Zamolodchikov periodicity and integrability

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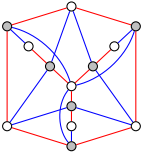
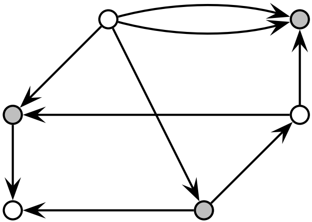
Joint work with Pavlo Pylyavskyy

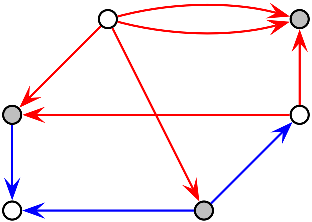


Part 1: T-systems

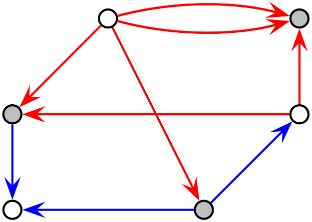
Bipartite quivers

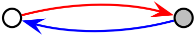


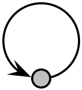
Bipartite quivers



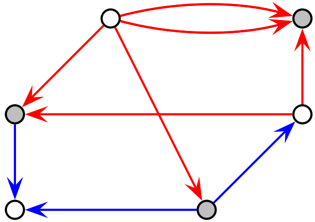
Bipartite quivers

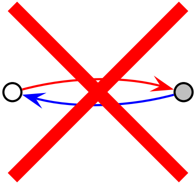


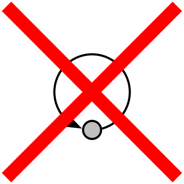




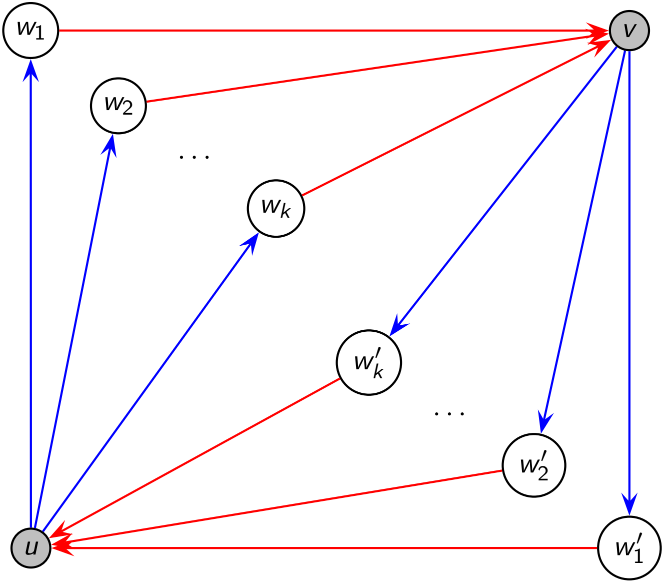
Bipartite quivers



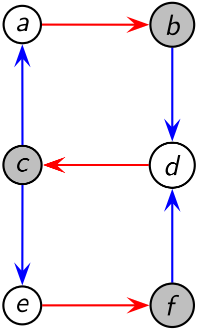




Bipartite **recurrent** quivers



T-system



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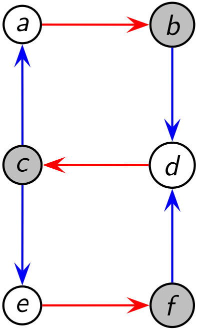
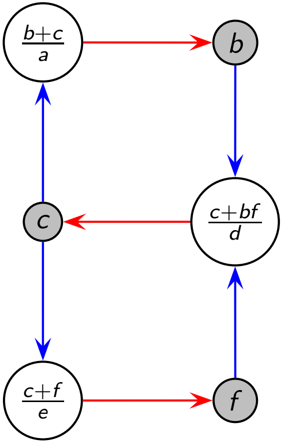


$\longrightarrow$



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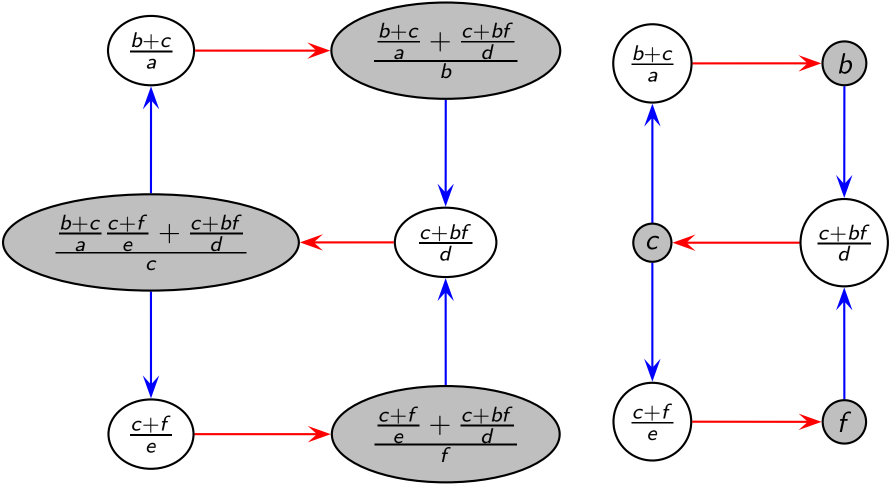
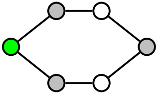
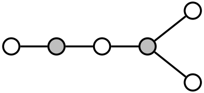
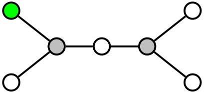
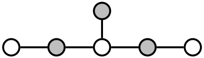
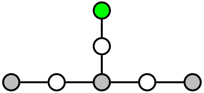
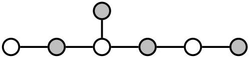
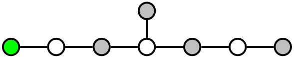
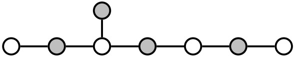
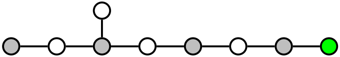


