Amplituhedra and origami

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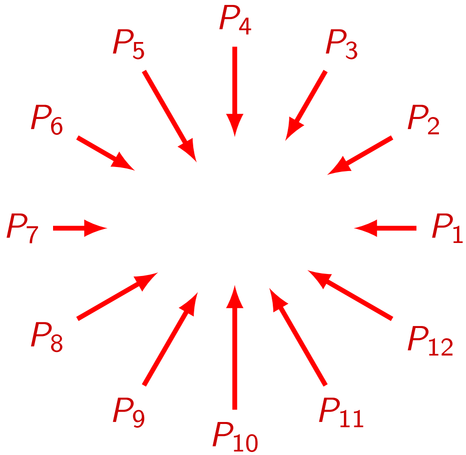
Cornell Discrete Geometry and Combinatorics Seminar
December 9, 2024

arXiv:2410.09574

Scattering amplitudes

- Consider incoming particles with momenta $P_1, P_2, \ldots, P_n$.



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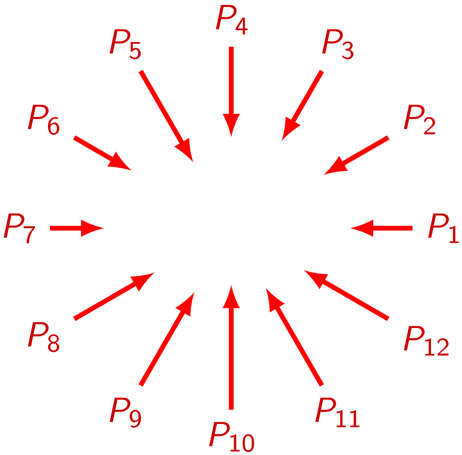

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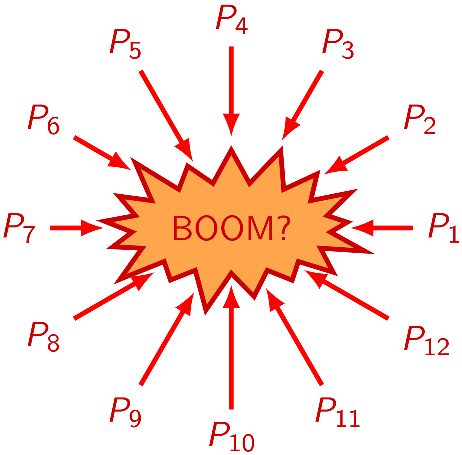

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Computing scattering amplitudes:

	Efficient?	Conceptual?
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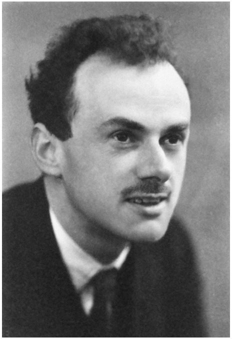

$$
\begin{aligned}
E_n^{(1)} &= V_{nn} \\
E_n^{(2)} &= \frac{|V_{nk_2}|^2}{E_{nk_2}} \\
E_n^{(3)} &= \frac{V_{nk_3}V_{k_3k_2}V_{k_2n}}{E_{nk_2}E_{nk_3}} - V_{nn}\frac{|V_{nk_3}|^2}{E_{nk_3}^2} \\
E_n^{(4)} &= \frac{V_{nk_4}V_{k_4k_3}V_{k_3k_2}V_{k_2n}}{E_{nk_2}E_{nk_3}E_{nk_4}} - \frac{|V_{nk_4}|^2}{E_{nk_4}^2}\frac{|V_{nk_2}|^2}{E_{nk_2}} - V_{nn}\frac{V_{nk_4}V_{k_4k_3}V_{k_3n}}{E_{nk_3}^2E_{nk_4}} - V_{nn}\frac{V_{nk_4}V_{k_4k_2}V_{k_2n}}{E_{nk_2}E_{nk_4}^2} + V_{nn}^2\frac{|V_{nk_4}|^2}{E_{nk_4}^3} \\
&= \frac{V_{nk_4}V_{k_4k_3}V_{k_3k_2}V_{k_2n}}{E_{nk_2}E_{nk_3}E_{nk_4}} - E_n^{(2)}\frac{|V_{nk_4}|^2}{E_{nk_4}^2} - 2V_{nn}\frac{V_{nk_4}V_{k_4k_3}V_{k_3n}}{E_{nk_3}^2E_{nk_4}} + V_{nn}^2\frac{|V_{nk_4}|^2}{E_{nk_4}^3} \\
E_n^{(5)} &= \frac{V_{nk_5}V_{k_5k_4}V_{k_4k_3}V_{k_3k_2}V_{k_2n}}{E_{nk_2}E_{nk_3}E_{nk_4}E_{nk_5}} - \frac{V_{nk_5}V_{k_5k_4}V_{k_4n}}{E_{nk_4}^2E_{nk_5}}\frac{|V_{nk_2}|^2}{E_{nk_2}} - \frac{V_{nk_5}V_{k_5k_2}V_{k_2n}}{E_{nk_2}E_{nk_5}^2}\frac{|V_{nk_2}|^2}{E_{nk_2}} - \frac{|V_{nk_5}|^2}{E_{nk_5}^2}\frac{V_{nk_3}V_{k_3k_2}V_{k_2n}}{E_{nk_2}E_{nk_3}} \\
&\quad - V_{nn}\frac{V_{nk_5}V_{k_5k_4}V_{k_4k_3}V_{k_3n}}{E_{nk_3}^2E_{nk_4}E_{nk_5}} - V_{nn}\frac{V_{nk_5}V_{k_5k_4}V_{k_4k_2}V_{k_2n}}{E_{nk_2}E_{nk_4}^2E_{nk_5}} - V_{nn}\frac{V_{nk_5}V_{k_5k_3}V_{k_3k_2}V_{k_2n}}{E_{nk_2}E_{nk_3}E_{nk_5}^2} + V_{nn}\frac{|V_{nk_5}|^2}{E_{nk_5}^2}\frac{|V_{nk_3}|^2}{E_{nk_3}^2} + 2V_{nn}\frac{|V_{nk_5}|^2}{E_{nk_5}^3}\frac{|V_{nk_2}|^2}{E_{nk_2}} \\
&\quad + V_{nn}^2\frac{V_{nk_5}V_{k_5k_4}V_{k_4n}}{E_{nk_4}^3E_{nk_5}} + V_{nn}^2\frac{V_{nk_5}V_{k_5k_3}V_{k_3n}}{E_{nk_3}^2E_{nk_5}^2} + V_{nn}^2\frac{V_{nk_5}V_{k_5k_2}V_{k_2n}}{E_{nk_2}E_{nk_5}^3} - V_{nn}^3\frac{|V_{nk_5}|^2}{E_{nk_5}^4} \\
&= \frac{V_{nk_5}V_{k_5k_4}V_{k_4k_3}V_{k_3k_2}V_{k_2n}}{E_{nk_2}E_{nk_3}E_{nk_4}E_{nk_5}} - 2E_n^{(2)}\frac{V_{nk_5}V_{k_5k_4}V_{k_4n}}{E_{nk_4}^2E_{nk_5}} - \frac{|V_{nk_5}|^2}{E_{nk_5}^2}\frac{V_{nk_3}V_{k_3k_2}V_{k_2n}}{E_{nk_2}E_{nk_3}} \\
&\quad - 2V_{nn}\left(\frac{V_{nk_5}V_{k_5k_4}V_{k_4k_3}V_{k_3n}}{E_{nk_3}^2E_{nk_4}E_{nk_5}} - \frac{V_{nk_5}V_{k_5k_4}V_{k_4k_2}V_{k_2n}}{E_{nk_2}E_{nk_4}^2E_{nk_5}} + \frac{|V_{nk_5}|^2}{E_{nk_5}^2}\frac{|V_{nk_3}|^2}{E_{nk_3}^2} + 2E_n^{(2)}\frac{|V_{nk_5}|^2}{E_{nk_5}^3}\right) \\
&\quad + V_{nn}^2\left(2\frac{V_{nk_5}V_{k_5k_4}V_{k_4n}}{E_{nk_4}^3E_{nk_5}} + \frac{V_{nk_5}V_{k_5k_3}V_{k_3n}}{E_{nk_3}^2E_{nk_5}^2}\right) - V_{nn}^3\frac{|V_{nk_5}|^2}{E_{nk_5}^4}
\end{aligned}
$$

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- Consider incoming particles with momenta $P_1, P_2, \ldots, P_n$.
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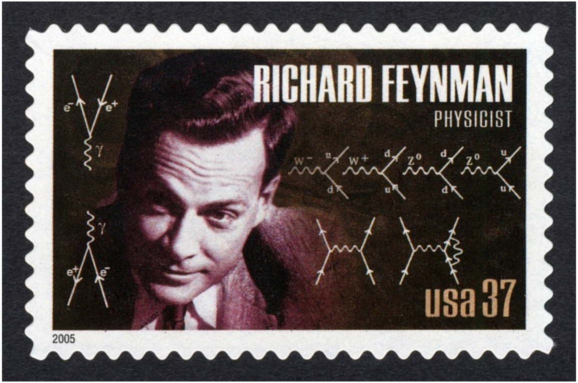

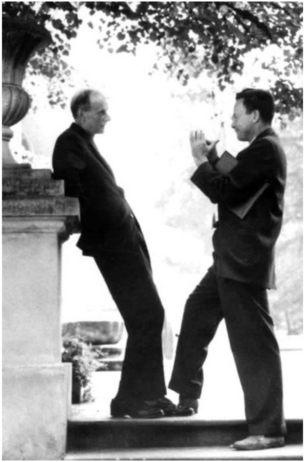

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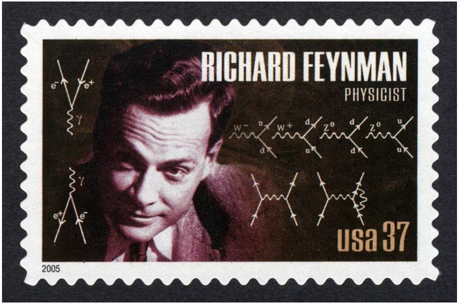

Each Feynman diagram is the sum of exponentially many old-fashioned terms.

