Zamolodchikov periodicity and integrability

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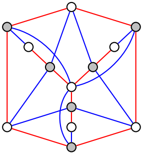
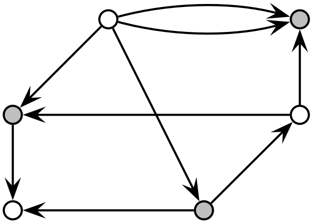
October 7, 2016

Joint work with Pavlo Pylyavskyy

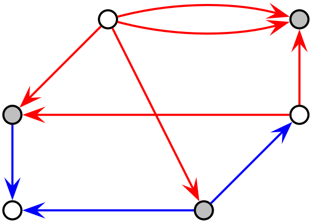


Part 1: T-systems

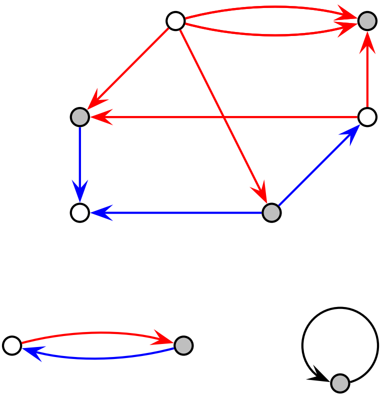
Bipartite quivers



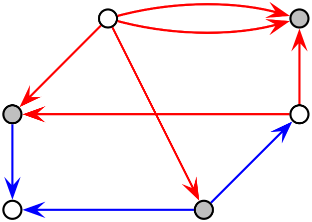
Bipartite quivers

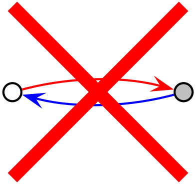


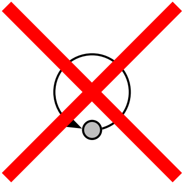
Bipartite quivers



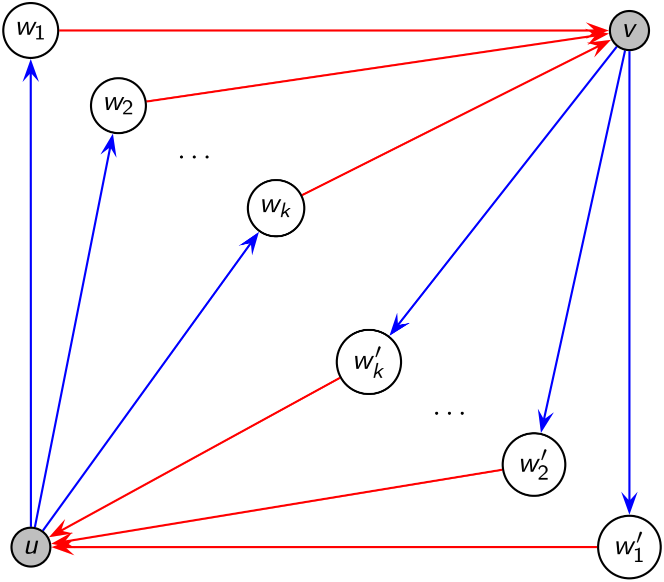
Bipartite quivers



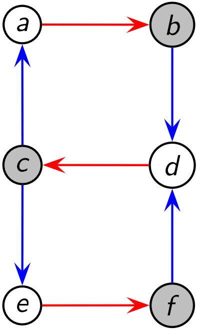




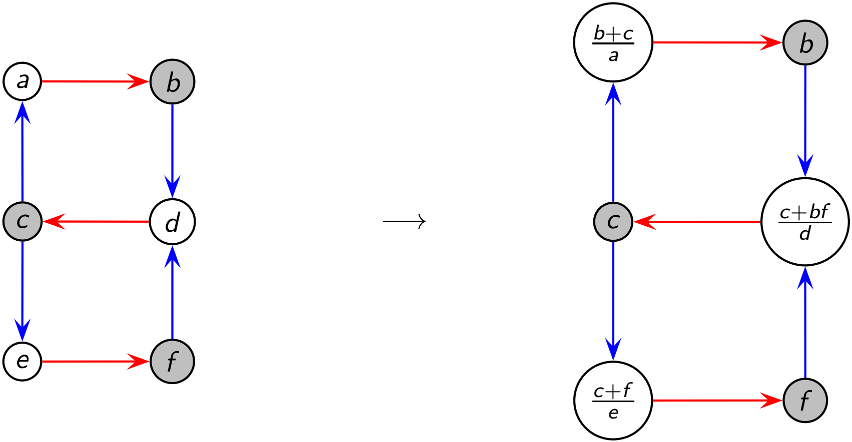
Bipartite **recurrent** quivers



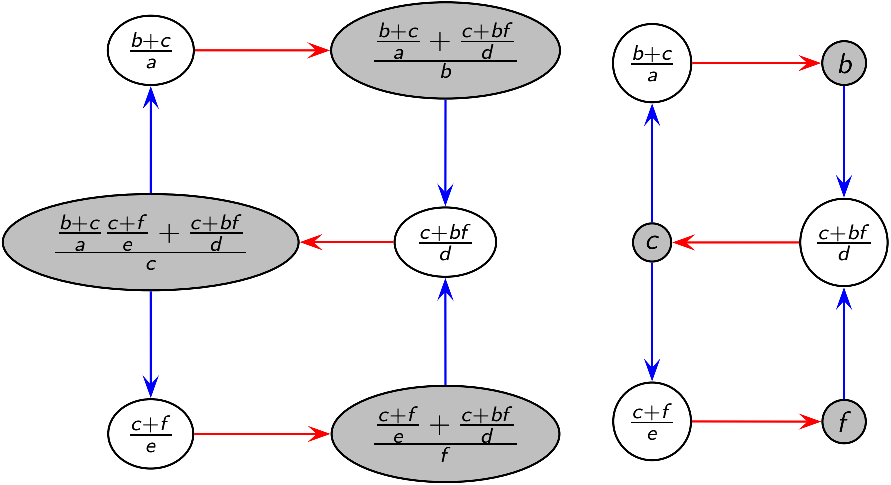
T-system



T-system

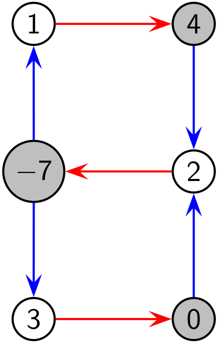


T-system

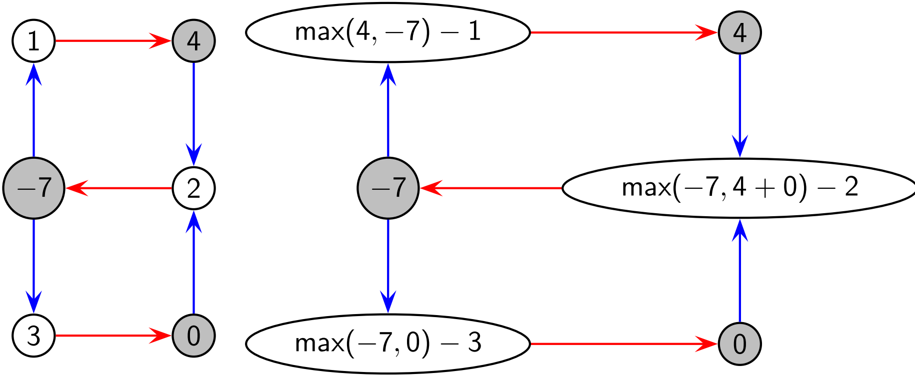


$\frac{b+c}{a}$
$\frac{\frac{b+c}{a}+\frac{c+bf}{d}}{b}$
$\frac{\frac{b+c}{a}\frac{c+f}{e}+\frac{c+bf}{d}}{c}$
$\frac{c+bf}{d}$
$\frac{c+f}{e}$
$\frac{\frac{c+f}{e}+\frac{c+bf}{d}}{f}$
$\frac{b+c}{a}$
b
c
$\frac{c+bf}{d}$
$\frac{c+f}{e}$
f

Tropical T-system

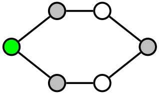
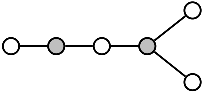
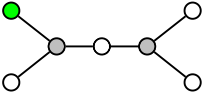
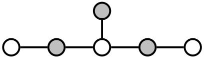
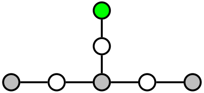
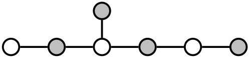
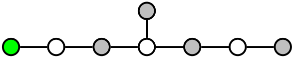
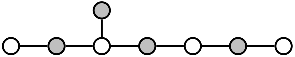
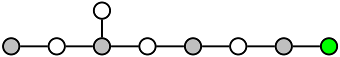


Tropical T-system

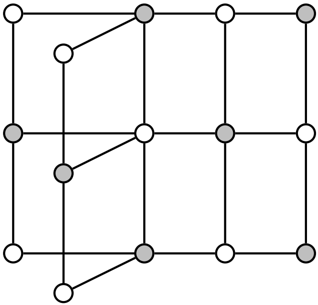


Part 2: Zamolodchikov periodicity

ADE Dynkin diagrams

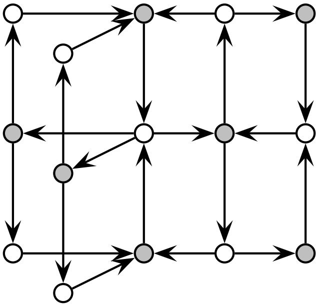
Name	Finite diagram	Affine diagram	Name
A_n			$\hat{A}_{n-1}$
D_n			$\hat{D}_{n-1}$
E_6			$\hat{E}_6$
E_7			$\hat{E}_7$
E_8			$\hat{E}_8$