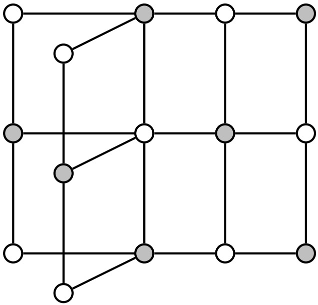
$\frac{b+c}{a}$
$\frac{\frac{b+c}{a}+\frac{c+bf}{d}}{b}$
$\frac{\frac{b+c}{a}\frac{c+f}{e}+\frac{c+bf}{d}}{c}$
$\frac{c+bf}{d}$
$\frac{c+f}{e}$
$\frac{\frac{c+f}{e}+\frac{c+bf}{d}}{f}$
$\frac{b+c}{a}$
b
c
$\frac{c+bf}{d}$
$\frac{c+f}{e}$
f

Part 2: Zamolodchikov periodicity

ADE Dynkin diagrams

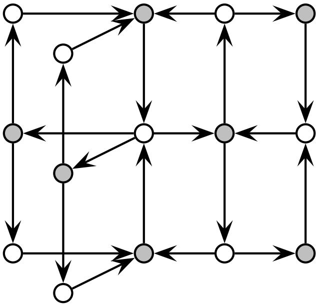
Name	Finite diagram	Affine diagram	Name
A_n			$\hat{A}_{n-1}$
D_n			$\hat{D}_{n-1}$
E_6			$\hat{E}_6$
E_7			$\hat{E}_7$
E_8			$\hat{E}_8$

Tensor product



$D_5 \otimes A_3$

Tensor product



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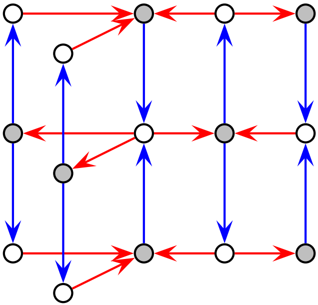


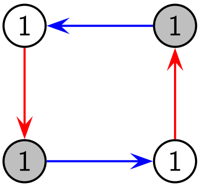
$D_5 \otimes A_3$

Zamolodchikov periodicity

Theorem (B. Keller, 2013)

Tensor product of **finite** *Dynkin diagrams* $\Longrightarrow$ *the* T*-system is periodic.*

Example: $A_2 \otimes A_2$



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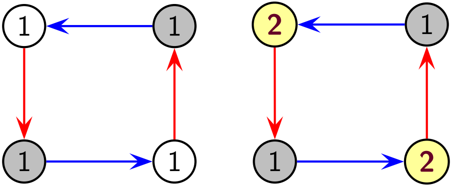

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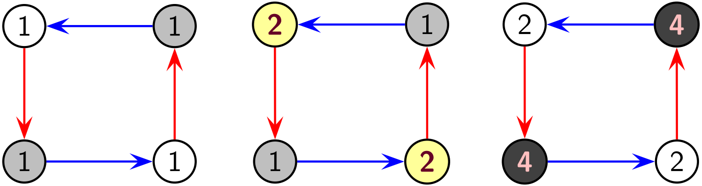

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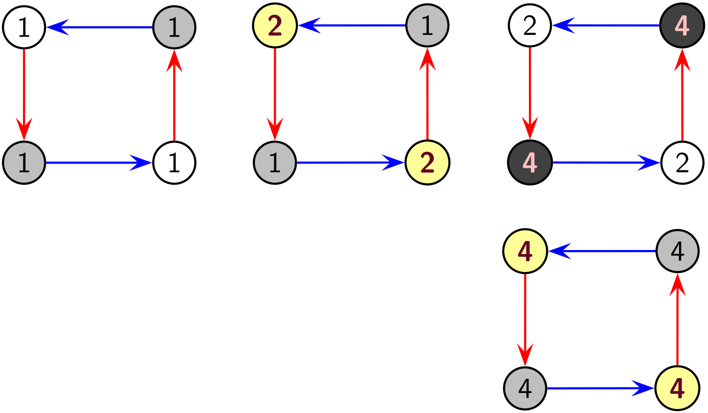

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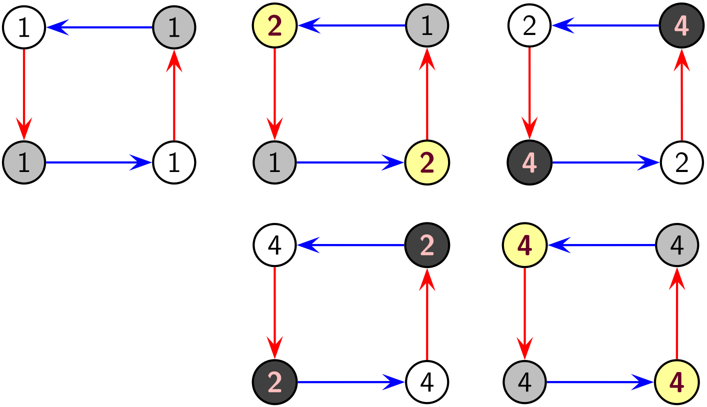