— [Wikipedia]

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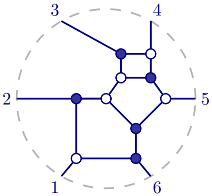

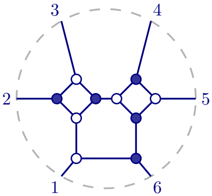

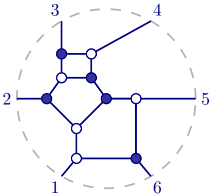

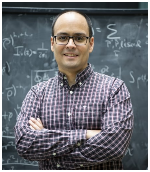

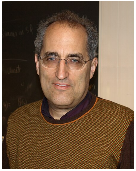

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Loops	# Feynman (super)graphs	# BCFW Cells
1	940	1
2	47,380	10
3	4,448,500	146
4	672,315,700	2,684
5	148,251,680,500	56,914
6	44,838,422,282,500	1,329,324
7	17,796,990,083,372,500	33,291,164
8	8,968,512,580,259,732,500	878,836,728
9	5,592,013,331,255,143,292,500	24,175,924,094
10	4,225,692,640,945,498,084,862,500	687,444,432,396
11	3,804,754,710,505,713,091,940,312,500	20,086,271,785,340

credit: J. Bourjaily

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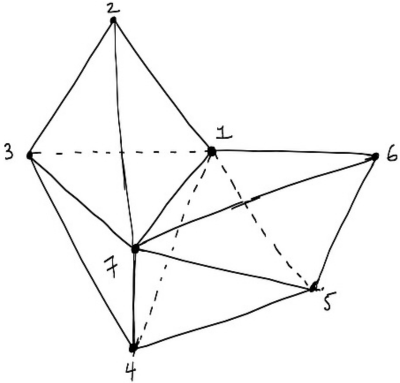

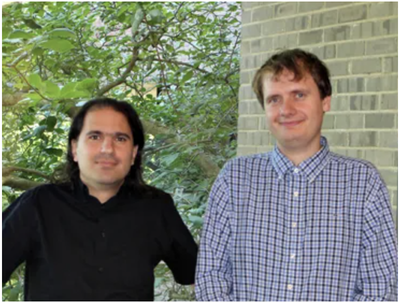

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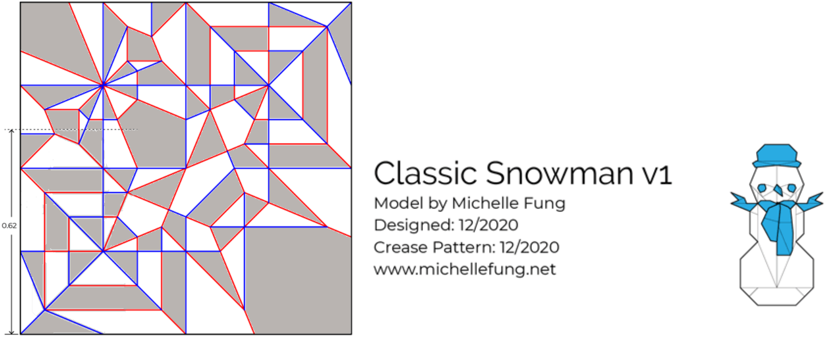
Today: origami [G., 2024]	?/10	?/10



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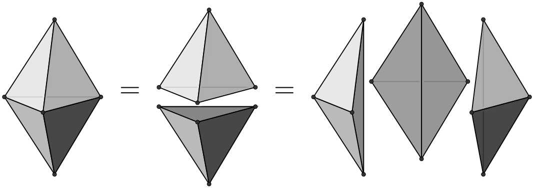

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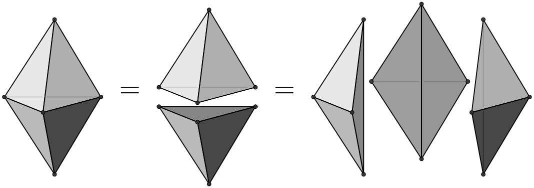

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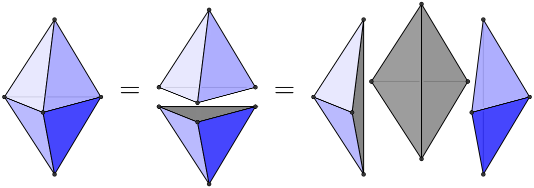

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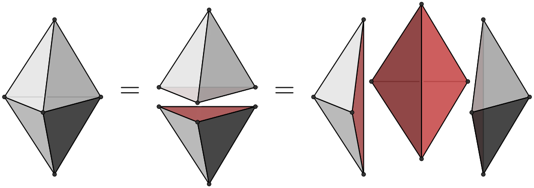
- Actual singularities $\longleftrightarrow$ boundaries of the amplituhedron

BCFW recurrence computes $\mathcal{A}(P_1^-, P_2^-, P_3^-, P_4^+, P_5^+, P_6^+)$ in two different ways:

$$\left(\frac{[4|5+6|1\rangle^3}{[34][23]\langle 56\rangle\langle 61\rangle[2|3+4|5\rangle S_{234}} + \frac{[6|1+2|3\rangle^3}{[61][12]\langle 34\rangle\langle 45\rangle[2|3+4|5\rangle S_{612}}\right) = \left(\frac{(S_{123})^3}{[12][23]\langle 45\rangle\langle 56\rangle[1|2+3|4\rangle[3|4+5|6\rangle} + \frac{\langle 12\rangle^3[45]^3}{\langle 16\rangle[34][3|4+5|6\rangle[5|6+1|2\rangle S_{612}} + \frac{\langle 23\rangle^3[56]^3}{\langle 34\rangle[16][1|2+3|4\rangle[5|6+1|2\rangle S_{234}}\right)$$



Amplituhedron philosophy: [Hodges '09] [Arkani-Hamed–Trnka '14]

- Ways to run BCFW recurrence $\longrightarrow$ triangulations of the amplituhedron
- Terms in BCFW recurrence $\longleftrightarrow$ pieces of a triangulation
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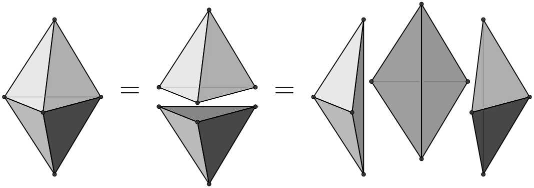

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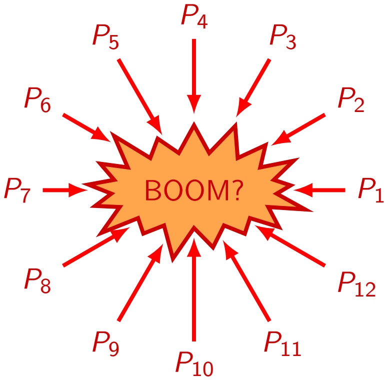
- $\mathcal{A}(P_1, P_2, \ldots, P_n)$ $\longleftrightarrow$ "volume" of the amplituhedron

Particle momenta

- Consider incoming particles with momenta $P_1, P_2, \ldots, P_n$.

- Scattering amplitude $\mathcal{A}(P_1, P_2, \ldots, P_n) =$ probability of 'scattering'.



Particle momenta

- Consider incoming particles with momenta $P_1, P_2, \ldots, P_n \in \mathbb{R}^{3,1}$ which are light-like ($P_i^2 = 0$) and satisfy $P_1 + P_2 + \cdots + P_n = 0$.

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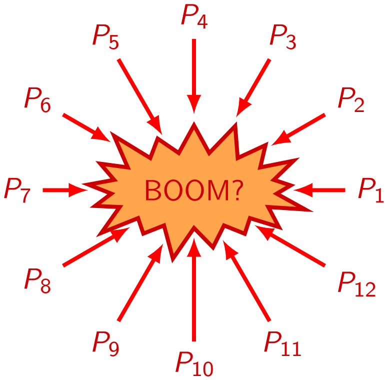

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- Here $P = (p_0, p_1, p_2, p_3)$ and $P^2 = p_0^2 + p_1^2 + p_2^2 - p_3^2$.
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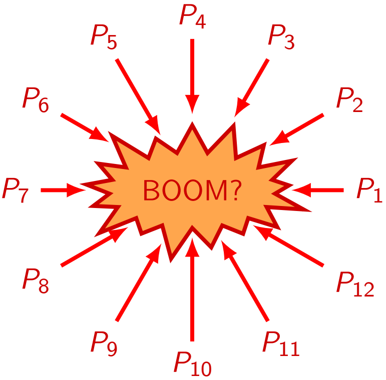

Particle momenta

- Consider incoming particles with momenta $P_1, P_2, \ldots, P_n \in \mathbb{R}^{2,2}$ which are light-like ($P_i^2 = 0$) and satisfy $P_1 + P_2 + \cdots + P_n = 0$.
- Here $P = (p_0, p_1, p_2, p_3)$ and $P^2 = p_0^2 + p_1^2 - p_2^2 - p_3^2$.
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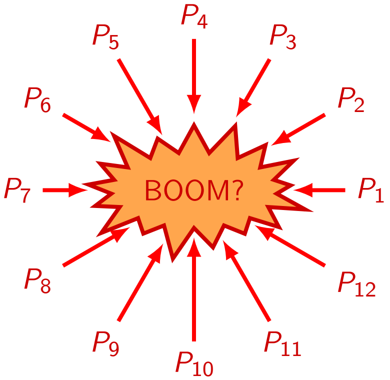

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On the contrary, for *complex-valued momenta* p^μ, the angle and square spinors are independent.[1] It may not seem physical to take p^μ complex, but it is a very very very useful strategy. We will see this repeatedly.

[1] One can keep p^μ real and change the spacetime signature to $(-,+,-,+)$; in that case, the angle and square spinors are real and independent.

[Elvang, Huang. *Scattering amplitudes in gauge theory and gravity.* Cambridge University Press, Cambridge, 2015.]

