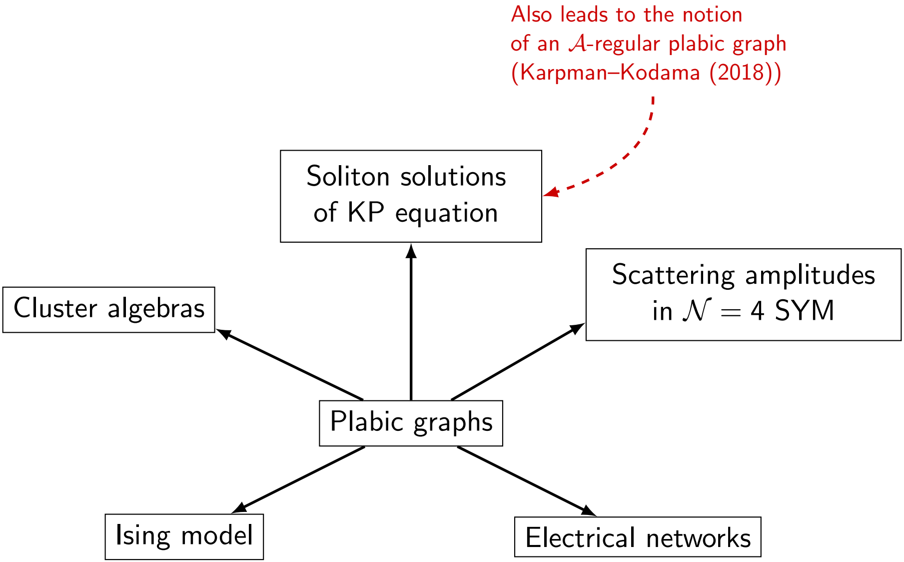
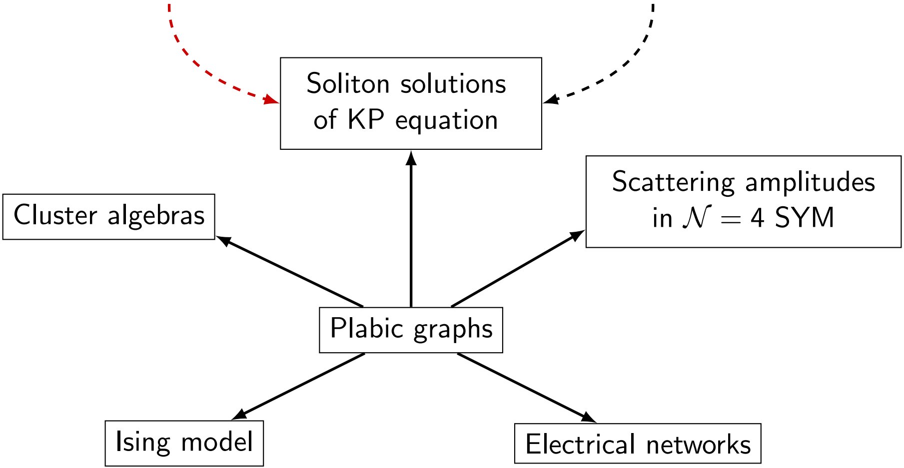


Soliton solutions of Kadomtsev-Petviashvili (KP) equation

The theory of higher associahedra can be applied to resolve a conjecture of Kodama–Williams (2011)

Also leads to the notion of an $\mathcal{A}$-regular plabic graph (Karpman–Kodama (2018))



Diameter of the higher associahedron

Confirming a conjecture of Sleator–Tarjan–Thurston (1980), Pournin showed

Theorem (Pournin (2014))

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- *For* $n = 2k$*, the diameter of the "middle" higher associahedron* $\widehat{\Sigma}_{\mathcal{A},k-1} = -\widehat{\Sigma}_{\mathcal{A},k-1}$ *equals* $\frac{1}{2}k(k-1)^2$.

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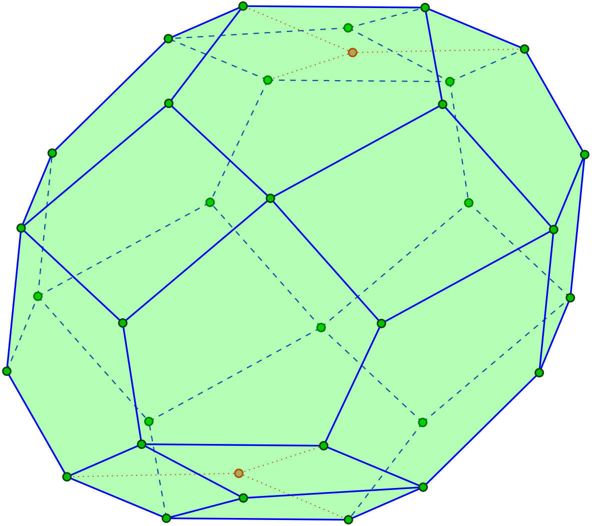
- *More generally, the square move distance between any bipartite $(k,2k)$-plabic graph and its "opposite" plabic graph equals $\frac{1}{2}k(k-1)^2$.*