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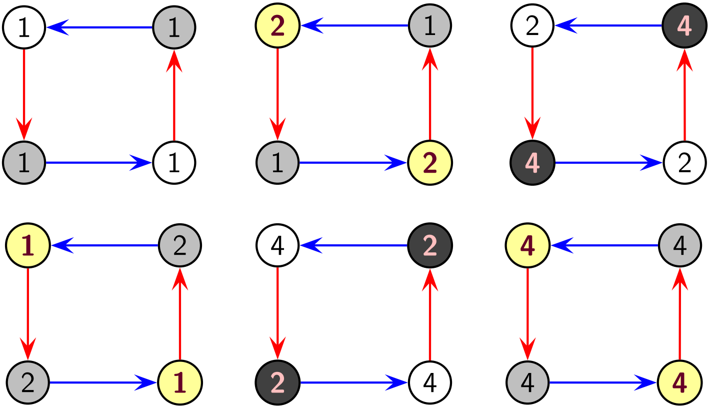
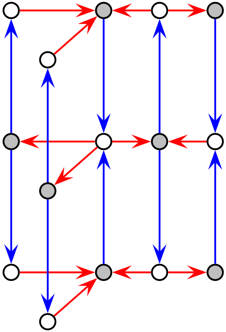


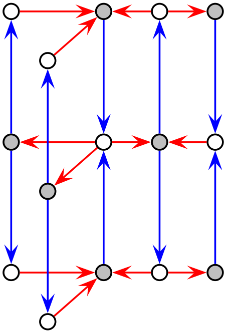
Theorem (B. Keller, 2013)

Tensor product of **finite** *Dynkin diagrams* $\Longrightarrow$ *the* T*-system is periodic.*

Finite ⊠ finite quivers

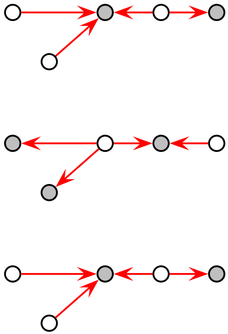


Finite ⊠ finite quivers



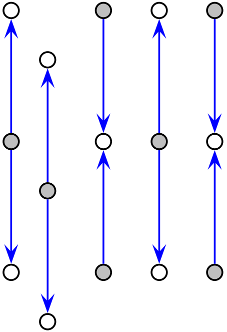
- Bipartite recurrent quiver

Finite ⊠ finite quivers



- Bipartite recurrent quiver
- All red components are **finite** Dynkin diagrams

Finite ⊠ finite quivers



- Bipartite recurrent quiver
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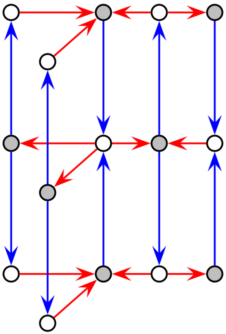
- All blue components are **finite** Dynkin diagrams

Finite ⊠ finite quivers



- Bipartite recurrent quiver
- All red components are **finite** Dynkin diagrams
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↑

"Finite ⊠ finite quiver"

The classification of Zamolodchikov periodic quivers

Theorem (G.-Pylyavskyy, 2016)

Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1. *The T-system associated with Q is periodic.*
2.

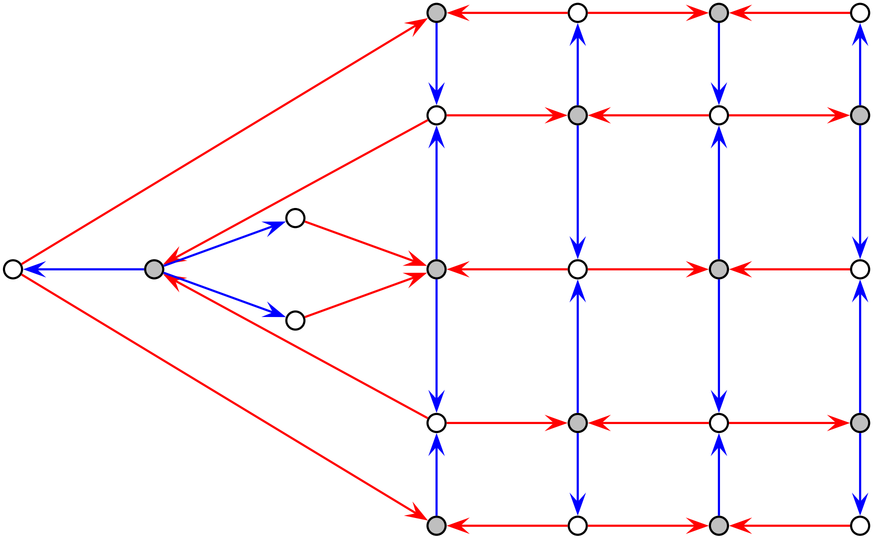
The classification of Zamolodchikov periodic quivers

Theorem (G.-Pylyavskyy, 2016)

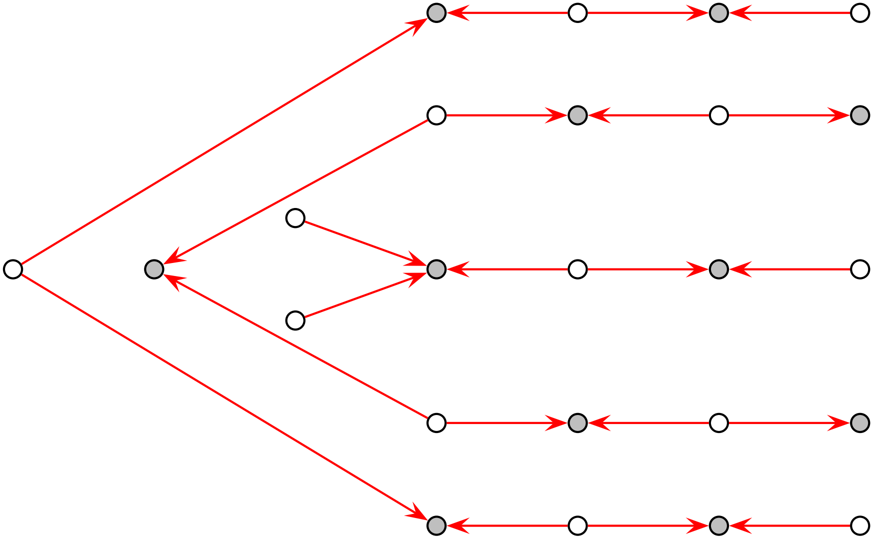
Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1. *The T-system associated with Q is periodic.*
2. **Q is a finite $\boxtimes$ finite quiver.**