Tensor product



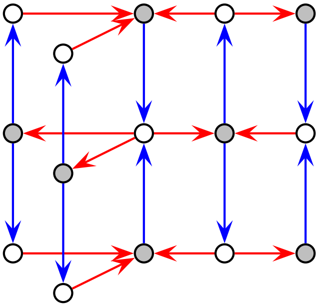
$D_5 \otimes A_3$

Tensor product



$D_5 \otimes A_3$

Tensor product



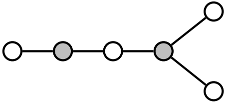
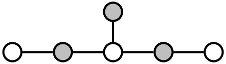
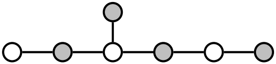
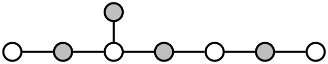
$D_5 \otimes A_3$

Zamolodchikov periodicity

Theorem (B. Keller, 2013)

Tensor product of **finite** *Dynkin diagrams* $\Longrightarrow$ *the* T*-system is periodic.*

Coxeter number

Name	**Picture**	**h**
A_n		$n+1$
D_n		$2n-2$
E_6		12
E_7		18
E_8		30

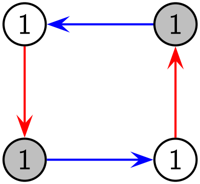
Zamolodchikov periodicity

Theorem (B. Keller, 2013)

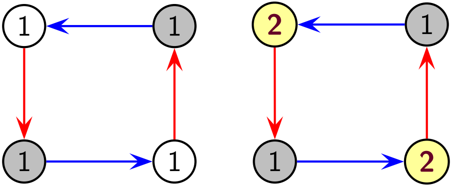
Tensor product of **finite** *Dynkin diagrams* $\Longrightarrow$ *the* T*-system is periodic* **with period dividing**

$$2(h + h').$$

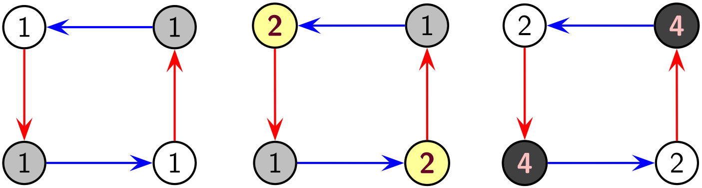
Example: $A_2 \otimes A_2$



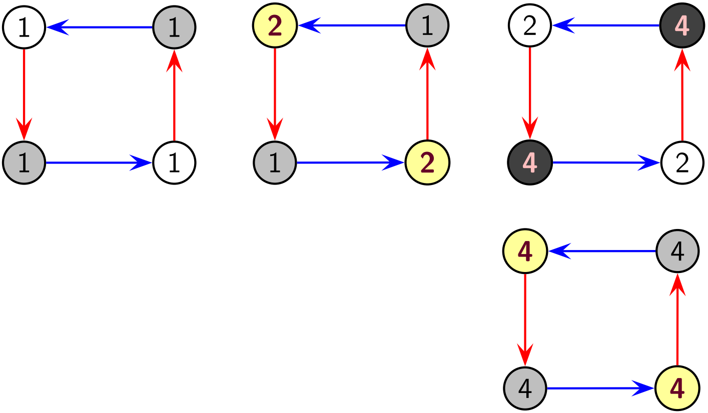
Example: $A_2 \otimes A_2$



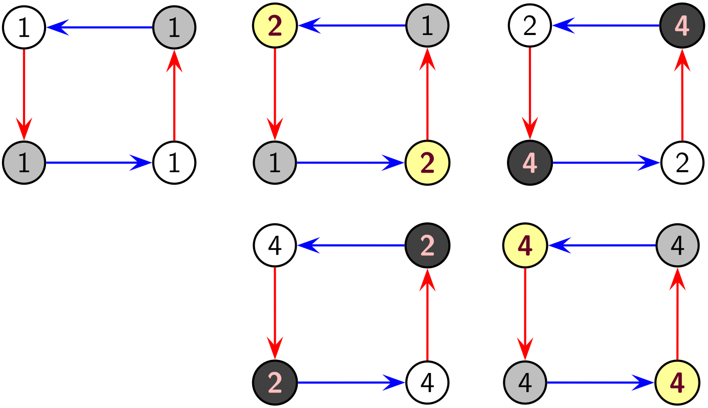
Example: $A_2 \otimes A_2$



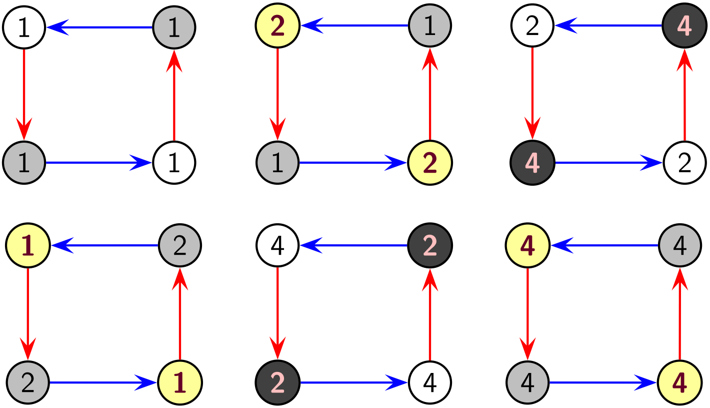
Example: $A_2 \otimes A_2$



Example: $A_2 \otimes A_2$



Example: $A_2 \otimes A_2$



Zamolodchikov periodicity

Tensor product of **finite** Dynkin diagrams $\Longrightarrow$ the T-system is periodic

$\Longleftarrow$ the T-system is periodic

Zamolodchikov periodicity

Tensor product of **finite** Dynkin diagrams	$\Longrightarrow$	the T-system is periodic
?	$\Longleftarrow$	the T-system is periodic

The result

Theorem

Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1.
2.
3.
4.
5. *The T-system associated with Q is periodic.*

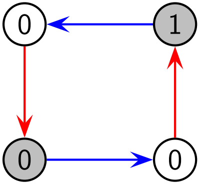
The result

Theorem

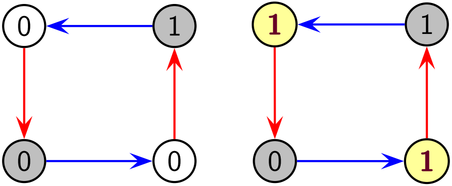
Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1.
2.
3.
4. **The tropical T-system is periodic for any initial value.**
5. *The T-system associated with Q is periodic.*