Particle momenta

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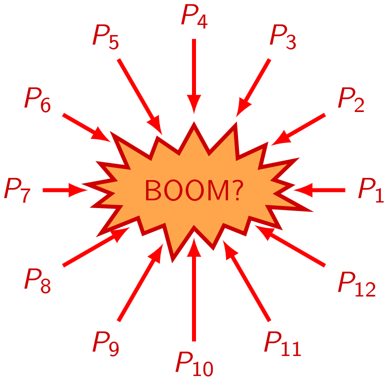

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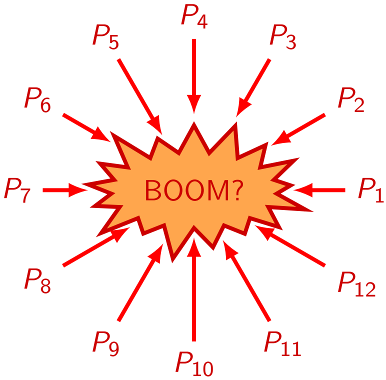
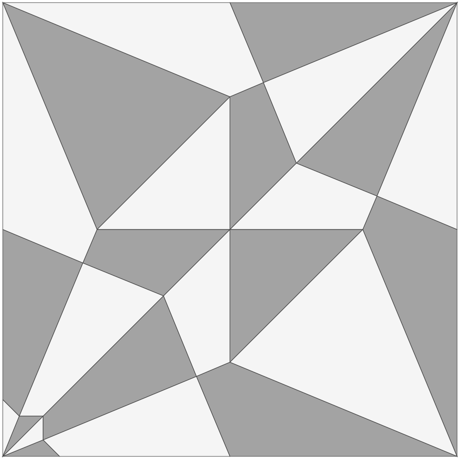


- Think of a light-like momentum vector $P \in \mathbb{R}^{2,2}$ as a pair $(P^{\mathcal{T}}, P^{\mathcal{O}}) \in \mathbb{C}^2$ of complex numbers satisfying $|P^{\mathcal{T}}| = |P^{\mathcal{O}}|$.

Origami crease patterns



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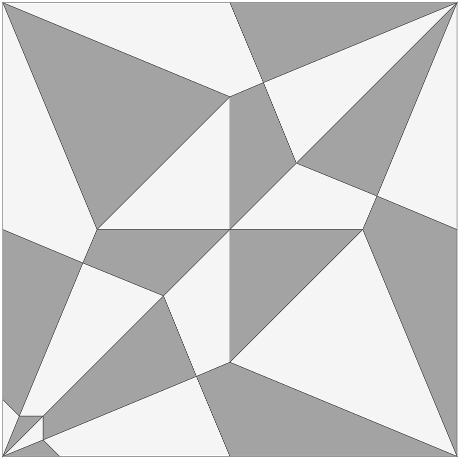
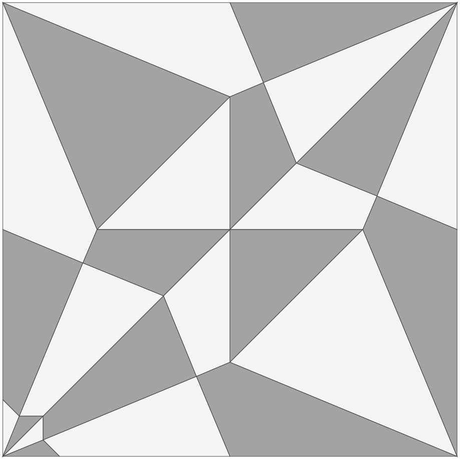


Faces: convex polygons colored black and white;

Origami crease patterns



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Angle condition:
sum(white angles) $= \pi$,
sum(black angles) $= \pi$,
around each interior vertex.

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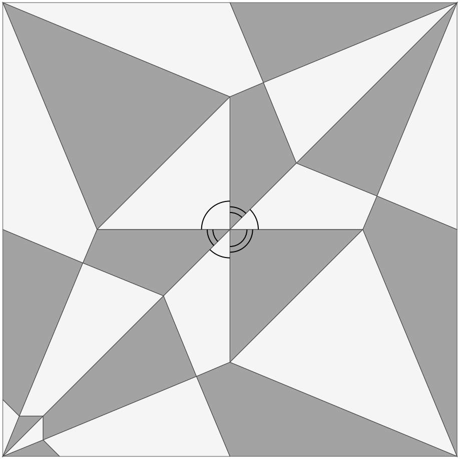

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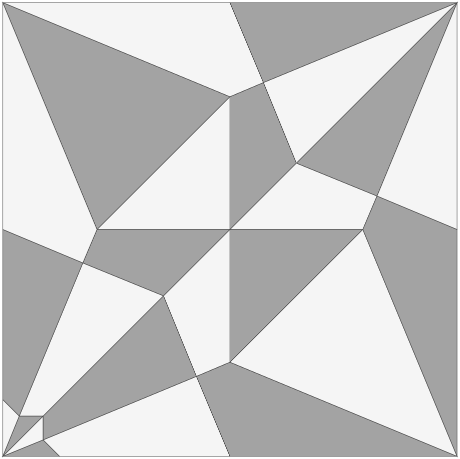

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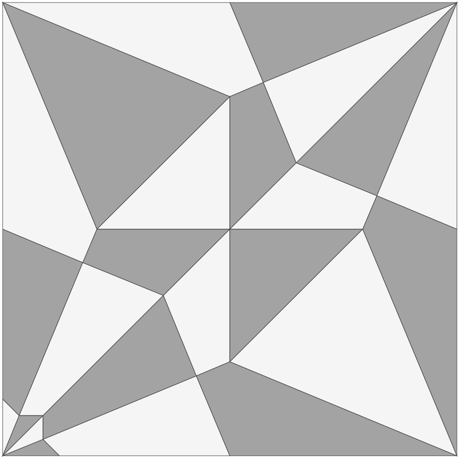
Origami map $\mathcal{O}$:
isometry on each face preserving/reversing the orientations of white/black faces.

Origami crease patterns



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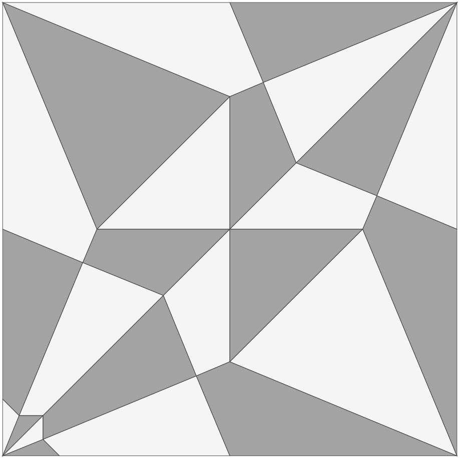

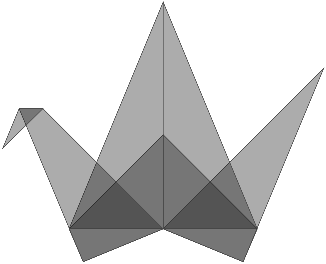

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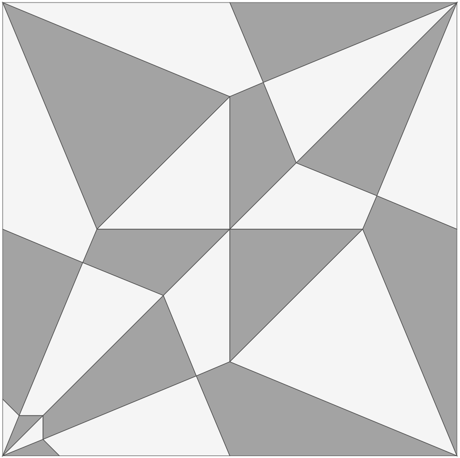
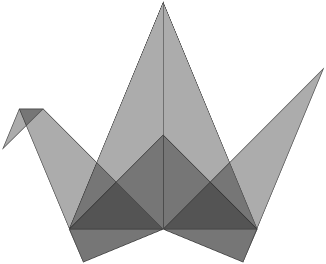
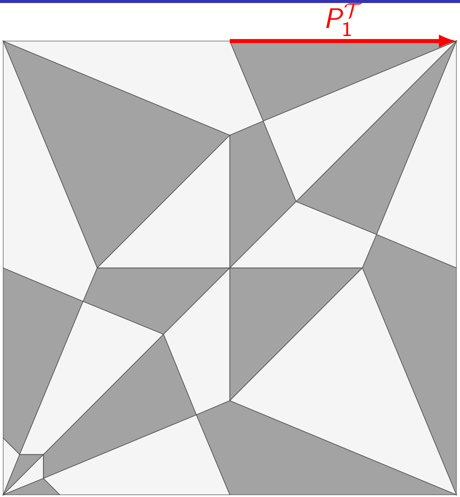


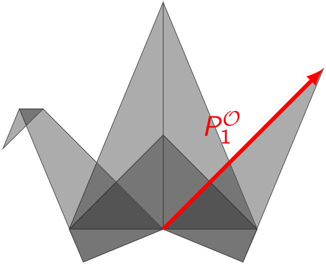


Origami map $\mathcal{O}$: isometry on each face preserving/reversing the orientations of white/black faces.

Boundary vectors $P_i^{\mathcal{T}}$ and their images $P_i^{\mathcal{O}}$ under $\mathcal{O}$ satisfy $|P_i^{\mathcal{T}}| = |P_i^{\mathcal{O}}|$!

Origami crease patterns





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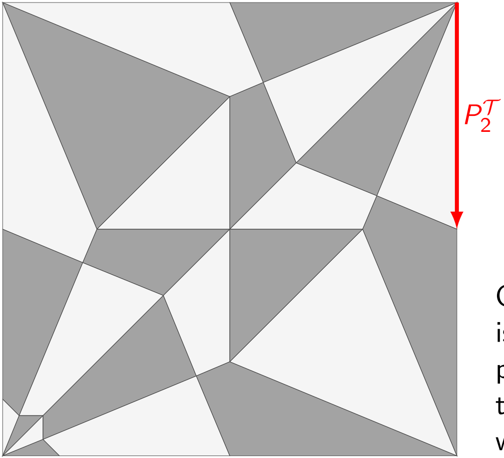

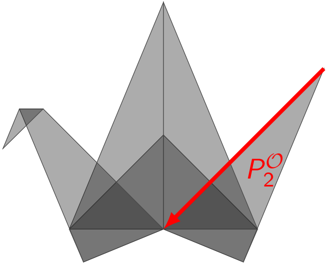

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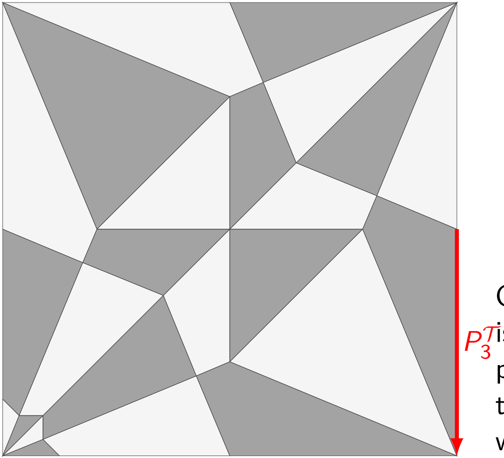