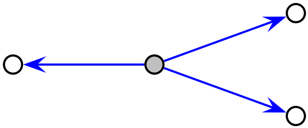
Finite ⊠ finite quivers

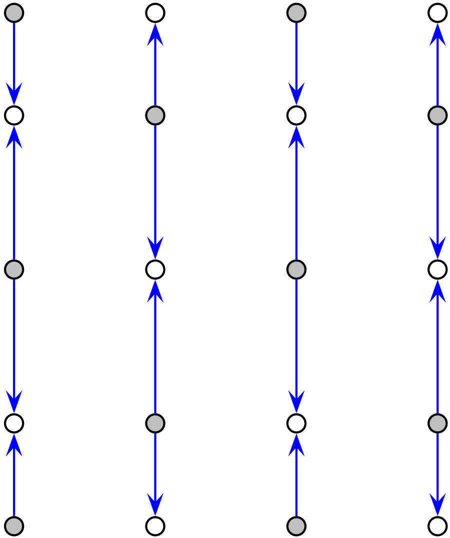


Finite ⊠ finite quivers

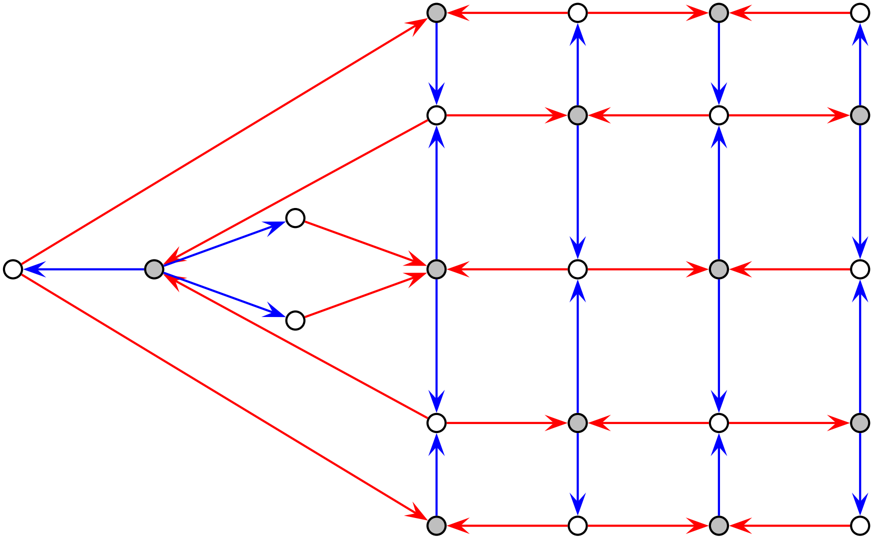


Finite ⊠ finite quivers

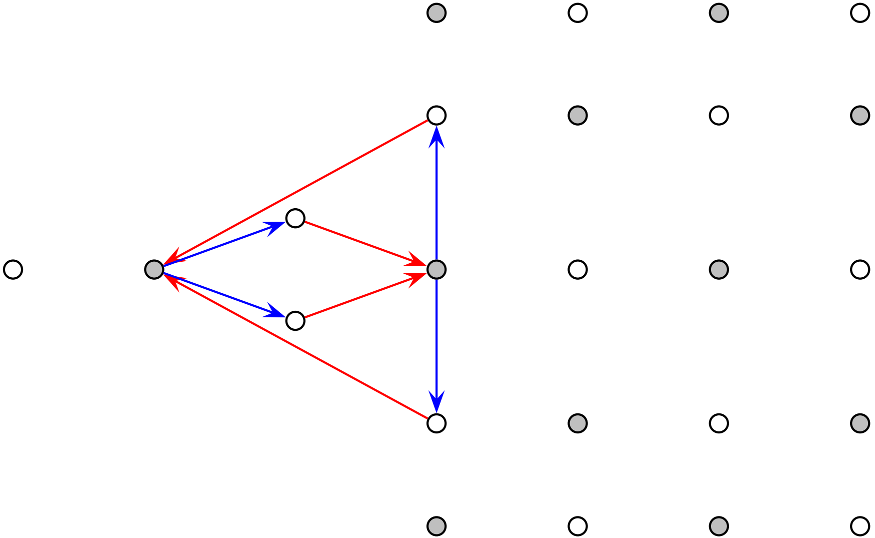




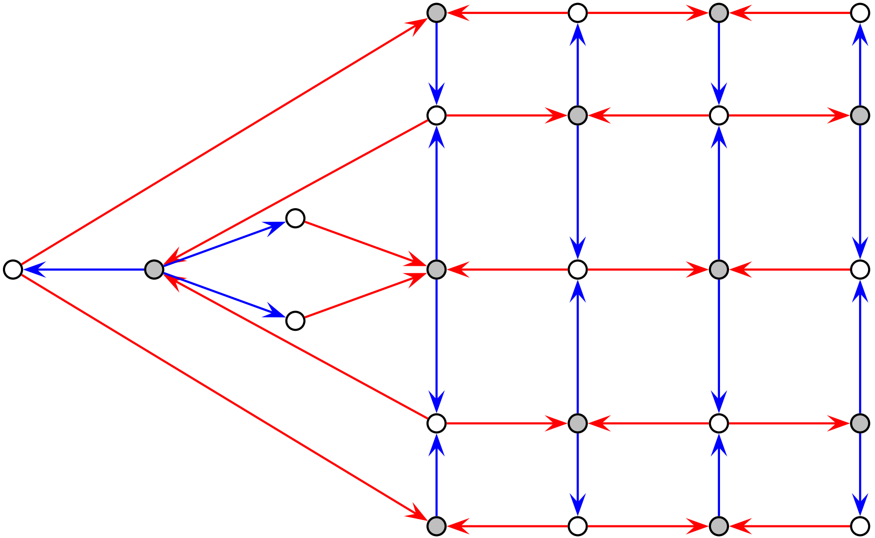
Finite ⊠ finite quivers



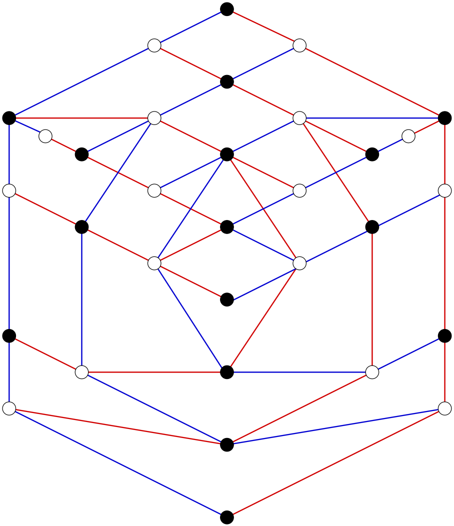
Finite ⊠ finite quivers



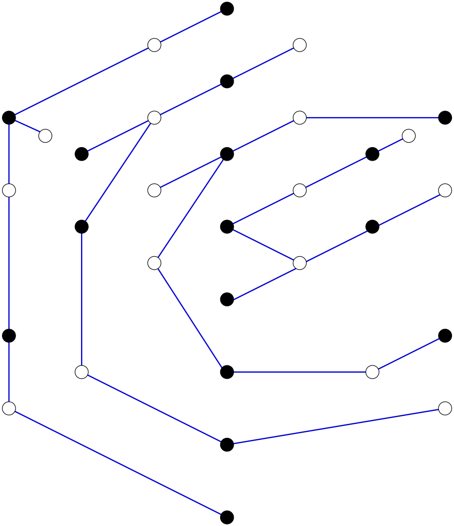
Finite ⊠ finite quivers



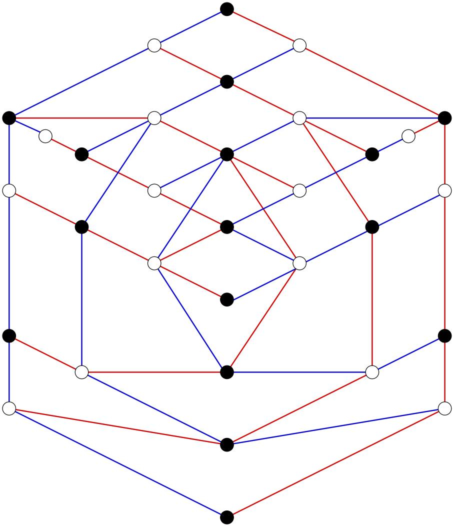
5 infinite families and 11 exceptional quivers



5 infinite families and 11 exceptional quivers