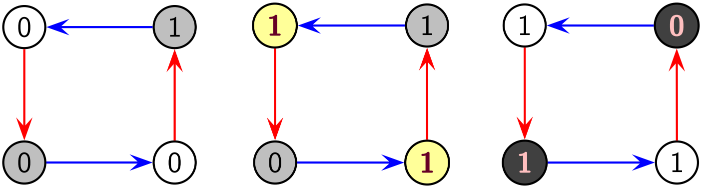
Example: $A_2 \otimes A_2$



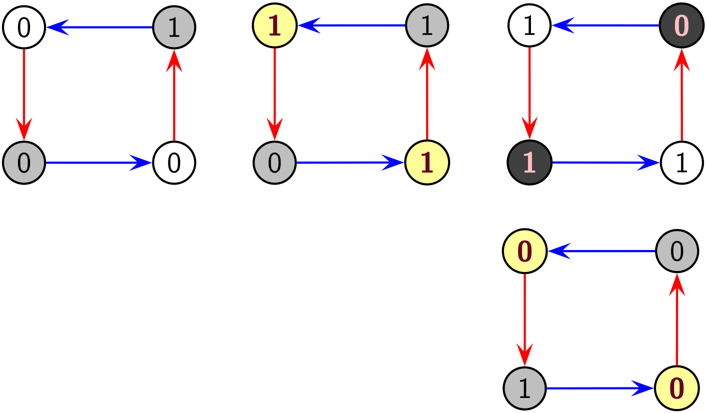
Example: $A_2 \otimes A_2$



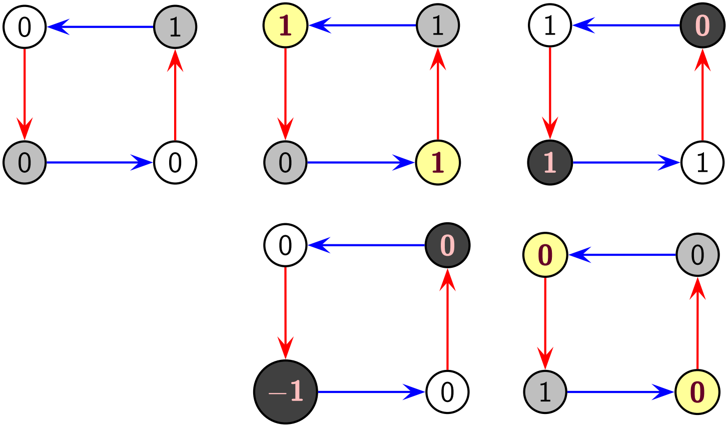
Example: $A_2 \otimes A_2$



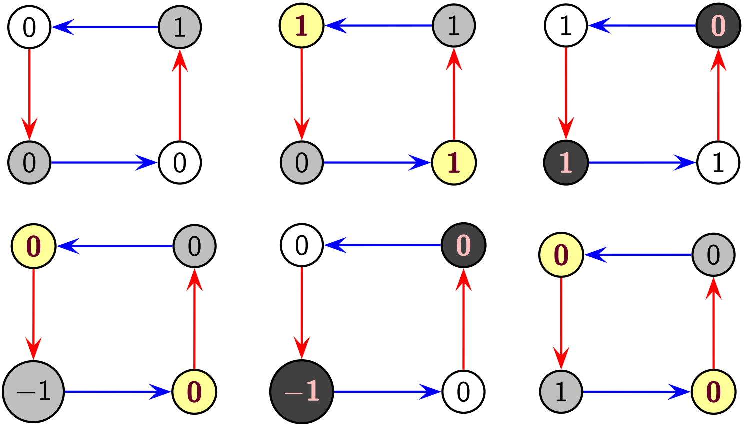
Example: $A_2 \otimes A_2$



Example: $A_2 \otimes A_2$



Example: $A_2 \otimes A_2$



The result

Theorem

Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1.
2.
3.
4. *The tropical T-system is periodic for any initial value.*
5. *The T-system associated with Q is periodic.*

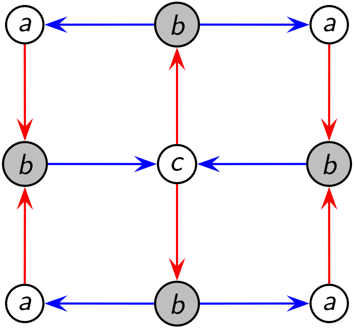
The result

Theorem

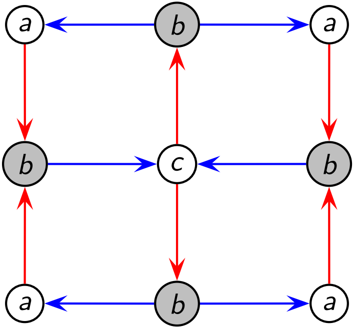
Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1.
2.
3. **Q has a fixed point.**
4. *The tropical T-system is periodic for any initial value.*
5. *The T-system associated with Q is periodic.*

Fixed point

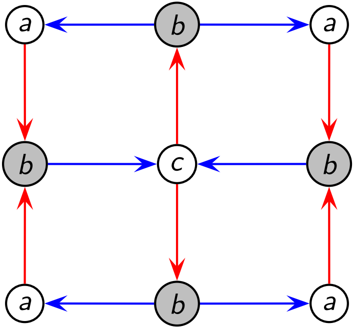


Fixed point



$$a^2 = b + b; \quad b^2 = a^2 + c; \quad c^2 = b^2 + b^2.$$

Fixed point



$$a^2 = b + b; \quad b^2 = a^2 + c; \quad c^2 = b^2 + b^2.$$

$$a = \sqrt{4 + 2\sqrt{2}}; \quad b = 2 + \sqrt{2}; \quad c = 2 + 2\sqrt{2}.$$

The result

Theorem

Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1.
2.
3. *Q has a fixed point.*
4. *The tropical T-system is periodic for any initial value.*
5. *The T-system associated with Q is periodic.*

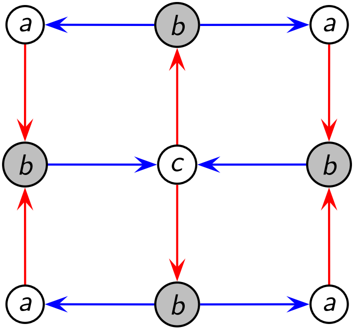
The result

Theorem

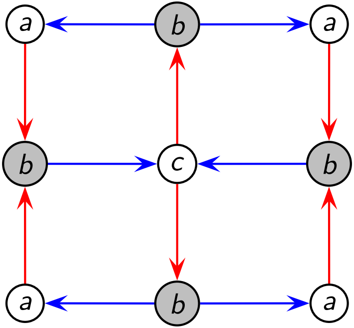
Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1.
2. Q **has a strictly subadditive labeling.**
3. *Q has a fixed point.*
4. *The tropical T-system is periodic for any initial value.*
5. *The T-system associated with Q is periodic.*

Strictly subadditive labeling

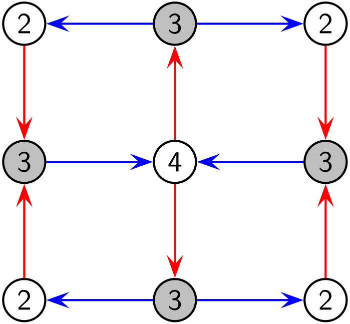


Strictly subadditive labeling



$$2a > \max(b, b); \quad 2b > \max(a + a, c); \quad 2c > \max(b + b, b + b).$$

Strictly subadditive labeling



$$2a > \max(b, b); \quad 2b > \max(a + a, c); \quad 2c > \max(b + b, b + b).$$

$$a = 2; \quad b = 3; \quad c = 4.$$

The result

Theorem

Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1.
2. *Q has a strictly subadditive labeling.*
3. *Q has a fixed point.*
4. *The tropical T-system is periodic for any initial value.*
5. *The T-system associated with Q is periodic.*

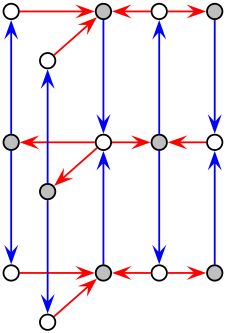
The result

Theorem

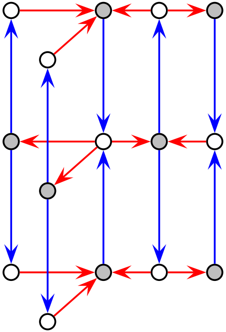
Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1. **Q is a finite $\boxtimes$ finite quiver.**
2. *Q has a strictly subadditive labeling.*
3. *Q has a fixed point.*
4. *The tropical T-system is periodic for any initial value.*
5. *The T-system associated with Q is periodic.*

Finite ⊠ finite quivers

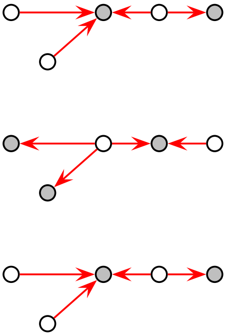


Finite ⊠ finite quivers



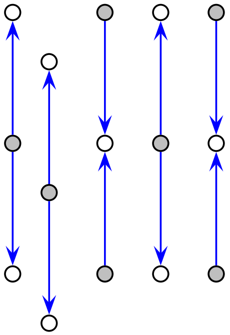
- Bipartite recurrent quiver

Finite ⊠ finite quivers



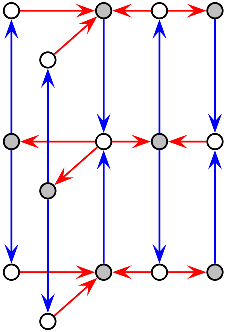
- Bipartite recurrent quiver
- All red components are **finite** Dynkin diagrams

Finite ⊠ finite quivers



- Bipartite recurrent quiver
- All red components are **finite** Dynkin diagrams
- All blue components are **finite** Dynkin diagrams

Finite ⊠ finite quivers



- Bipartite recurrent quiver
- All red components are **finite** Dynkin diagrams
- All blue components are **finite** Dynkin diagrams

↑

"Finite ⊠ finite quiver"

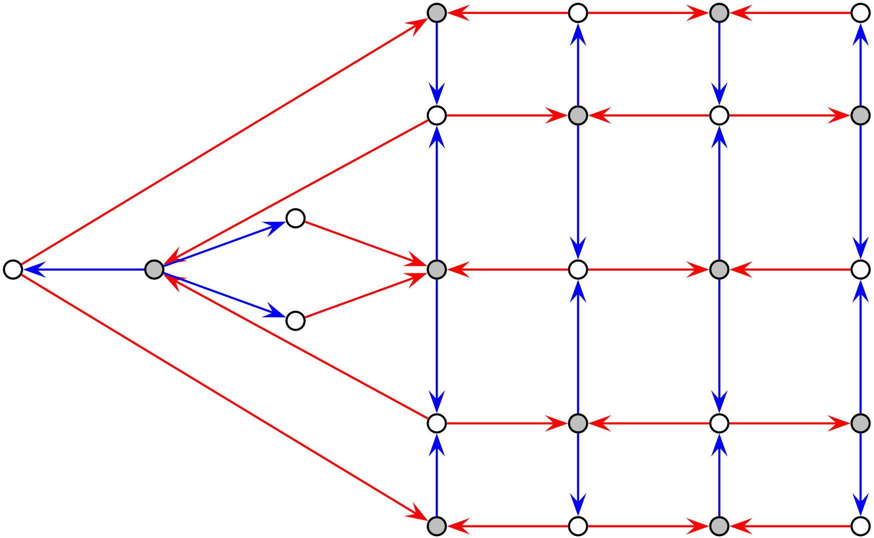
The classification of Zamolodchikov periodic quivers