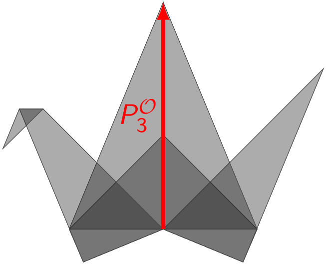

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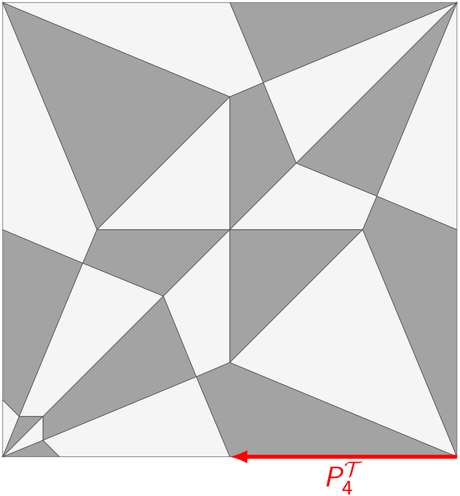

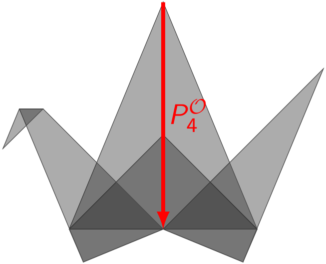

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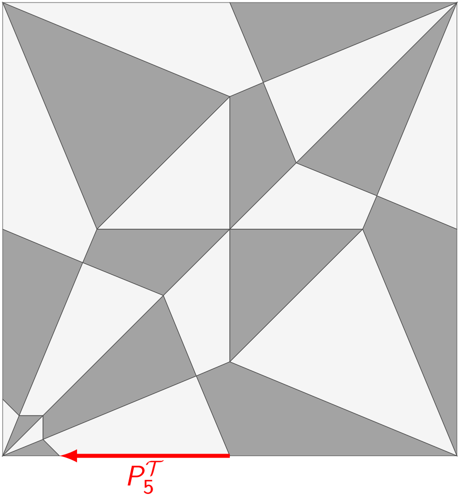

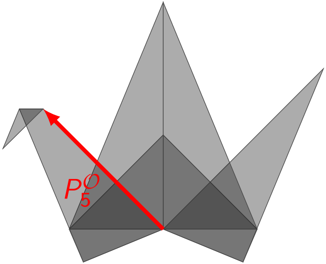

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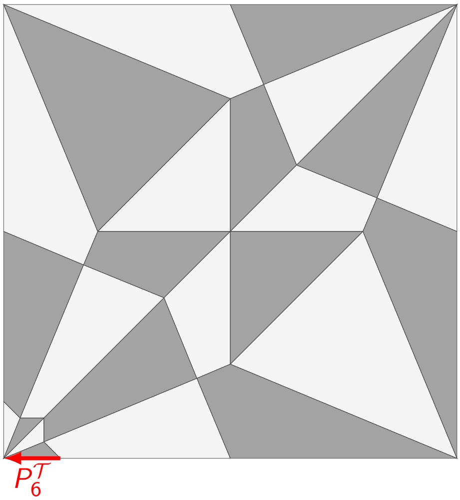

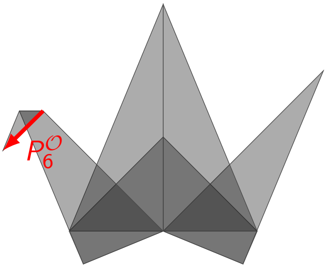

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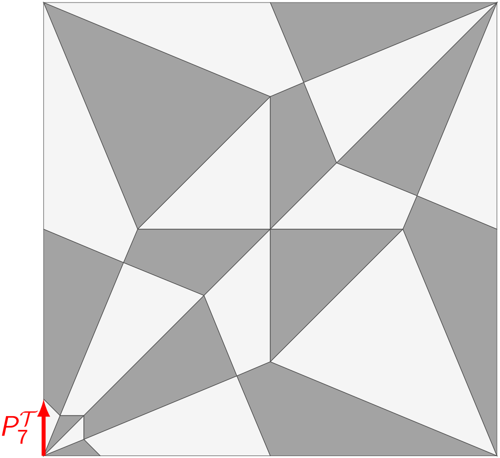

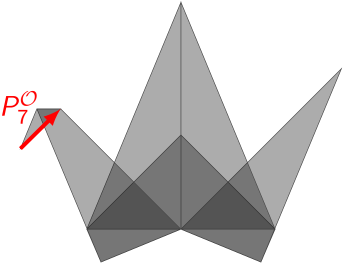

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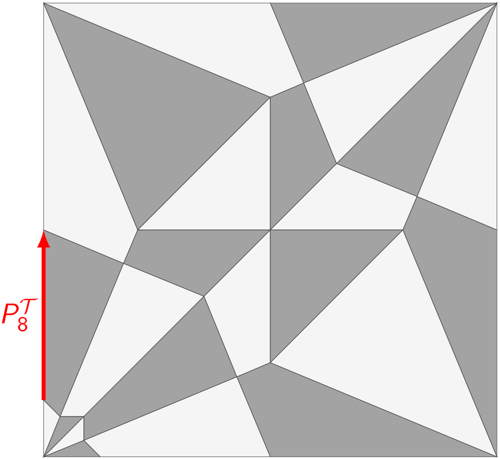

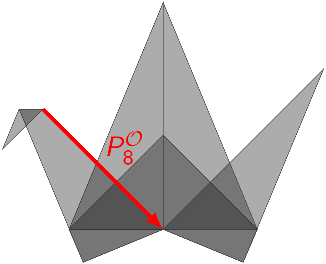

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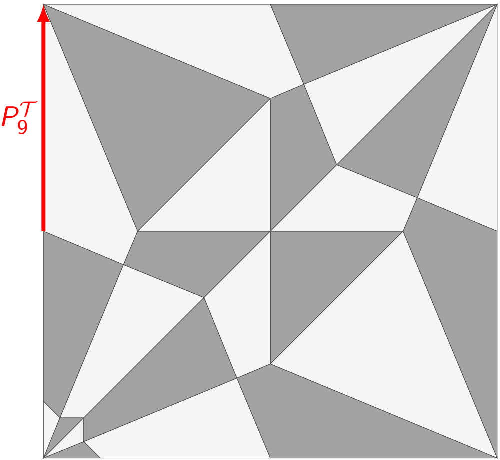

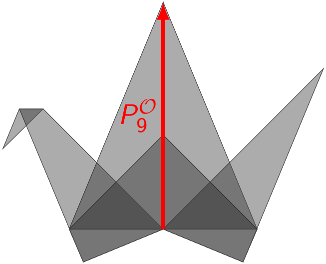

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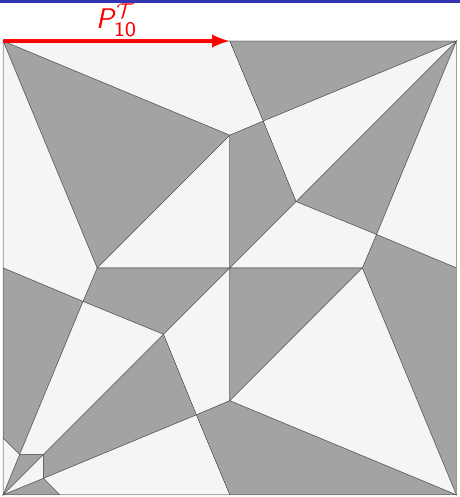

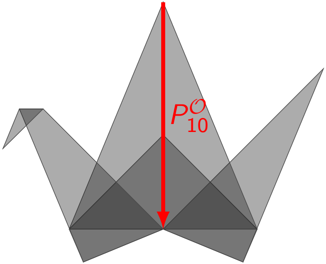

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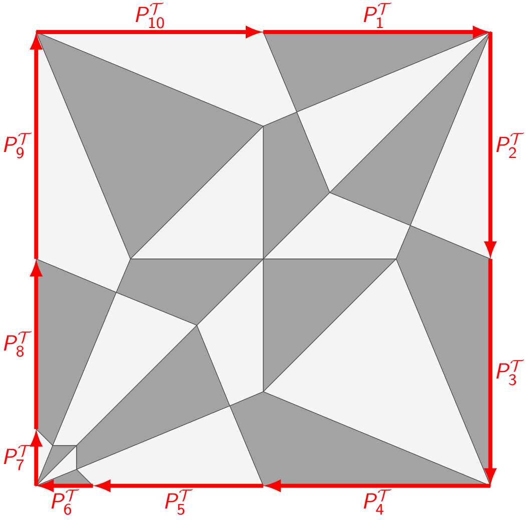

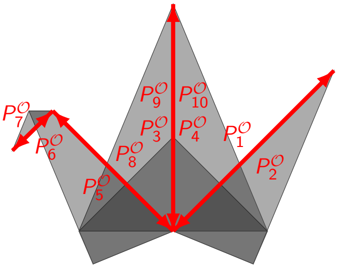

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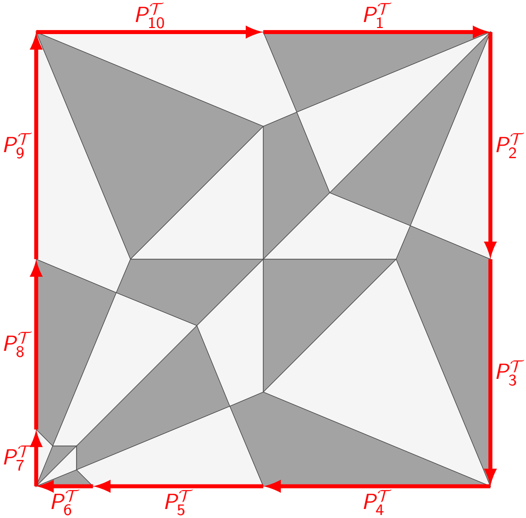

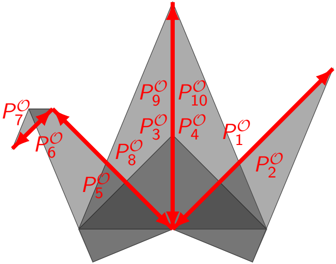

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Main result (preview):

$\mathcal{A}(P_1, \ldots, P_n)$ = integral over origami crease patterns with boundary $P_1, \ldots, P_n$.