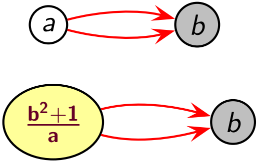
5 infinite families and 11 exceptional quivers



Part 3: Zamolodchikov integrability

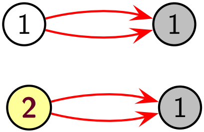
Example

Example

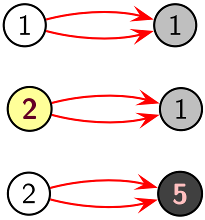


Example

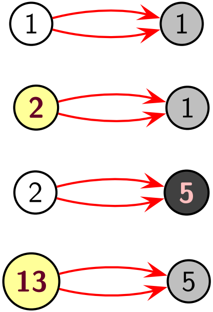
Example



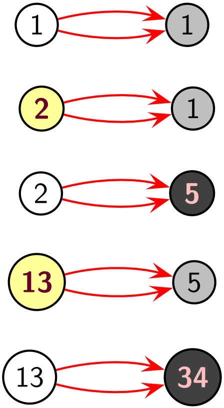
Example



Example



Example



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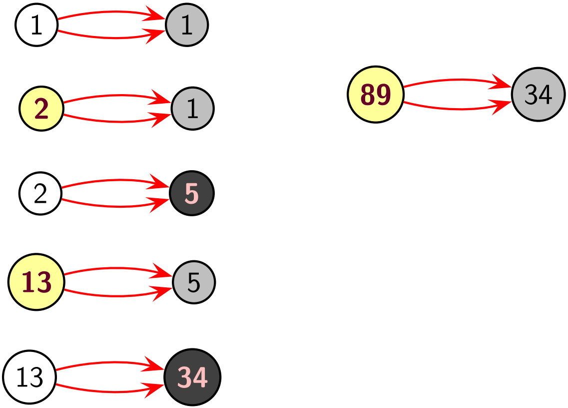

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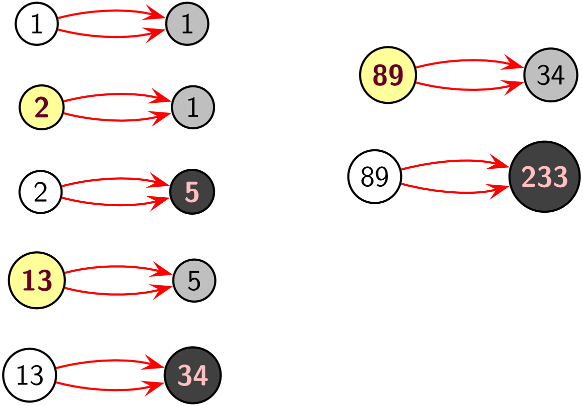