Diameter of the higher associahedron for $k = 3$, $n = 6$



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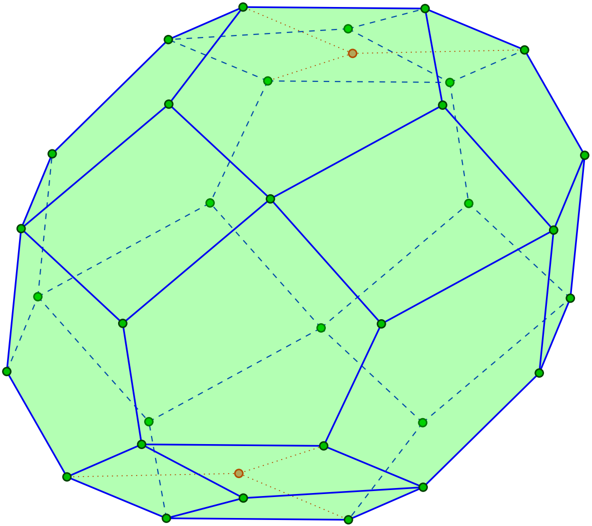
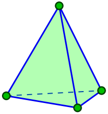
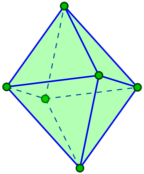
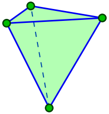
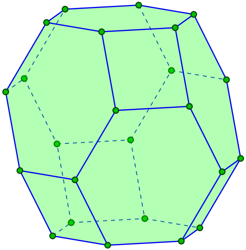


The distance between any two antipodal points equals $\frac{1}{2}k(k-1)^2 = 6$.

Case $d = 2$?

$d = 1$:  $+$  $+$  $=$ 

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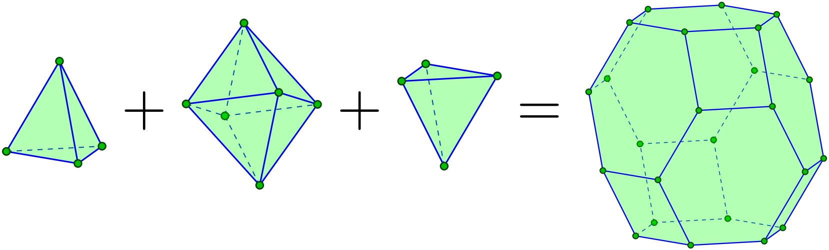

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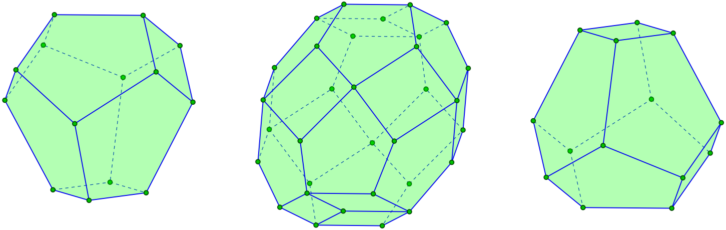

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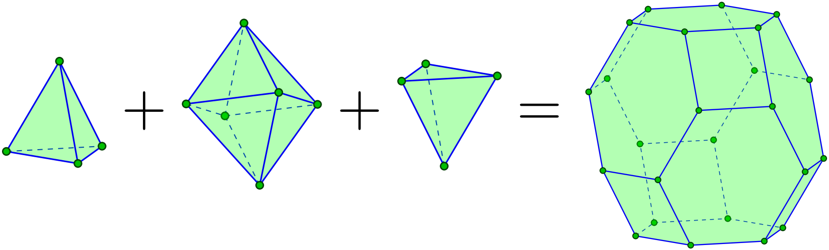

Question

What happens for $d = 2$?

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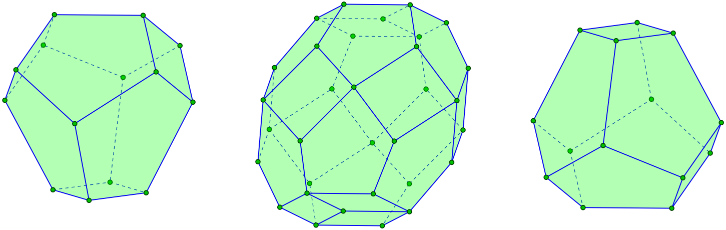

Thank you!