Grassmannian

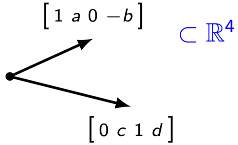
$k \times n$ matrix C

$$C = \begin{bmatrix} 1 & a & 0 & -b \\ 0 & c & 1 & d \end{bmatrix}$$

Grassmannian

$k \times n$ matrix $C \quad \leftrightarrow \quad$ k row vectors in $\mathbb{R}^n$

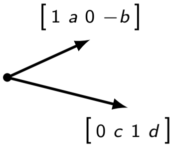
$C = \begin{bmatrix} 1 & a & 0 & -b \\ 0 & c & 1 & d \end{bmatrix} \quad \leftrightarrow$



Grassmannian

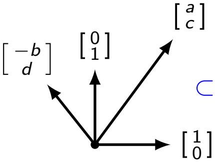
$k \times n$ matrix C $\quad \leftrightarrow \quad$ k row vectors in $\mathbb{R}^n$ $\quad \leftrightarrow$ n column vectors in $\mathbb{R}^k$

$C = \begin{bmatrix} 1 & a & 0 & -b \\ 0 & c & 1 & d \end{bmatrix}$ $\leftrightarrow$



$\subset \mathbb{R}^4$

$\leftrightarrow$

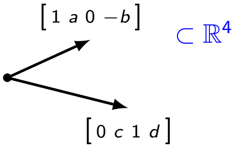


$\subset \mathbb{R}^2$

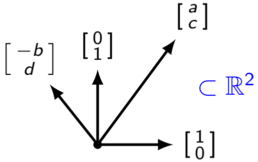
Grassmannian

$k \times n$ matrix C mod row operations $\leftrightarrow$ k row vectors in $\mathbb{R}^n$ $\leftrightarrow$ n column vectors in $\mathbb{R}^k$

$C = \begin{bmatrix} 1 & a & 0 & -b \\ 0 & c & 1 & d \end{bmatrix}$ $\leftrightarrow$



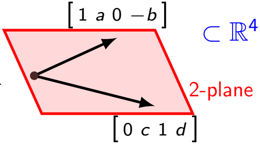
$\leftrightarrow$



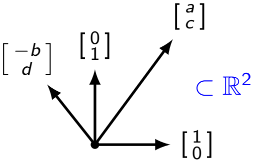
Grassmannian

$k \times n$ matrix C mod row operations $\leftrightarrow$ ~~k row vectors in $\mathbb{R}^n$~~ k-plane inside $\mathbb{R}^n$ $\leftrightarrow$ n column vectors in $\mathbb{R}^k$

$$C = \begin{bmatrix} 1 & a & 0 & -b \\ 0 & c & 1 & d \end{bmatrix} \leftrightarrow$$



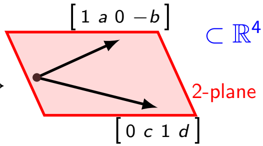
$\leftrightarrow$



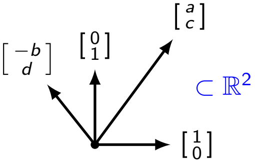
Grassmannian

$k \times n$ matrix C mod row operations $\leftrightarrow$ ~~k row vectors in $\mathbb{R}^n$~~ k-plane inside $\mathbb{R}^n$ $\leftrightarrow$ n column vectors in $\mathbb{R}^k$ mod $\mathrm{GL}_k(\mathbb{R})$-action

$$C = \begin{bmatrix} 1 & a & 0 & -b \\ 0 & c & 1 & d \end{bmatrix} \leftrightarrow$$



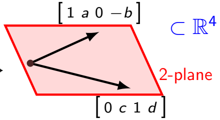
$\leftrightarrow$



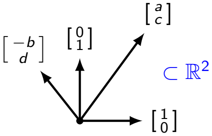
Grassmannian

$k \times n$ matrix C mod row operations	$\leftrightarrow$	~~k row vectors in $\mathbb{R}^n$~~ k-plane inside $\mathbb{R}^n$	$\leftrightarrow$	n column vectors in $\mathbb{R}^k$ mod $\mathrm{GL}_k(\mathbb{R})$-action

$$C = \begin{bmatrix} 1 & a & 0 & -b \\ 0 & c & 1 & d \end{bmatrix} \leftrightarrow$$



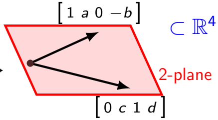
$\leftrightarrow$



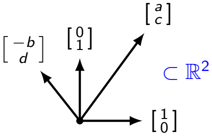
Grassmannian

$k \times n$ matrix C mod row operations $\leftrightarrow$ ~~k row vectors in $\mathbb{R}^n$~~ k-plane inside $\mathbb{R}^n$ $\leftrightarrow$ n column vectors in $\mathbb{R}^k$ mod $\mathrm{GL}_k(\mathbb{R})$-action $\leftrightarrow$ $\binom{n}{k}$ maximal minors mod rescaling

$$C = \begin{bmatrix} 1 & a & 0 & -b \\ 0 & c & 1 & d \end{bmatrix} \leftrightarrow$$



$\leftrightarrow$



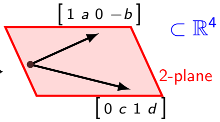
$\leftrightarrow$

$\Delta_{12} = c$
$\Delta_{13} = 1$
$\Delta_{14} = d$
$\Delta_{23} = a$
$\Delta_{24} = ad + bc$
$\Delta_{34} = b$

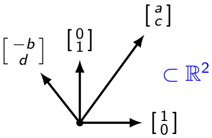
Grassmannian

$k \times n$ matrix C mod row operations	$\leftrightarrow$	~~k row vectors in $\mathbb{R}^n$~~ k-plane inside $\mathbb{R}^n$	$\leftrightarrow$	n column vectors in $\mathbb{R}^k$ mod $\mathrm{GL}_k(\mathbb{R})$-action	$\leftrightarrow$	$\binom{n}{k}$ maximal minors mod rescaling

$C = \begin{bmatrix} 1 & a & 0 & -b \\ 0 & c & 1 & d \end{bmatrix}$ $\leftrightarrow$



$\leftrightarrow$



$\leftrightarrow$

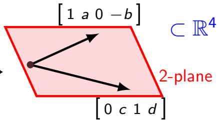
$\Delta_{12} = c$
$\Delta_{13} = 1$
$\Delta_{14} = d$
$\Delta_{23} = a$
$\Delta_{24} = ad + bc$
$\Delta_{34} = b$

$\mathrm{Gr}(k, n) := \{C \in \mathrm{Mat}(k, n)\}/(\text{row operations}) = \{k\text{-planes inside } \mathbb{R}^n\};$

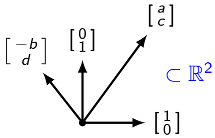
Positive Grassmannian

$k \times n$ matrix C mod row operations $\leftrightarrow$ ~~k row vectors in $\mathbb{R}^n$~~ k-plane inside $\mathbb{R}^n$ $\leftrightarrow$ n column vectors in $\mathbb{R}^k$ mod $\mathrm{GL}_k(\mathbb{R})$-action $\leftrightarrow \binom{n}{k}$ maximal minors mod rescaling

$C = \begin{bmatrix} 1 & a & 0 & -b \\ 0 & c & 1 & d \end{bmatrix}$ $\leftrightarrow$



$\leftrightarrow$



$\leftrightarrow$

$\Delta_{12} = c$
$\Delta_{13} = 1$
$\Delta_{14} = d$
$\Delta_{23} = a$
$\Delta_{24} = ad + bc$
$\Delta_{34} = b$

$$\mathrm{Gr}(k, n) := \{C \in \mathrm{Mat}(k, n)\}/(\text{row operations}) = \{k\text{-planes inside } \mathbb{R}^n\};$$

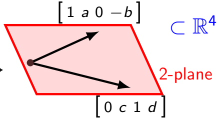
$$\mathrm{Gr}_{\geqslant 0}(k, n) := \{C \in \mathrm{Mat}(k, n) \mid \Delta_I(C) \geqslant 0 \text{ for all } I \text{ of size } k\}/(\text{row operations}).$$

[Lusztig '94], [Postnikov '06].

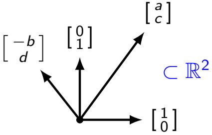
Positive Grassmannian

$k \times n$ matrix C mod row operations $\leftrightarrow$ ~~k row vectors in $\mathbb{R}^n$~~ k-plane inside $\mathbb{R}^n$ $\leftrightarrow$ n column vectors in $\mathbb{R}^k$ mod $\mathrm{GL}_k(\mathbb{R})$-action $\leftrightarrow$ $\binom{n}{k}$ maximal minors mod rescaling

$C = \begin{bmatrix} 1 & a & 0 & -b \\ 0 & c & 1 & d \end{bmatrix}$ $\leftrightarrow$



$\leftrightarrow$



$\leftrightarrow$ $\Delta_{12} = c$, $\Delta_{13} = 1$, $\Delta_{14} = d$, $\Delta_{23} = a$, $\Delta_{24} = ad + bc$, $\Delta_{34} = b$

$$\mathrm{Gr}(k, n) := \{C \in \mathrm{Mat}(k, n)\}/(\text{row operations}) = \{k\text{-planes inside } \mathbb{R}^n\};$$

$$\mathrm{Gr}_{\geqslant 0}(k, n) := \{C \in \mathrm{Mat}(k, n) \mid \Delta_I(C) \geqslant 0 \text{ for all } I \text{ of size } k\}/(\text{row operations}).$$

[Lusztig '94], [Postnikov '06].

Cyclic symmetry: $[C_1|C_2|\cdots|C_n] \mapsto [C_2|\cdots|C_n|(-1)^{k-1}C_1]$ preserves $\mathrm{Gr}_{\geqslant 0}(k, n)$.

Positive kinematic space

- Spinor-helicity formalism: Since $P_i = (P_i^{\mathcal{T}}, P_i^{\mathcal{O}}) \in \mathbb{C}^2$ with $|P_i^{\mathcal{T}}| = |P_i^{\mathcal{O}}|$, can choose $\lambda_i, \tilde{\lambda}_i \in \mathbb{C}$ such that $P_i^{\mathcal{T}} = \lambda_i \tilde{\lambda}_i$ and $P_i^{\mathcal{O}} = \bar{\lambda}_i \tilde{\lambda}_i$.