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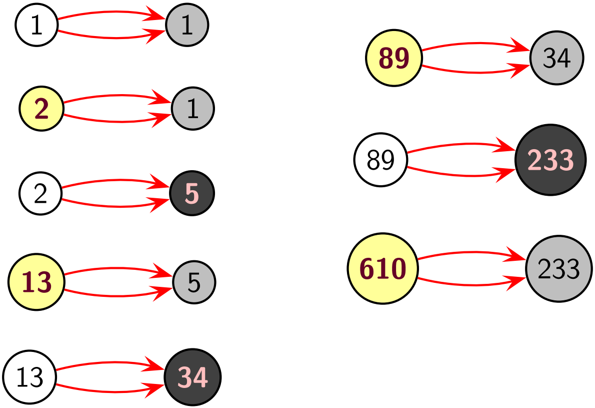

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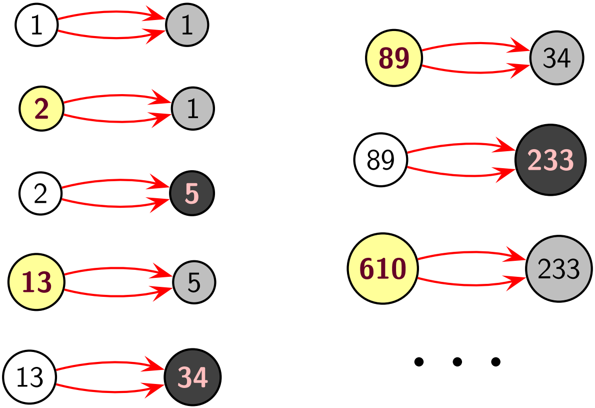

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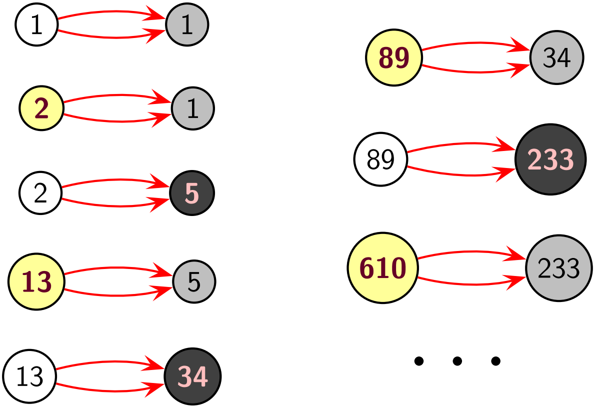
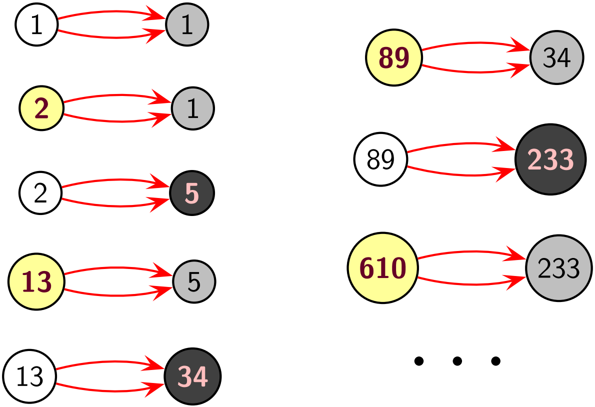