Theorem (G.-Pylyavskyy, 2016)

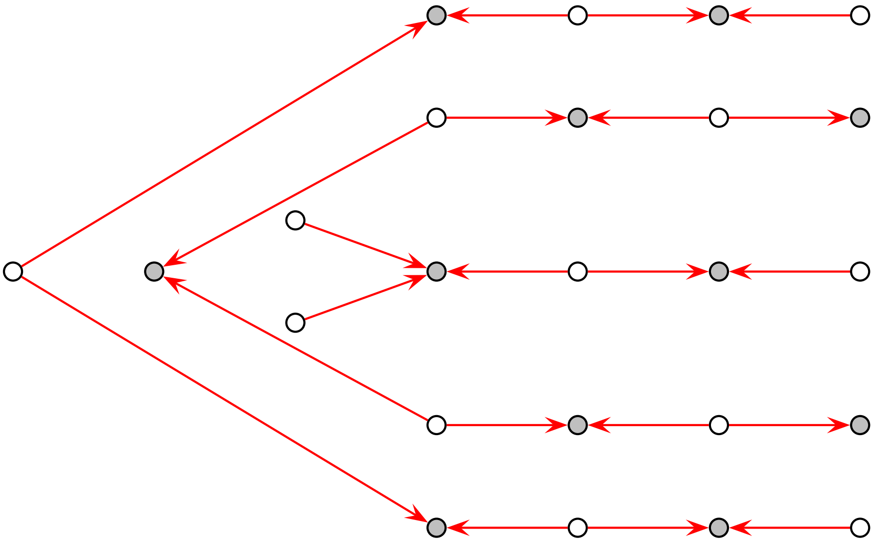
Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1. *Q is a finite $\boxtimes$ finite quiver.*
2. *Q has a strictly subadditive labeling.*
3. *Q has a fixed point.*
4. *The tropical T-system is periodic for any initial value.*
5. *The T-system associated with Q is periodic.*

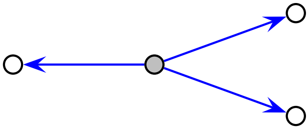
Finite ⊠ finite quivers

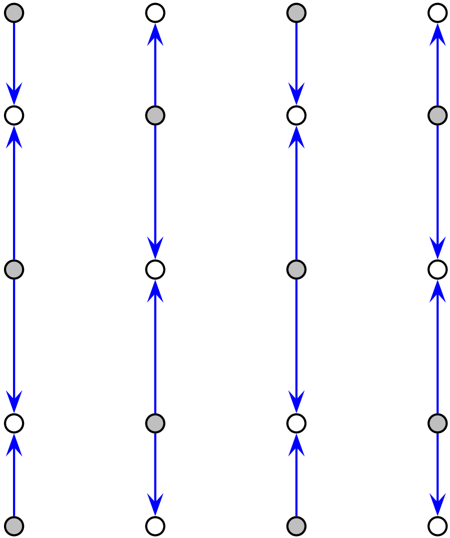


Finite ⊠ finite quivers

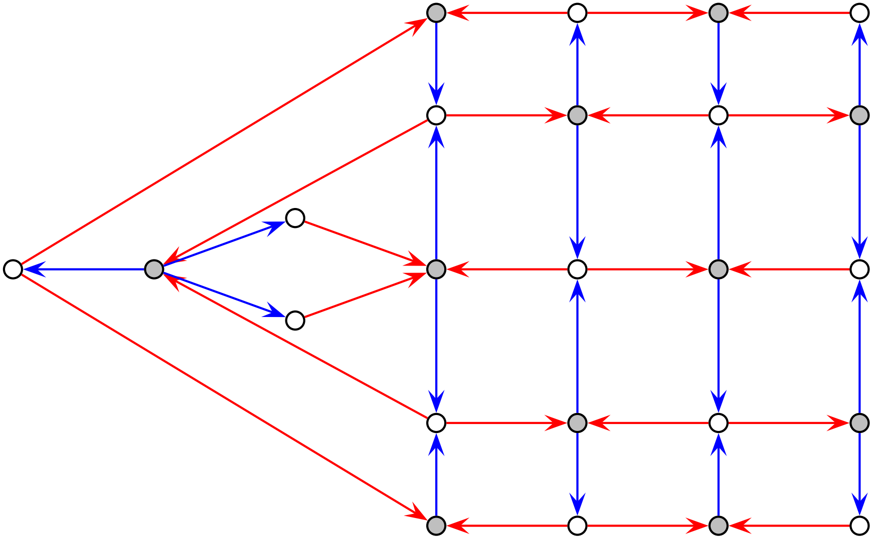


Finite ⊠ finite quivers

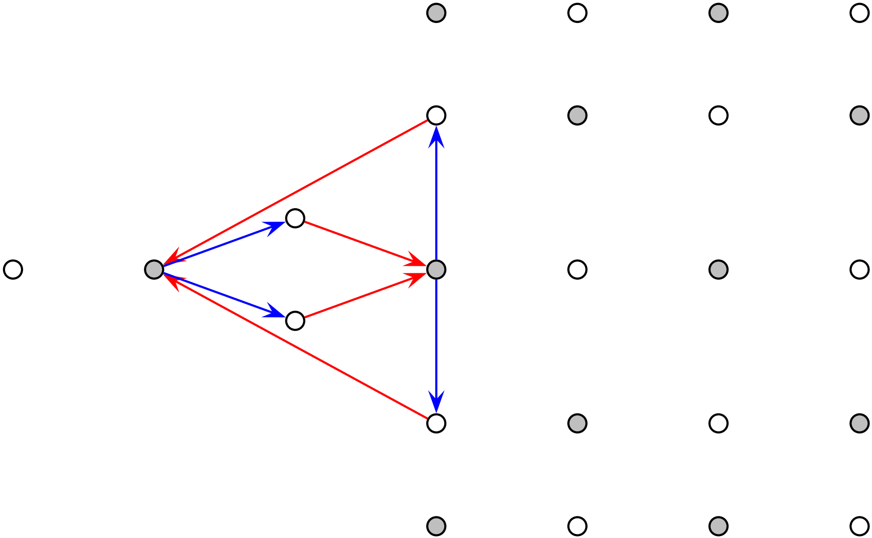




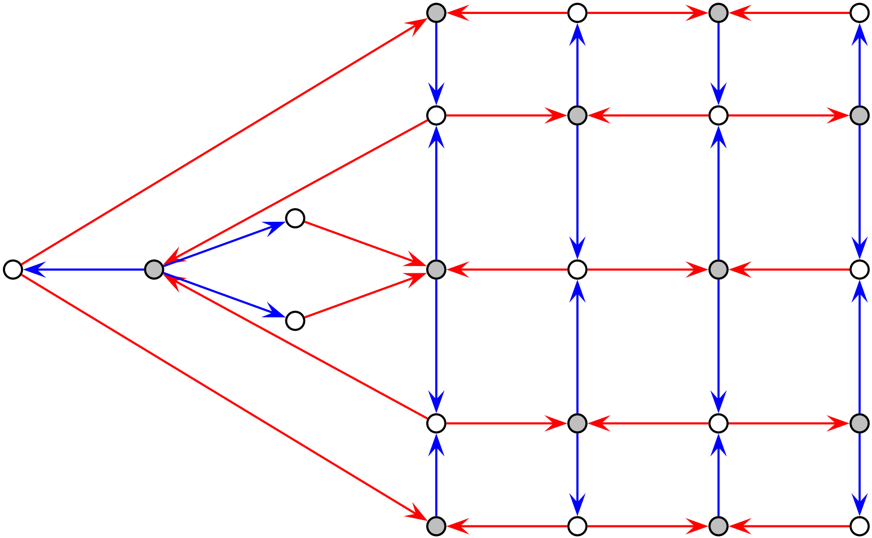
Finite ⊠ finite quivers



Finite ⊠ finite quivers



Finite ⊠ finite quivers



The classification of Zamolodchikov periodic quivers

Theorem (G.-Pylyavskyy, 2016)

Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1. *Q is a finite $\boxtimes$ finite quiver.*
2. *Q has a strictly subadditive labeling.*
3. *Q has a fixed point.*
4. *The tropical T-system is periodic for any initial value.*
5. *The T-system associated with Q is periodic.*

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In all cases, both the T-system and its tropicalization have period dividing

$$2(h+h').$$

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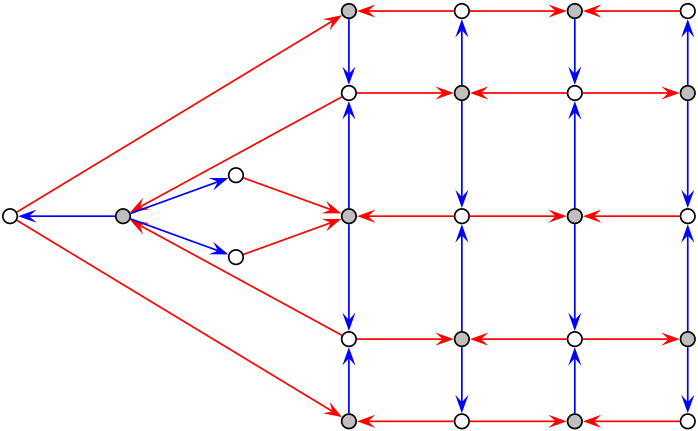