Positive kinematic space

- Spinor-helicity formalism: Since $P_i = (P_i^{\mathcal{T}}, P_i^{\mathcal{O}}) \in \mathbb{C}^2$ with $|P_i^{\mathcal{T}}| = |P_i^{\mathcal{O}}|$, can choose $\lambda_i, \tilde{\lambda}_i \in \mathbb{R}^2$ such that $P_i^{\mathcal{T}} = \lambda_i \tilde{\lambda}_i$ and $P_i^{\mathcal{O}} = \bar{\lambda}_i \tilde{\lambda}_i$.

Positive kinematic space

- Spinor-helicity formalism: Since $P_i = (P_i^{\mathcal{T}}, P_i^{\mathcal{O}}) \in \mathbb{C}^2$ with $|P_i^{\mathcal{T}}| = |P_i^{\mathcal{O}}|$, can choose $\lambda_i, \tilde{\lambda}_i \in \mathbb{R}^2$ such that $P_i^{\mathcal{T}} = \lambda_i \tilde{\lambda}_i$ and $P_i^{\mathcal{O}} = \bar{\lambda}_i \tilde{\lambda}_i$.
- 2-planes $\lambda, \tilde{\lambda} \in \mathrm{Gr}(2, n)$ with columns $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^2$, $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n \in \mathbb{R}^2$.

Positive kinematic space

- Spinor-helicity formalism: Since $P_i = (P_i^{\mathcal{T}}, P_i^{\mathcal{O}}) \in \mathbb{C}^2$ with $|P_i^{\mathcal{T}}| = |P_i^{\mathcal{O}}|$, can choose $\lambda_i, \tilde{\lambda}_i \in \mathbb{R}^2$ such that $P_i^{\mathcal{T}} = \lambda_i \tilde{\lambda}_i$ and $P_i^{\mathcal{O}} = \bar{\lambda}_i \tilde{\lambda}_i$.
- 2-planes $\lambda, \tilde{\lambda} \in \mathrm{Gr}(2, n)$ with columns $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^2$, $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n \in \mathbb{R}^2$.
- Momentum conservation $P_1 + \cdots + P_n = 0$ is equivalent to $\lambda \perp \tilde{\lambda}$.

Positive kinematic space

- Spinor-helicity formalism: Since $P_i = (P_i^{\mathcal{T}}, P_i^{\mathcal{O}}) \in \mathbb{C}^2$ with $|P_i^{\mathcal{T}}| = |P_i^{\mathcal{O}}|$, can choose $\lambda_i, \tilde{\lambda}_i \in \mathbb{R}^2$ such that $P_i^{\mathcal{T}} = \lambda_i \tilde{\lambda}_i$ and $P_i^{\mathcal{O}} = \bar{\lambda}_i \tilde{\lambda}_i$.
- 2-planes $\lambda, \tilde{\lambda} \in \mathrm{Gr}(2, n)$ with columns $\lambda_1, \dots, \lambda_n \in \mathbb{R}^2$, $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n \in \mathbb{R}^2$.
- Momentum conservation $P_1 + \cdots + P_n = 0$ is equivalent to $\lambda \perp \tilde{\lambda}$.
- Positive kinematic space: [He–Zhang '18]

$$\mathcal{K}_{k,n}^+ := \left\{ \lambda \perp \tilde{\lambda} \,\middle|\, \begin{array}{l} \langle i\, i+1 \rangle > 0,\ [i\, i+1] > 0 \text{ for } i = 1, \dots, n, \\ \mathrm{wind}(\lambda) = (k-1)\pi, \text{ and } \mathrm{wind}(\tilde{\lambda}) = (k+1)\pi \end{array} \right\}.$$

Positive kinematic space

- Spinor-helicity formalism: Since $P_i = (P_i^{\mathcal{T}}, P_i^{\mathcal{O}}) \in \mathbb{C}^2$ with $|P_i^{\mathcal{T}}| = |P_i^{\mathcal{O}}|$, can choose $\lambda_i, \tilde{\lambda}_i \in \mathbb{R}^2$ such that $P_i^{\mathcal{T}} = \lambda_i \tilde{\lambda}_i$ and $P_i^{\mathcal{O}} = \bar{\lambda}_i \tilde{\lambda}_i$.
- 2-planes $\lambda, \tilde{\lambda} \in \mathrm{Gr}(2, n)$ with columns $\lambda_1, \dots, \lambda_n \in \mathbb{R}^2$, $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n \in \mathbb{R}^2$.
- Momentum conservation $P_1 + \cdots + P_n = 0$ is equivalent to $\lambda \perp \tilde{\lambda}$.
- Positive kinematic space: [He–Zhang '18]

$$\mathcal{K}_{k,n}^{+} := \left\{ \lambda \perp \tilde{\lambda} \;\middle|\; \begin{array}{l} \langle i\, i+1 \rangle > 0,\ [i\, i+1] > 0 \text{ for } i = 1, \dots, n, \\ \mathrm{wind}(\lambda) = (k-1)\pi, \text{ and } \mathrm{wind}(\tilde{\lambda}) = (k+1)\pi \end{array} \right\}.$$

- Brackets: $\langle i\, j \rangle := \det(\lambda_i | \lambda_j)$ and $[i\, j] := \det(\tilde{\lambda}_i | \tilde{\lambda}_j)$.

Positive kinematic space

- Spinor-helicity formalism: Since $P_i = (P_i^{\mathcal{T}}, P_i^{\mathcal{O}}) \in \mathbb{C}^2$ with $|P_i^{\mathcal{T}}| = |P_i^{\mathcal{O}}|$, can choose $\lambda_i, \tilde{\lambda}_i \in \mathbb{R}^2$ such that $P_i^{\mathcal{T}} = \lambda_i \tilde{\lambda}_i$ and $P_i^{\mathcal{O}} = \bar{\lambda}_i \tilde{\lambda}_i$.
- 2-planes $\lambda, \tilde{\lambda} \in \mathrm{Gr}(2,n)$ with columns $\lambda_1, \dots, \lambda_n \in \mathbb{R}^2$, $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n \in \mathbb{R}^2$.
- Momentum conservation $P_1 + \cdots + P_n = 0$ is equivalent to $\lambda \perp \tilde{\lambda}$.
- Positive kinematic space: [He–Zhang '18]

$$\mathcal{K}_{k,n}^{+} := \left\{ \lambda \perp \tilde{\lambda} \;\middle|\; \begin{array}{l} \langle i\, i+1 \rangle > 0,\ [i\, i+1] > 0 \text{ for } i = 1, \dots, n, \\ \mathrm{wind}(\lambda) = (k-1)\pi, \text{ and } \mathrm{wind}(\tilde{\lambda}) = (k+1)\pi \end{array} \right\}.$$

- Brackets: $\langle i\, j \rangle := \det(\lambda_i | \lambda_j)$ and $[i\, j] := \det(\tilde{\lambda}_i | \tilde{\lambda}_j)$.

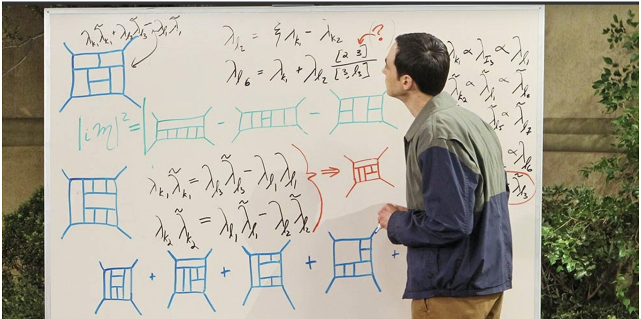
$$\left(\frac{[4|5+6|1\rangle^3}{[34][23]\langle 56\rangle\langle 61\rangle[2|3+4|5\rangle S_{234}} + \frac{[6|1+2|3\rangle^3}{[61][12]\langle 34\rangle\langle 45\rangle[2|3+4|5\rangle S_{612}} \right) = \left(\frac{(S_{123})^3}{[12][23]\langle 45\rangle\langle 56\rangle[1|2+3|4\rangle[3|4+5|6\rangle} + \frac{\langle 12\rangle^3[45]^3}{\langle 16\rangle[34][3|4+5|6\rangle[5|6+1|2\rangle S_{612}} + \frac{\langle 23\rangle^3[56]^3}{\langle 34\rangle[16][1|2+3|4\rangle[5|6+1|2\rangle S_{234}} \right)$$

Positive kinematic space

- Spinor-helicity formalism: Since $P_i = (P_i^{\mathcal{T}}, P_i^{\mathcal{O}}) \in \mathbb{C}^2$ with $|P_i^{\mathcal{T}}| = |P_i^{\mathcal{O}}|$, can choose $\lambda_i, \tilde{\lambda}_i \in \mathbb{R}^2$ such that $P_i^{\mathcal{T}} = \lambda_i \tilde{\lambda}_i$ and $P_i^{\mathcal{O}} = \bar{\lambda}_i \tilde{\lambda}_i$.
- 2-planes $\lambda, \tilde{\lambda} \in \mathrm{Gr}(2, n)$ with columns $\lambda_1, \dots, \lambda_n \in \mathbb{R}^2$, $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n \in \mathbb{R}^2$.
- Momentum conservation $P_1 + \cdots + P_n = 0$ is equivalent to $\lambda \perp \tilde{\lambda}$.
- Positive kinematic space: [He–Zhang '18]

$$\mathcal{K}_{k,n}^{+} := \left\{ \lambda \perp \tilde{\lambda} \;\middle|\; \begin{array}{l} \langle i\, i+1 \rangle > 0,\ [i\, i+1] > 0 \text{ for } i = 1, \dots, n, \\ \mathrm{wind}(\lambda) = (k-1)\pi, \text{ and } \mathrm{wind}(\tilde{\lambda}) = (k+1)\pi \end{array} \right\}.$$

- Brackets: $\langle i\, j \rangle := \det(\lambda_i | \lambda_j)$ and $[i\, j] := \det(\tilde{\lambda}_i | \tilde{\lambda}_j)$.