$$x_{n+1} - 3x_n + x_{n-1} = 0$$

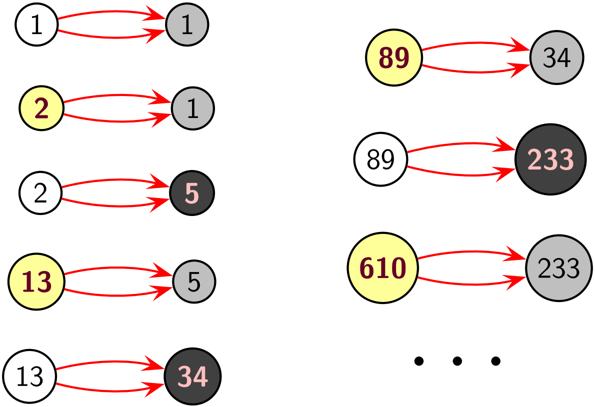
Example



$$x_{n+1} - 3x_n + x_{n-1} = 0$$

$$5 - 3 \cdot 2 + 1 = 0$$

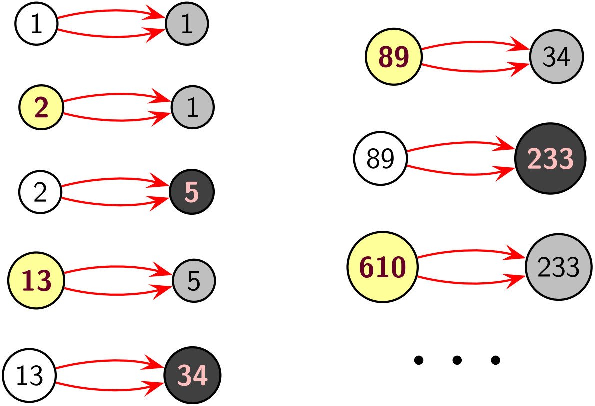
Example



$$x_{n+1} - 3x_n + x_{n-1} = 0$$

$$13 - 3 \cdot 5 + 2 = 0$$

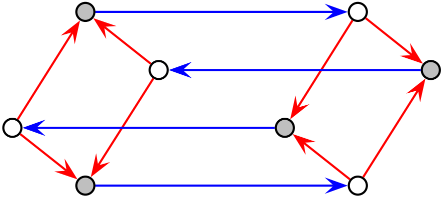
Example



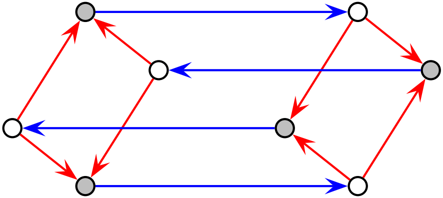
$$x_{n+1} - 3x_n + x_{n-1} = 0$$

$$34 - 3 \cdot 13 + 5 = 0$$

Affine ⊠ finite quivers

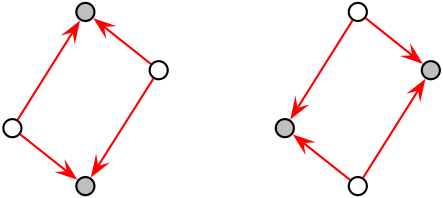


Affine ⊠ finite quivers



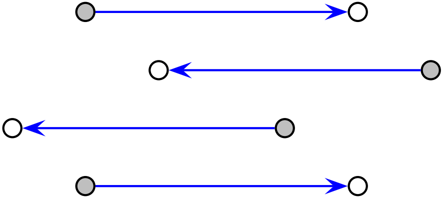
- Bipartite recurrent quiver

Affine ⊠ finite quivers



- Bipartite recurrent quiver
- All red components are **affine** Dynkin diagrams

Affine ⊠ finite quivers



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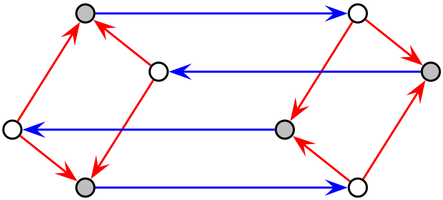
Affine ⊠ finite quivers



- Bipartite recurrent quiver
- All red components are **affine** Dynkin diagrams
- All blue components are **finite** Dynkin diagrams

↑

"Affine ⊠ finite quiver"

A necessary condition

Theorem (G.-Pylyavskyy, 2016)

Let Q be a bipartite recurrent quiver. Then:

1. **IF** *the T-system associated with Q is linearizable,*
2.

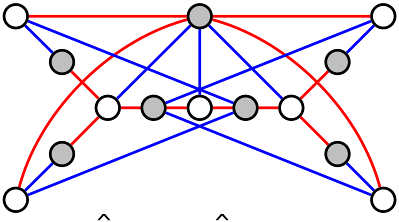
A necessary condition

Theorem (G.-Pylyavskyy, 2016)

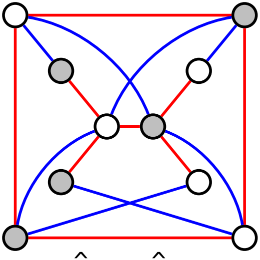
Let Q be a bipartite recurrent quiver. Then:

1. **IF** *the T-system associated with Q is linearizable,*
2. **THEN** *Q is an affine $\boxtimes$ finite quiver.*

15 infinite families and 4 exceptional cases

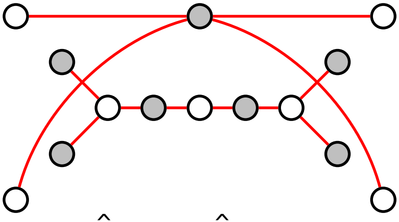


$\hat{D}_{n+1} * \hat{D}_{3n-1}$
for $n = 3$



$\hat{A}_3 * \hat{D}_5$

15 infinite families and 4 exceptional cases



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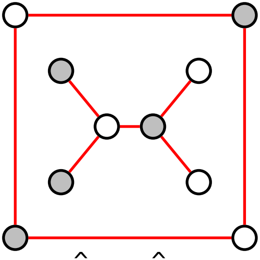

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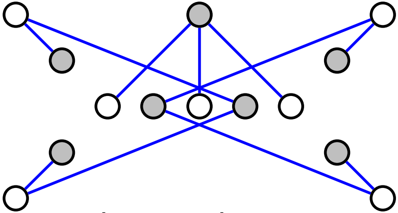

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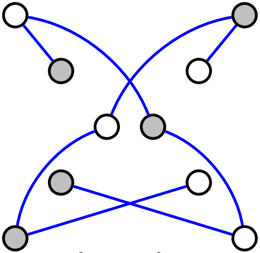

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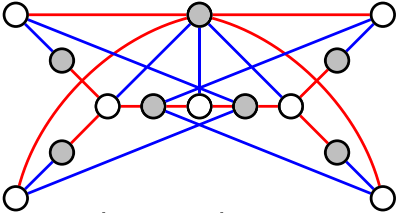

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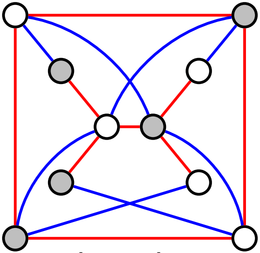

$\hat{A}_3 * \hat{D}_5$

Type $A_m \otimes \hat{A}_{2n-1}$

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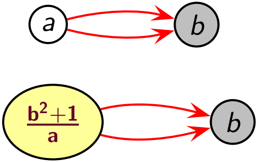

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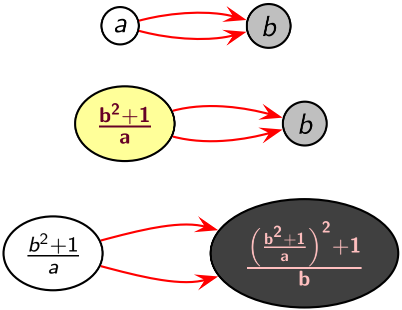

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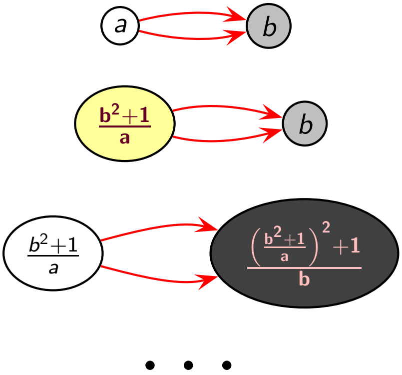