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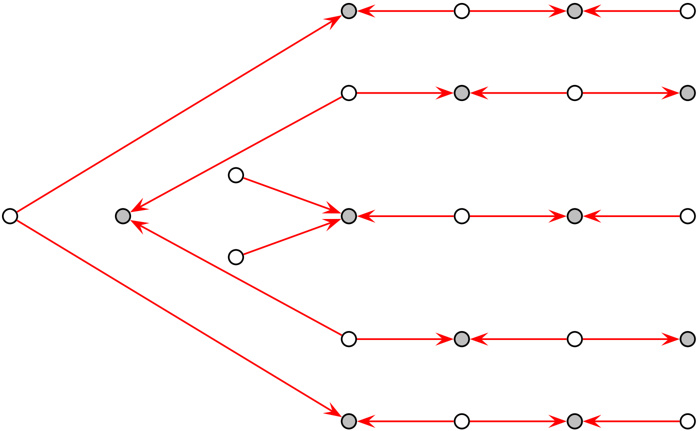
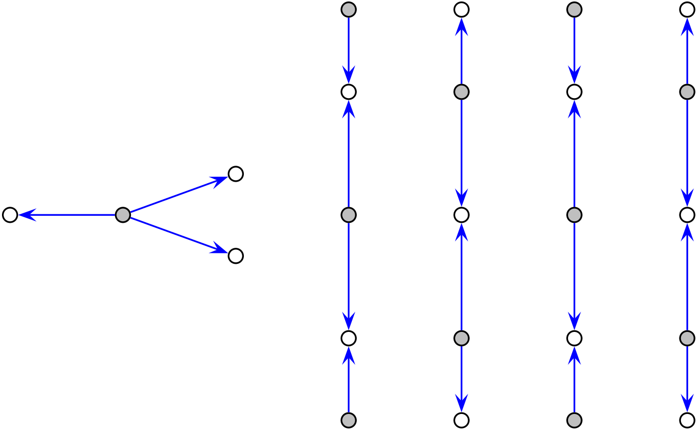


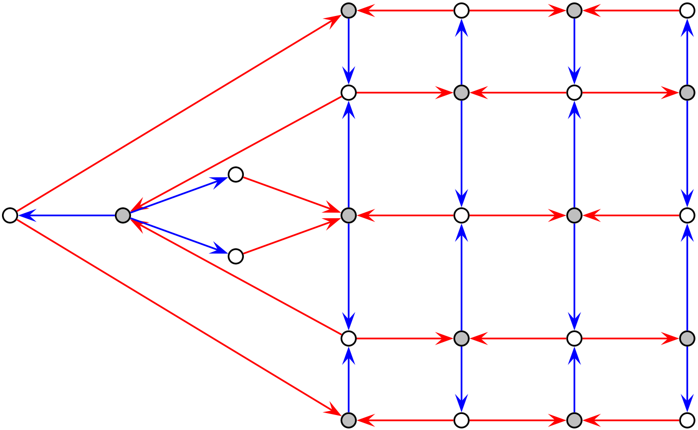
$\mathbf{h = 9 + 1 = 12 - 2 = 10}$;

Finite ⊠ finite quivers



$h = 9 + 1 = 12 - 2 = 10;$ $\quad \mathbf{h' = 5 + 1 = 8 - 2 = 6};$

Finite ⊠ finite quivers



$h = 9 + 1 = 12 - 2 = 10$; $h' = 5 + 1 = 8 - 2 = 6$; Period = **32**

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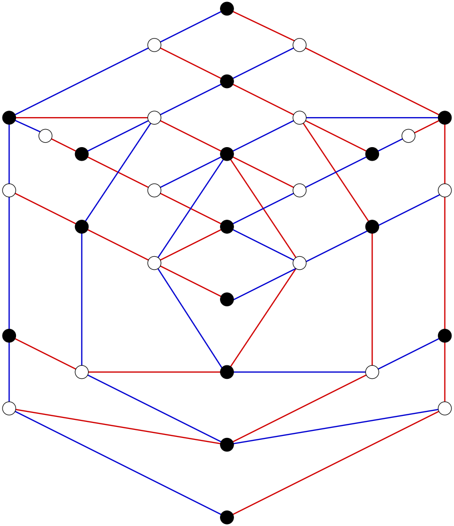
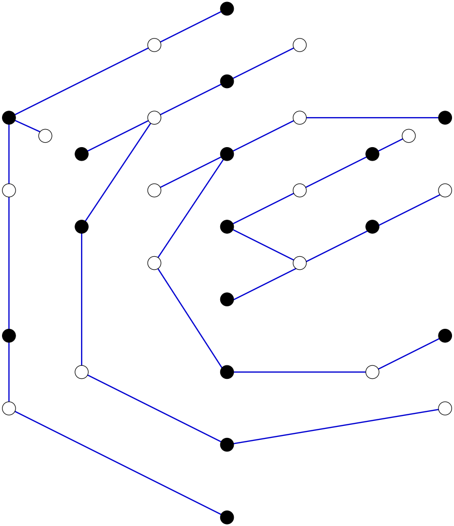
5 infinite families and 11 exceptional quivers



5 infinite families and 11 exceptional quivers



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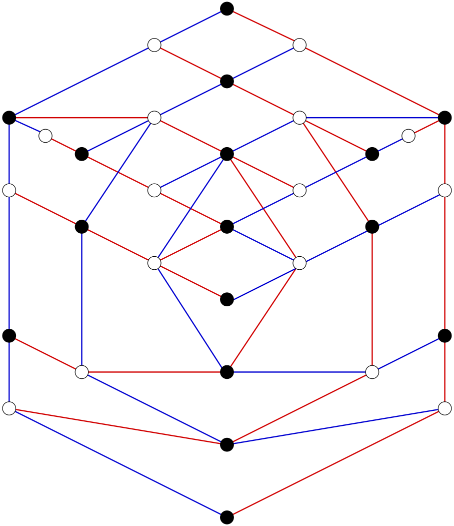

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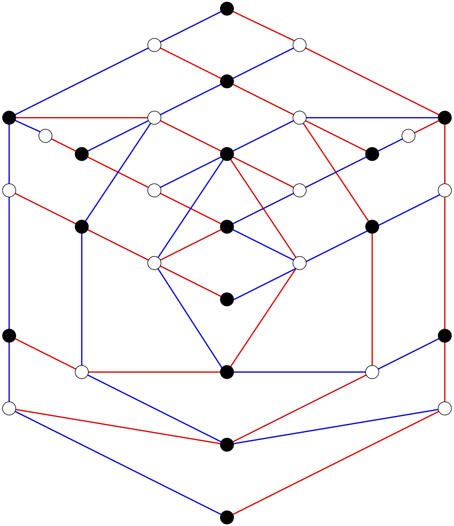
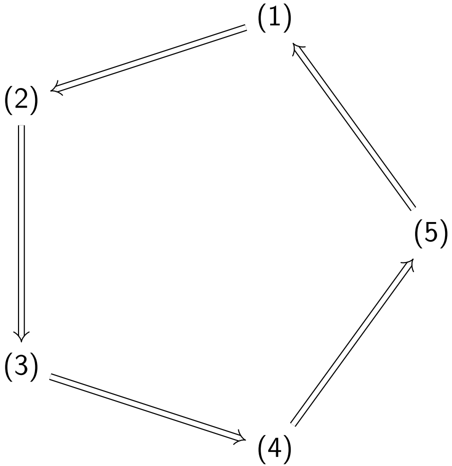


$$2(h + h') = 120$$

Plan of the proof



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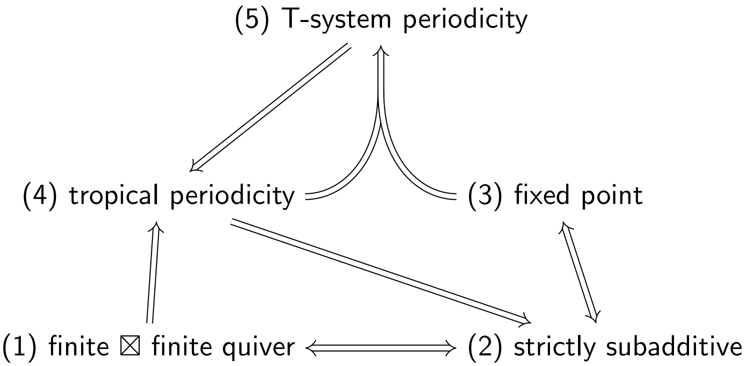

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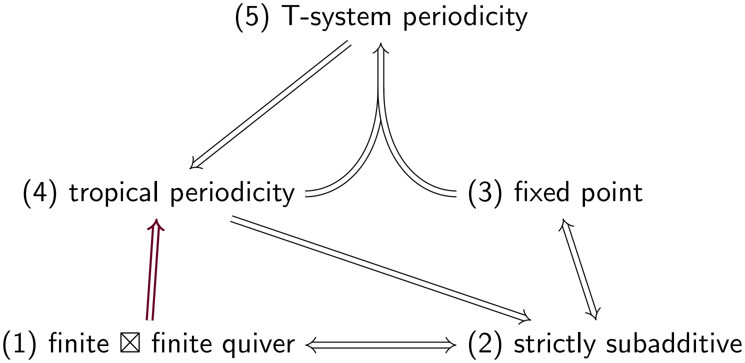
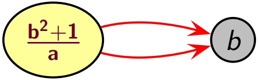