Positive kinematic space

- Spinor-helicity formalism: Since $P_i = (P_i^{\mathcal{T}}, P_i^{\mathcal{O}}) \in \mathbb{C}^2$ with $|P_i^{\mathcal{T}}| = |P_i^{\mathcal{O}}|$, can choose $\lambda_i, \tilde{\lambda}_i \in \mathbb{R}^2$ such that $P_i^{\mathcal{T}} = \lambda_i \tilde{\lambda}_i$ and $P_i^{\mathcal{O}} = \bar{\lambda}_i \tilde{\lambda}_i$.
- 2-planes $\lambda, \tilde{\lambda} \in \mathrm{Gr}(2, n)$ with columns $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^2$, $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n \in \mathbb{R}^2$.
- Momentum conservation $P_1 + \cdots + P_n = 0$ is equivalent to $\lambda \perp \tilde{\lambda}$.
- Positive kinematic space: [He–Zhang '18]

$$\mathcal{K}_{k,n}^{+} := \left\{ \lambda \perp \tilde{\lambda} \;\middle|\; \begin{array}{l} \langle i\, i+1 \rangle > 0,\ [i\, i+1] > 0 \text{ for } i = 1, \ldots, n, \\ \mathrm{wind}(\lambda) = (k-1)\pi, \text{ and } \mathrm{wind}(\tilde{\lambda}) = (k+1)\pi \end{array} \right\}.$$

- Brackets: $\langle i\, j \rangle := \det(\lambda_i | \lambda_j)$ and $[i\, j] := \det(\tilde{\lambda}_i | \tilde{\lambda}_j)$.

Positive kinematic space

- Spinor-helicity formalism: Since $P_i = (P_i^{\mathcal{T}}, P_i^{\mathcal{O}}) \in \mathbb{C}^2$ with $|P_i^{\mathcal{T}}| = |P_i^{\mathcal{O}}|$, can choose $\lambda_i, \tilde{\lambda}_i \in \mathbb{R}^2$ such that $P_i^{\mathcal{T}} = \lambda_i \tilde{\lambda}_i$ and $P_i^{\mathcal{O}} = \bar{\lambda}_i \tilde{\lambda}_i$.
- 2-planes $\lambda, \tilde{\lambda} \in \mathrm{Gr}(2, n)$ with columns $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^2$, $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n \in \mathbb{R}^2$.
- Momentum conservation $P_1 + \cdots + P_n = 0$ is equivalent to $\lambda \perp \tilde{\lambda}$.
- Positive kinematic space: [He–Zhang '18]

$$\mathcal{K}_{k,n}^{+} := \left\{ \lambda \perp \tilde{\lambda} \;\middle|\; \begin{array}{l} \langle i\, i+1 \rangle > 0,\ [i\, i+1] > 0 \text{ for } i = 1, \ldots, n, \\ \mathrm{wind}(\lambda) = (k-1)\pi, \text{ and } \mathrm{wind}(\tilde{\lambda}) = (k+1)\pi \end{array} \right\}.$$

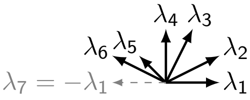
- Brackets: $\langle i\, j \rangle := \det(\lambda_i | \lambda_j)$ and $[i\, j] := \det(\tilde{\lambda}_i | \tilde{\lambda}_j)$.
- wind(λ): total turning angle of $\lambda_1 \to \lambda_2 \to \cdots \to \lambda_{n+1} := (-1)^{k-1} \lambda_1$.

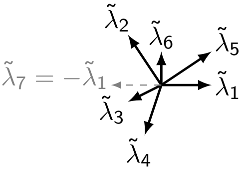
Positive kinematic space

- Spinor-helicity formalism: Since $P_i = (P_i^{\mathcal{T}}, P_i^{\mathcal{O}}) \in \mathbb{C}^2$ with $|P_i^{\mathcal{T}}| = |P_i^{\mathcal{O}}|$, can choose $\lambda_i, \tilde{\lambda}_i \in \mathbb{R}^2$ such that $P_i^{\mathcal{T}} = \lambda_i \tilde{\lambda}_i$ and $P_i^{\mathcal{O}} = \bar{\lambda}_i \tilde{\lambda}_i$.
- 2-planes $\lambda, \tilde{\lambda} \in \mathrm{Gr}(2,n)$ with columns $\lambda_1, \dots, \lambda_n \in \mathbb{R}^2$, $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n \in \mathbb{R}^2$.
- Momentum conservation $P_1 + \cdots + P_n = 0$ is equivalent to $\lambda \perp \tilde{\lambda}$.
- Positive kinematic space: [He–Zhang '18]

$$\mathcal{K}_{k,n}^{+} := \left\{ \lambda \perp \tilde{\lambda} \;\middle|\; \begin{array}{l} \langle i\, i+1 \rangle > 0,\ [i\, i+1] > 0 \text{ for } i = 1, \dots, n, \\ \mathrm{wind}(\lambda) = (k-1)\pi, \text{ and } \mathrm{wind}(\tilde{\lambda}) = (k+1)\pi \end{array} \right\}.$$

- Brackets: $\langle i\, j \rangle := \det(\lambda_i | \lambda_j)$ and $[i\, j] := \det(\tilde{\lambda}_i | \tilde{\lambda}_j)$.
- wind(λ): total turning angle of $\lambda_1 \to \lambda_2 \to \cdots \to \lambda_{n+1} := (-1)^{k-1}\lambda_1$.





wind(λ) $= \pi$

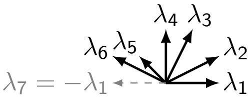
wind($\tilde{\lambda}$) $= 3\pi$

Positive kinematic space

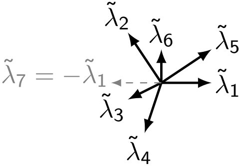
- Spinor-helicity formalism: Since $P_i = (P_i^{\mathcal{T}}, P_i^{\mathcal{O}}) \in \mathbb{C}^2$ with $|P_i^{\mathcal{T}}| = |P_i^{\mathcal{O}}|$, can choose $\lambda_i, \tilde{\lambda}_i \in \mathbb{R}^2$ such that $P_i^{\mathcal{T}} = \lambda_i \tilde{\lambda}_i$ and $P_i^{\mathcal{O}} = \bar{\lambda}_i \tilde{\lambda}_i$.
- 2-planes $\lambda, \tilde{\lambda} \in \mathrm{Gr}(2,n)$ with columns $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^2$, $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n \in \mathbb{R}^2$.
- Momentum conservation $P_1 + \cdots + P_n = 0$ is equivalent to $\lambda \perp \tilde{\lambda}$.
- Positive kinematic space: [He–Zhang '18]

$$\mathcal{K}_{k,n}^{+} := \left\{ \lambda \perp \tilde{\lambda} \;\middle|\; \begin{array}{l} \langle i\, i+1 \rangle > 0,\ [i\, i+1] > 0 \text{ for } i = 1, \ldots, n, \\ \mathrm{wind}(\lambda) = (k-1)\pi, \text{ and } \mathrm{wind}(\tilde{\lambda}) = (k+1)\pi \end{array} \right\}.$$

- Brackets: $\langle i\, j \rangle := \det(\lambda_i | \lambda_j)$ and $[i\, j] := \det(\tilde{\lambda}_i | \tilde{\lambda}_j)$.
- $\mathrm{wind}(\lambda)$: total turning angle of $\lambda_1 \to \lambda_2 \to \cdots \to \lambda_{n+1} := (-1)^{k-1}\lambda_1$.



$\mathrm{wind}(\lambda) = \pi$



$\mathrm{wind}(\tilde{\lambda}) = 3\pi$

$\in \mathcal{K}^{+}_{k=2,n=6}$

Main bijection

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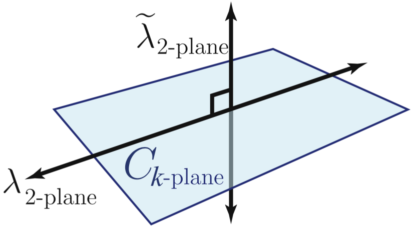
- $(\lambda, \tilde{\lambda})$ determine the (4-dimensional) boundary of the origami crease pattern.
- $\mathcal{A}(P_1, \dots, P_n) = \text{integral over } \{C \in \mathrm{Gr}_{\geqslant 0}(k,n) \mid \lambda \subset C \subset \tilde{\lambda}^{\perp}\}$ [ABCGPT '16].

As we saw in section 7, this can also be written as a residue of the top-form,

$$f_\sigma^{(k)} = \oint_{C\subset\Gamma_\sigma} \frac{d^{k\times n}C}{\text{vol}(GL(k))}\, \frac{\delta^{k\times 4}(C\cdot\widetilde{\eta})}{(1\cdots k)\cdots(n\cdots k{-}1)}\, \delta^{k\times 2}(C\cdot\widetilde{\lambda})\,\delta^{2\times(n-k)}(\lambda\cdot C^{\perp}). \quad (8.2)$$

Recall from section 4, the (ordinary) δ-functions in (8.2) have the geometric interpretation of constraining the k-plane C to be *orthogonal to* the 2-plane $\widetilde{\lambda}$ and to *contain* the 2-plane λ, [14]:



(8.3)

[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka. *Grassmannian Geometry of Scattering Amplitudes*. Cambridge University Press, Cambridge, 2016.]

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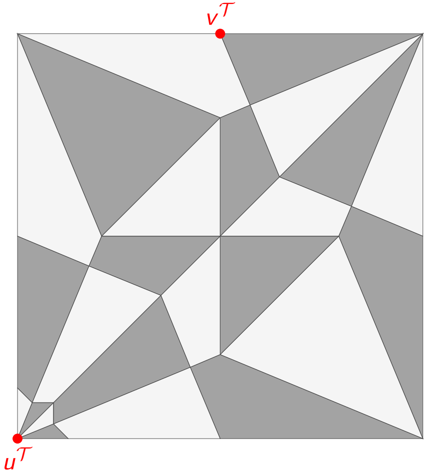
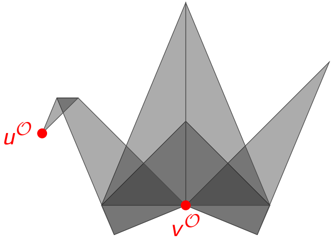
- $\mathcal{A}(P_1,\dots,P_n) =$ integral over $\{C \in \mathrm{Gr}_{\geqslant 0}(k,n) \mid \lambda \subset C \subset \tilde\lambda^\perp\}$ [ABCGPT '16].
- Corollary: BCFW cells triangulate (Mandelstam-positive region of) $\mathcal{K}^+_{k,n}$.