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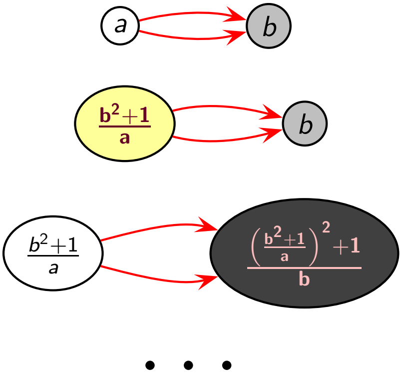
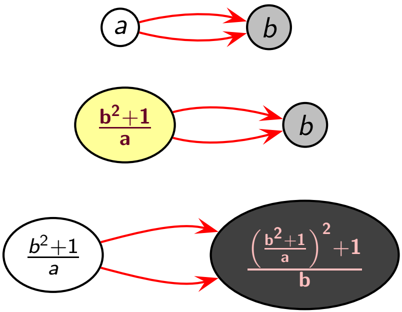


$$x_{n+1} - 3x_n + x_{n-1} = 0$$

Type $A_m \otimes \hat{A}_{2n-1}$

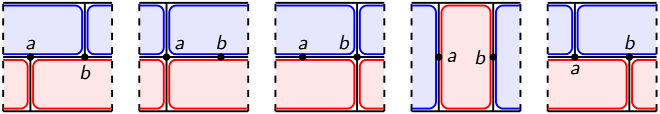


• • •

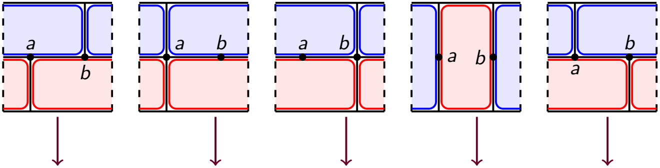
$$x_{n+1} - 3x_n + x_{n-1} = 0$$

$$\mathbf{1} \cdot x_{n+1} - \left(\frac{\mathbf{a}}{\mathbf{b}} + \frac{\mathbf{b}}{\mathbf{a}} + \frac{\mathbf{1}}{\mathbf{ab}} \right) \cdot x_n + \mathbf{1} \cdot x_{n-1} = 0$$

Domino tilings of the cylinder

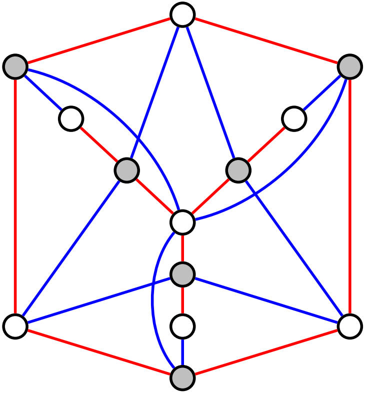


Domino tilings of the cylinder



$$\mathbf{1}\cdot x_{n+1} - \left(\frac{\mathbf{a}}{\mathbf{b}} + \frac{\mathbf{b}}{\mathbf{a}} + \frac{\mathbf{1}}{\mathbf{ab}}\right)\cdot x_n + \mathbf{1}\cdot x_{n-1} = 0$$

Affine ⊠ affine quivers?



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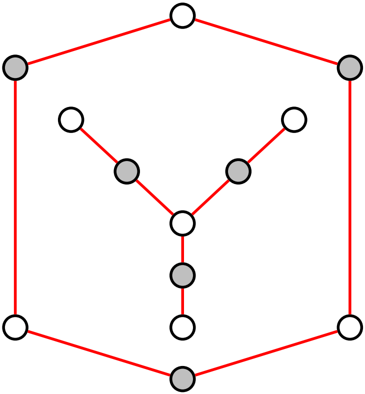

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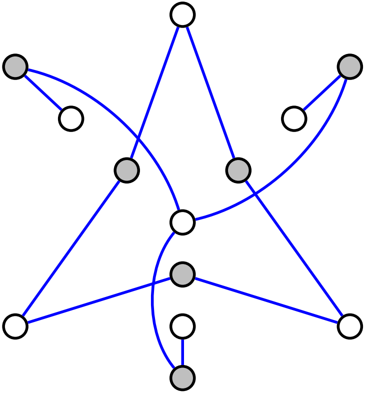

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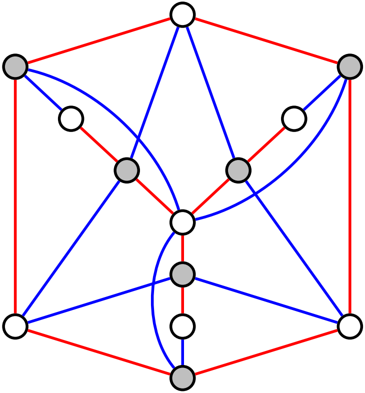

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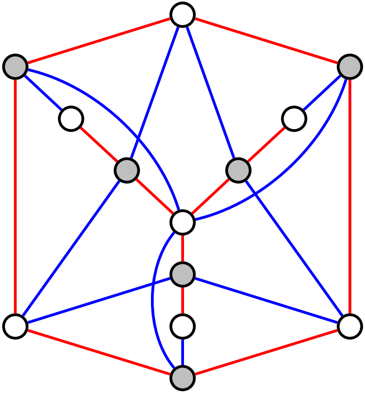

Thank you!

Bibliography

Sergey Fomin and Andrei Zelevinsky.
Y-systems and generalized associahedra.
Ann. of Math. (2), 158(3):977–1018, 2003.

Bernhard Keller.
The periodicity conjecture for pairs of Dynkin diagrams.
Ann. of Math. (2), 177(1):111–170, 2013.

John R. Stembridge.
Admissible W-graphs and commuting Cartan matrices.
Adv. in Appl. Math., 44(3):203–224, 2010.

Pavel Galashin and Pavlo Pylyavskyy
The classification of Zamolodchikov periodic quivers.
arXiv:1603.03942 (2016).

Pavel Galashin and Pavlo Pylyavskyy
Quivers with subadditive labelings: classification and integrability
arXiv:1606.04878 (2016).