Part 3: Zamolodchikov integrability

Example

Example



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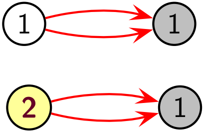

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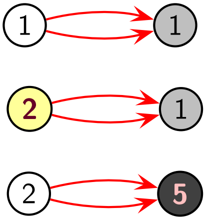
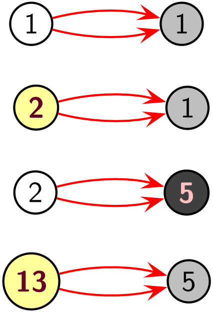
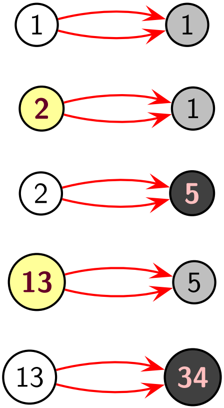


Example



Example



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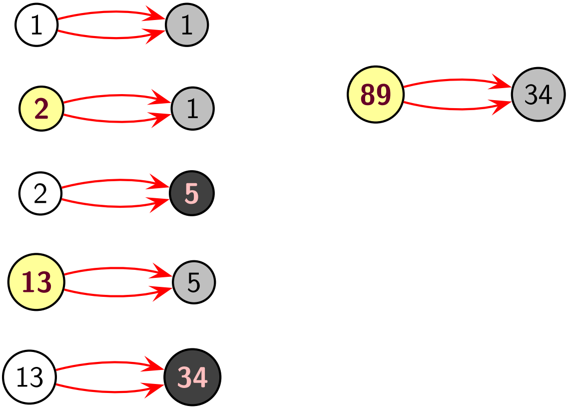

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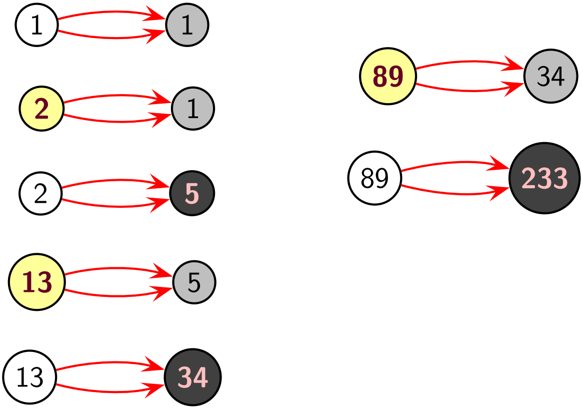

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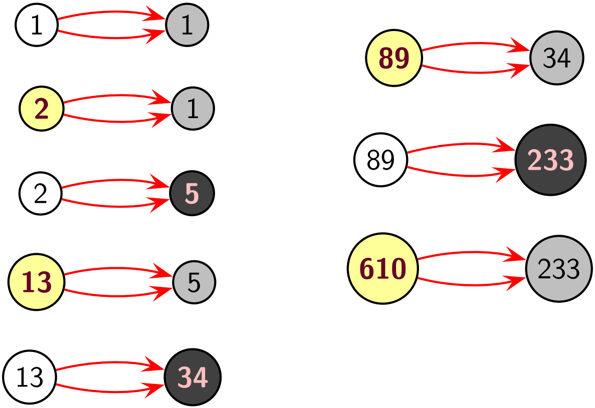
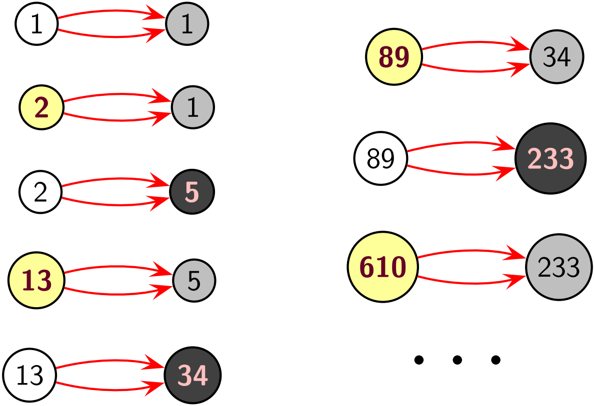
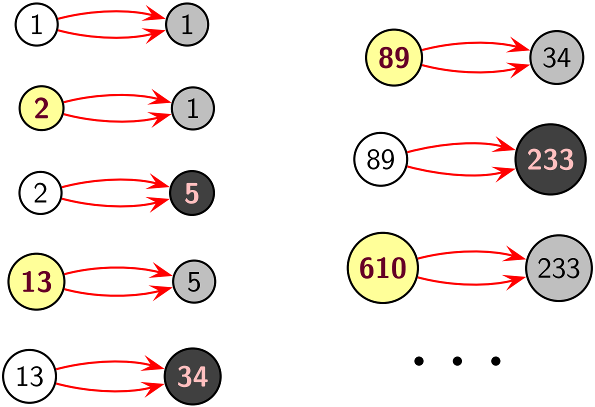


Example

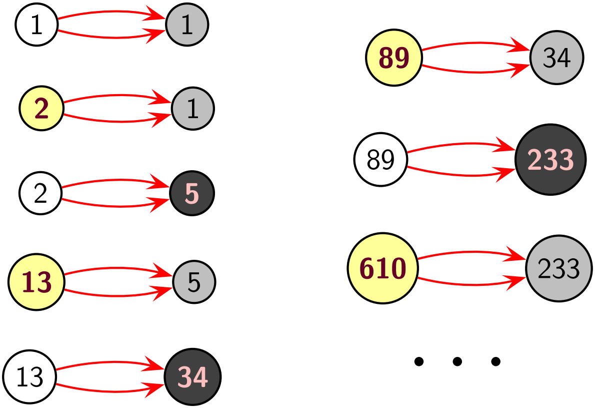


Example



$$x_{n+1} - 3x_n + x_{n-1} = 0$$

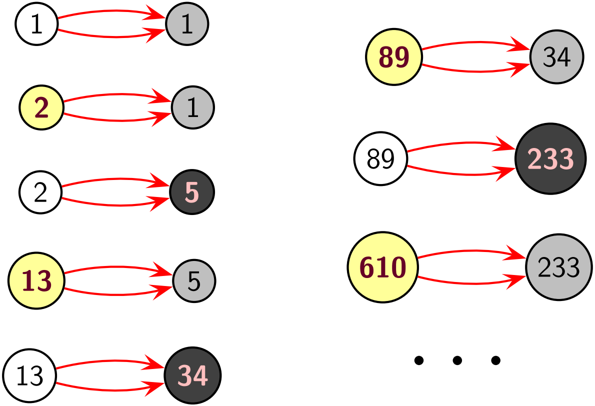
Example



$$x_{n+1} - 3x_n + x_{n-1} = 0$$

$$5 - 3 \cdot 2 + 1 = 0$$

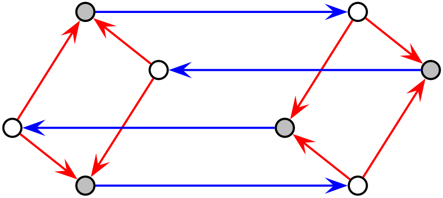
Example



$$x_{n+1} - 3x_n + x_{n-1} = 0$$

$$13 - 3 \cdot 5 + 2 = 0$$

Affine ⊠ finite quivers



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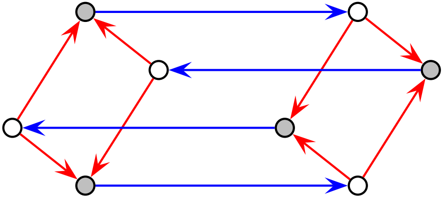
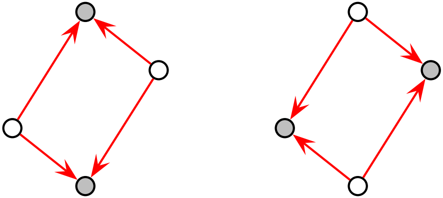


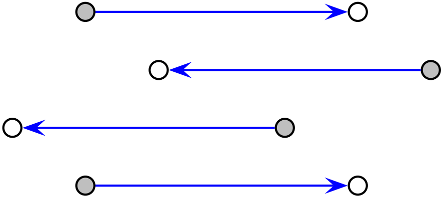
- Bipartite recurrent quiver

Affine ⊠ finite quivers



- Bipartite recurrent quiver
- All red components are **affine** Dynkin diagrams

Affine ⊠ finite quivers



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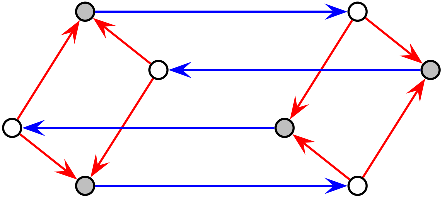

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↑

"Affine ⊠ finite quiver"

A necessary condition

Theorem (G.-Pylyavskyy, 2016)

Let Q be a bipartite recurrent quiver. Then:

1. **IF** *the T-system associated with Q is linearizable,*
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Let Q be a bipartite recurrent quiver. Then:

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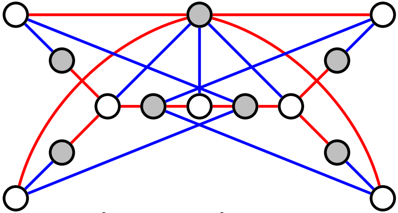
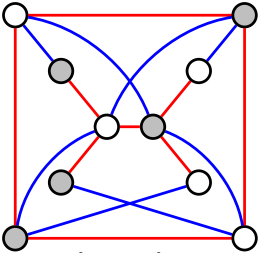
2. **THEN** *Q is an affine $\boxtimes$ finite quiver.*

15 infinite families and 4 exceptional cases



$\hat{D}_{n+1} * \hat{D}_{3n-1}$
for $n = 3$



$\hat{A}_3 * \hat{D}_5$

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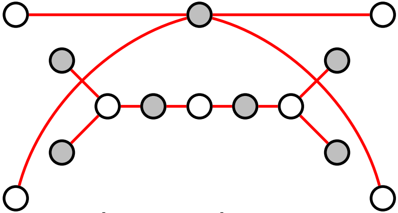

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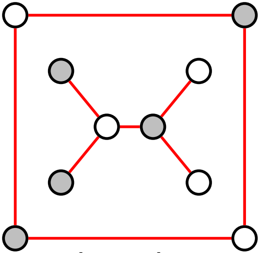

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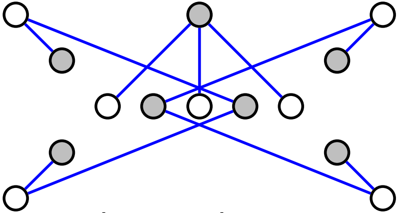

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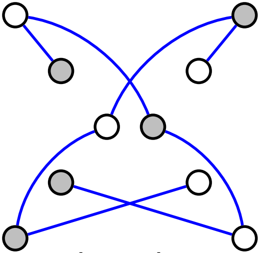

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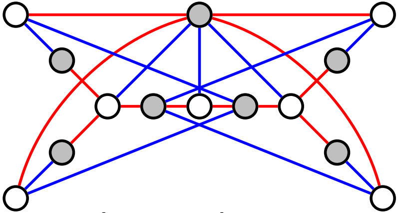

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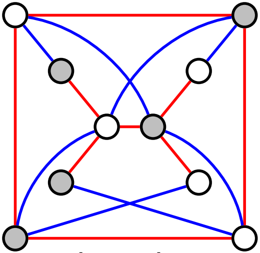

$\hat{A}_3 * \hat{D}_5$

Type $A_m \otimes \hat{A}_{2n-1}$

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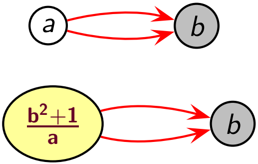

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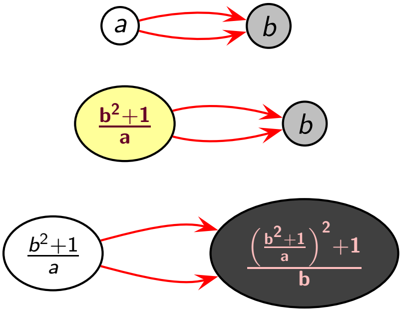

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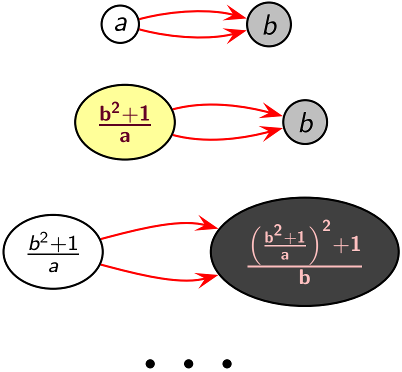

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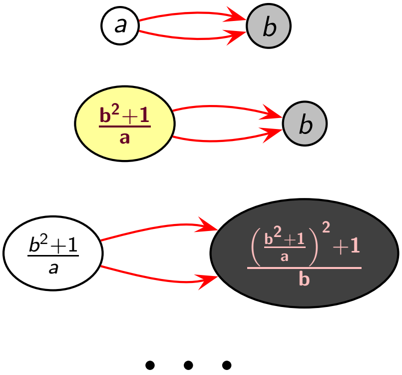

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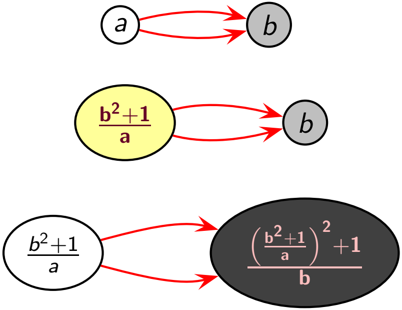
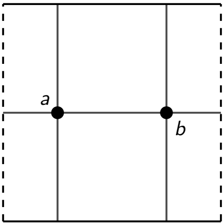


• • •

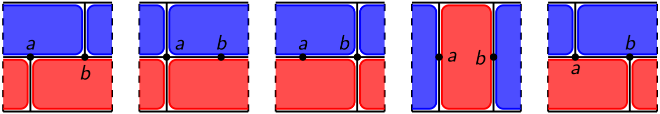
$$x_{n+1} - 3x_n + x_{n-1} = 0$$

$$\mathbf{1}\cdot x_{n+1} - \left(\frac{\mathbf{a}}{\mathbf{b}} + \frac{\mathbf{b}}{\mathbf{a}} + \frac{\mathbf{1}}{\mathbf{ab}}\right)\cdot x_n + \mathbf{1}\cdot x_{n-1} = 0$$

Domino tilings of the cylinder



Domino tilings of the cylinder



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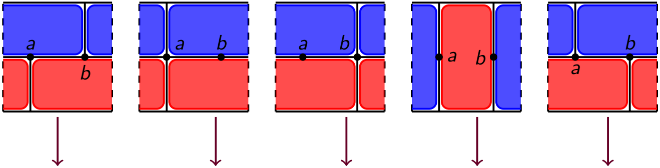

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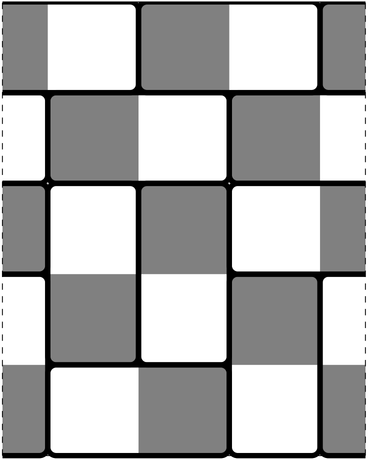
Thurston height



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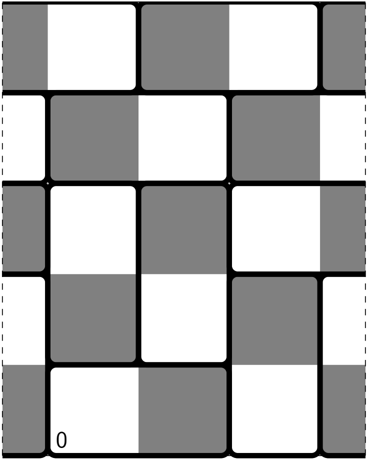

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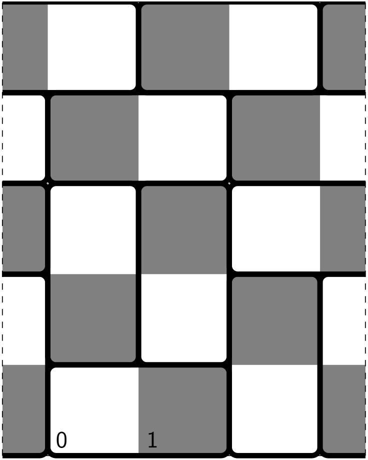

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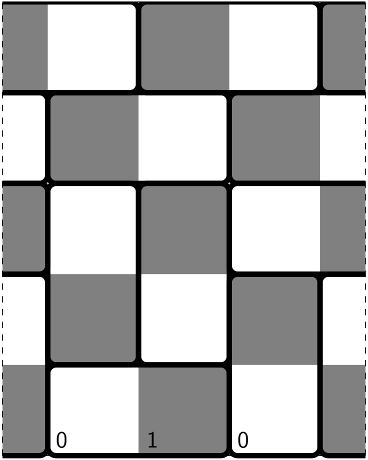

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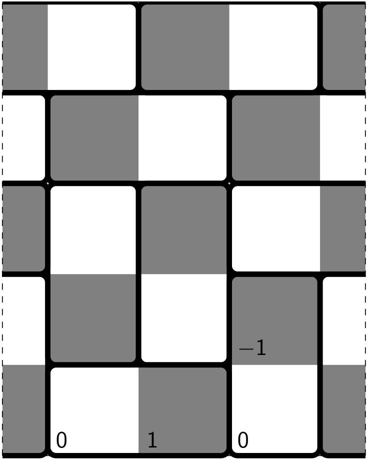

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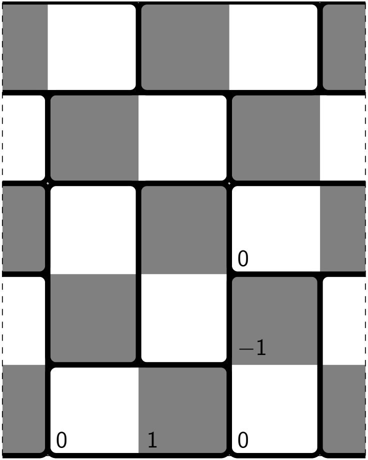
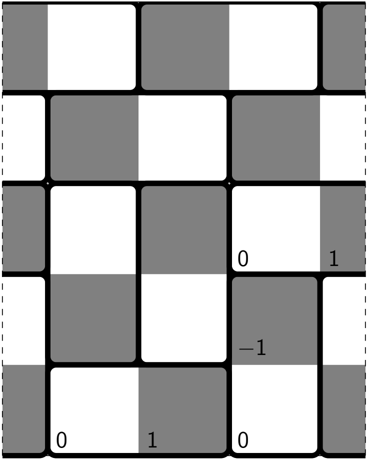


Thurston height



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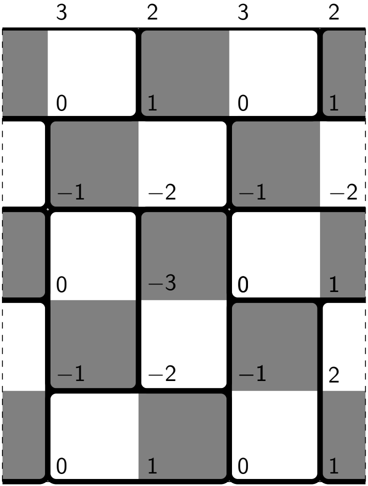

Type $A_m \otimes \hat{A}_{2n-1}$

Theorem (G.-Pylyavskyy, 2016)

- *Recurrence for* **boundary slice***:*

$$x_{t+(m+1)n} - H_1 x_{t+mn} + \ldots \pm H_m x_{t+n} \mp x_t = 0.$$

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$$H_i = \sum_{\substack{T \text{ -- cylinder tiling} \\ \text{of Thurston height } i}} \mathrm{wt}(T).$$

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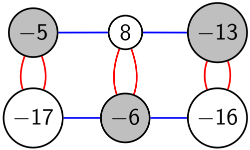
"Goncharov-Kenyon Hamiltonians"

- *Recurrence for* **r-th slice**: *express* $e_j[e_r]$ *in* e_i*'s and send* $e_i \mapsto H_i$.

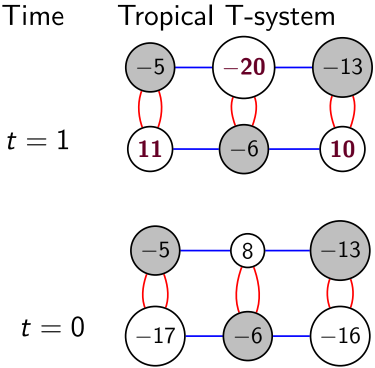
Solitonic behavior

Time Tropical T-system

$t = 0$



Solitonic behavior



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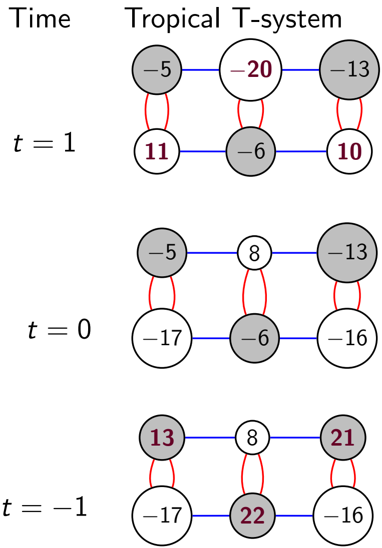
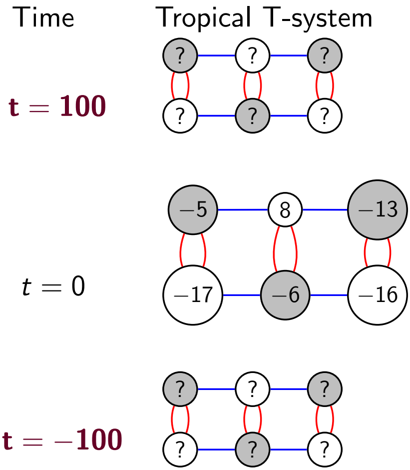
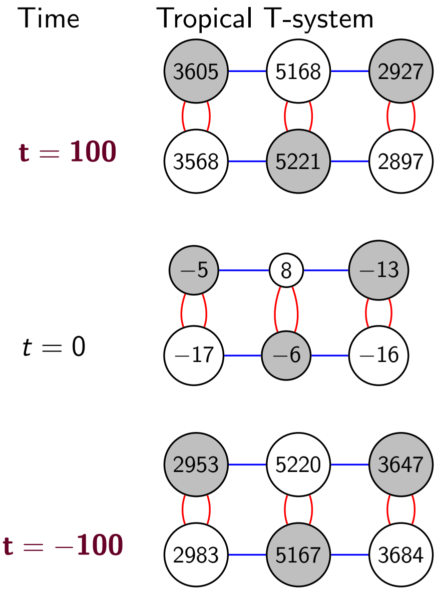


Solitonic behavior



Solitonic behavior



Tropical T-system of type $A_m \otimes \hat{A}_{2n-1}$: solitonic behavior

Theorem (G.-Pylyavskyy, 2016)

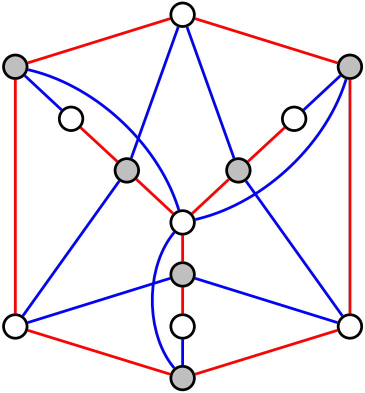
- **("soliton resolution")** *t sufficiently large $\Longrightarrow$ each affine slice moves independently with constant speed*

Tropical T-system of type $A_m \otimes \hat{A}_{2n-1}$: solitonic behavior

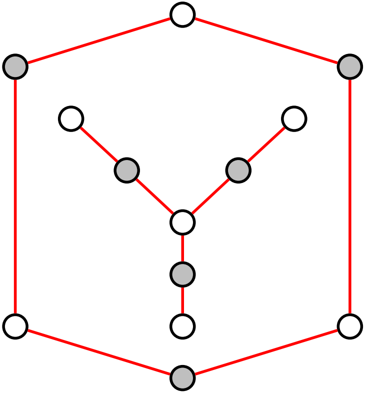
Theorem (G.-Pylyavskyy, 2016)

- **("soliton resolution")** *t sufficiently large $\implies$ each affine slice moves independently with constant speed*
- **("speed conservation")** *the speed of some slice at $t \to +\infty$ equals the speed of its mirror image at $t \to -\infty$*

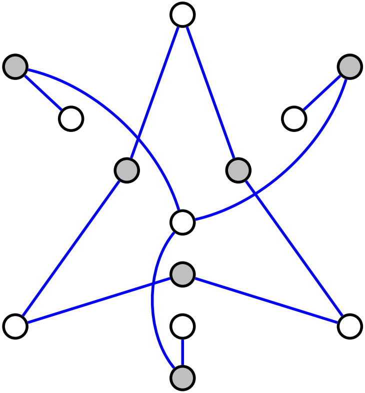
Affine ⊠ affine quivers?



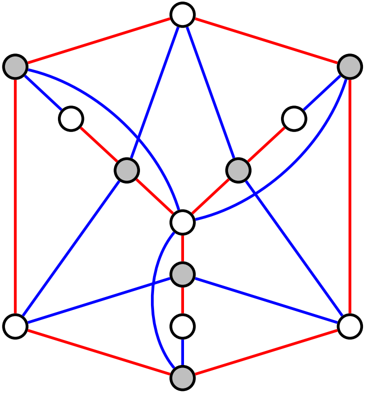
Affine ⊠ affine quivers?



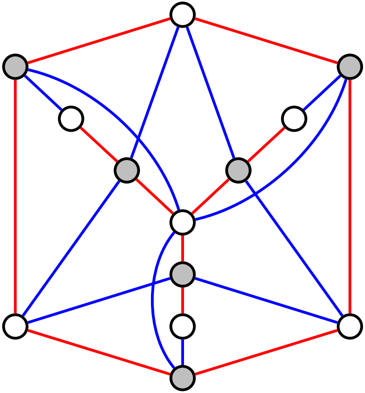
Affine ⊠ affine quivers?



Affine ⊠ affine quivers?



Affine ⊠ affine quivers?



Thank you!

Bibliography

Sergey Fomin and Andrei Zelevinsky.
Y-systems and generalized associahedra.
Ann. of Math. (2), 158(3):977–1018, 2003.

Bernhard Keller.
The periodicity conjecture for pairs of Dynkin diagrams.
Ann. of Math. (2), 177(1):111–170, 2013.

John R. Stembridge.
Admissible W-graphs and commuting Cartan matrices.
Adv. in Appl. Math., 44(3):203–224, 2010.

Pavel Galashin and Pavlo Pylyavskyy
The classification of Zamolodchikov periodic quivers.
arXiv:1603.03942 (2016).

Pavel Galashin and Pavlo Pylyavskyy
Quivers with subadditive labelings: classification and integrability
arXiv:1606.04878 (2016).