- Let $u^{\mathcal{T}}, v^{\mathcal{T}}$ be two vertices of an origami crease pattern and $u^{\mathcal{O}}, v^{\mathcal{O}}$ be their images under the origami map.





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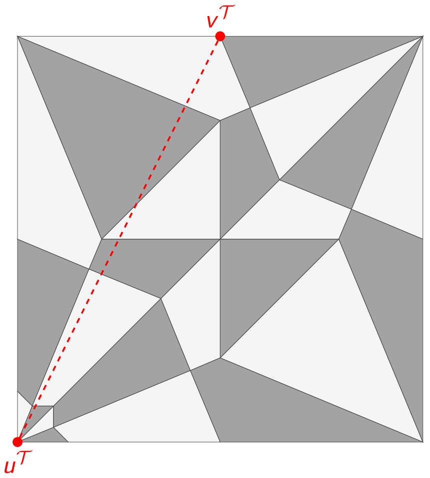

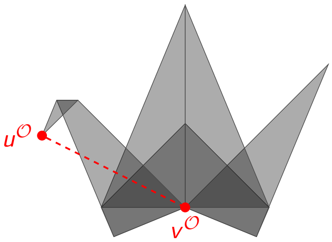

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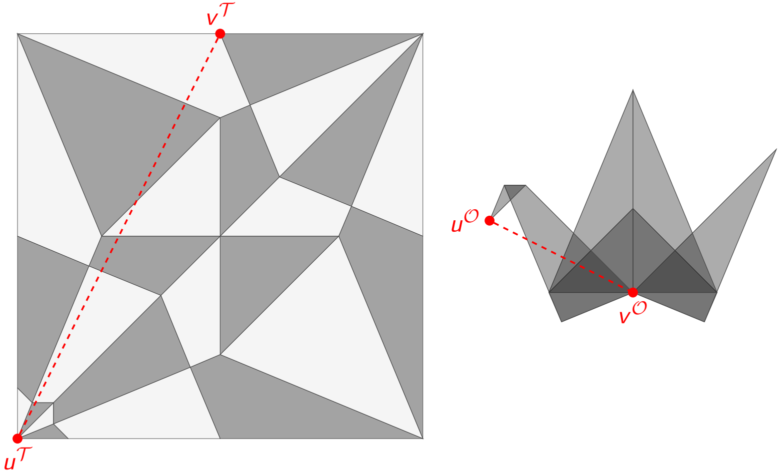

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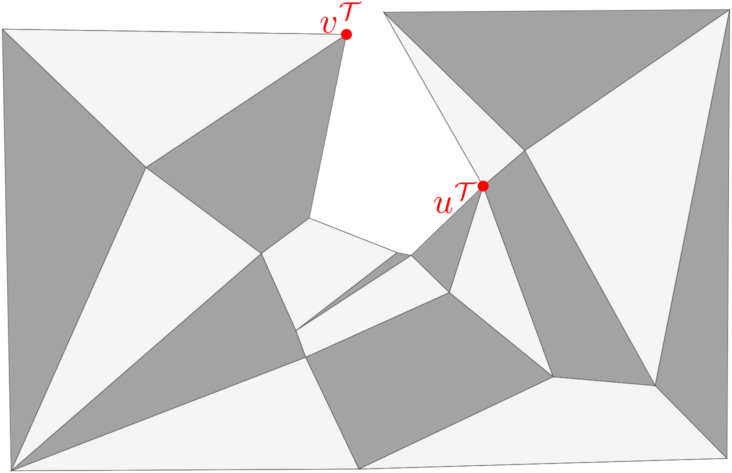

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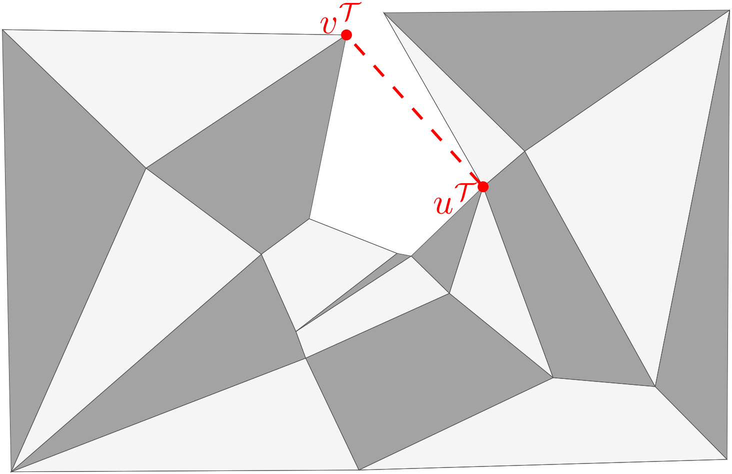

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$$\left(\frac{[4|5{+}6|1\rangle^3}{[34][23]\langle 56\rangle\langle 61\rangle[2|3{+}4|5\rangle S_{234}} + \frac{[6|1{+}2|3\rangle^3}{[61][12]\langle 34\rangle\langle 45\rangle[2|3{+}4|5\rangle S_{612}} \right) = \left(\frac{(S_{123})^3}{[12][23]\langle 45\rangle\langle 56\rangle[1|2{+}3|4\rangle[3|4{+}5|6\rangle} + \frac{\langle 12\rangle^3[45]^3}{\langle 16\rangle[34][3|4{+}5|6\rangle[5|6{+}1|2\rangle S_{612}} + \frac{\langle 23\rangle^3[56]^3}{\langle 34\rangle[16][1|2{+}3|4\rangle[5|6{+}1|2\rangle S_{234}} \right)$$

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Theorem (G. (2024))

BCFW cells triangulate $\mathcal{K}_{k,n}^{\mathrm{M}^+}$.

See also: [Arkani-Hamed–Trnka '14], [Even-Zohar–Lakrec–Tessler '21], [Even-Zohar–Lakrec–Parisi–Tessler–Sherman-Bennett–Williams '23].

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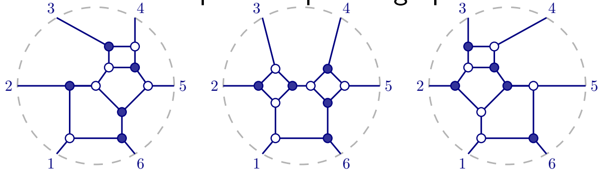
$\mathbf{\Gamma}_{3,6}^{\mathrm{BCFW}} =$



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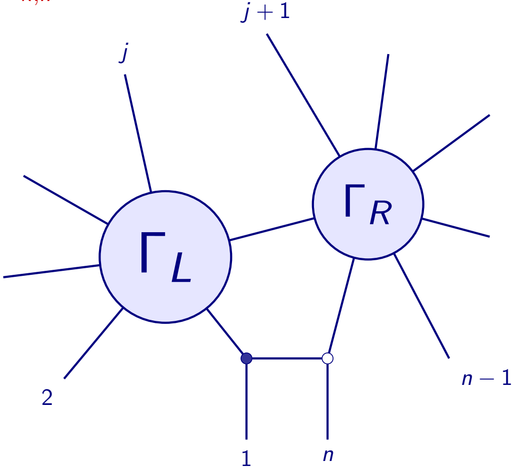

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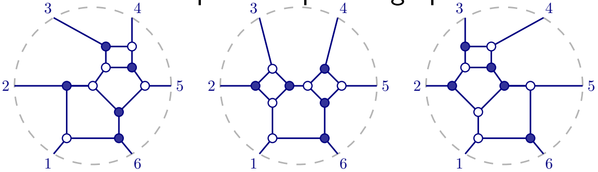
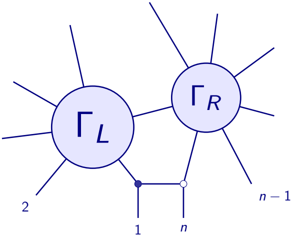
$\boldsymbol{\Gamma}_{3,6}^{\mathrm{BCFW}} =$





$\mathcal{K}^{\mathrm{M}^+}_{k,n} := \{(\lambda, \tilde{\lambda}) \in \mathcal{K}^+_{k,n} \mid S_{(i,j]} > 0 \text{ for all } i, j\}.$

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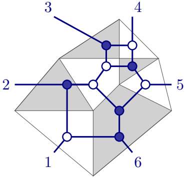
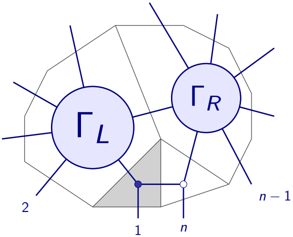
$\mathbf{\Gamma}^{\mathrm{BCFW}}_{3,6} =$ 



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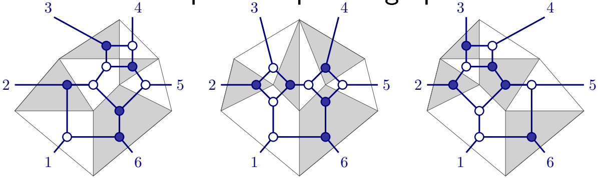
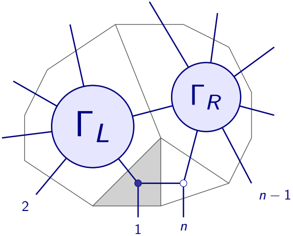


BCFW cell $\mathcal{K}^{\mathrm{M}+}_{\Gamma} := \left\{ (\lambda, \tilde{\lambda}) \in \mathcal{K}^{\mathrm{M}+}_{k,n} \;\middle|\; \begin{array}{l} \text{there exists an origami crease pattern} \\ \text{with boundary } (\lambda, \tilde{\lambda}) \text{ and planar dual } \Gamma \end{array} \right\}$.



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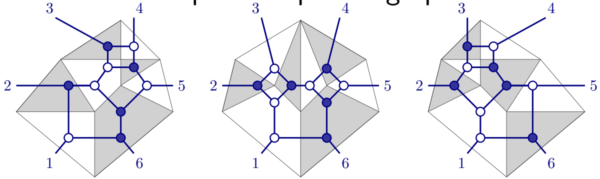

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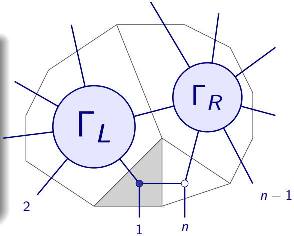

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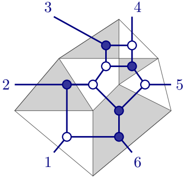

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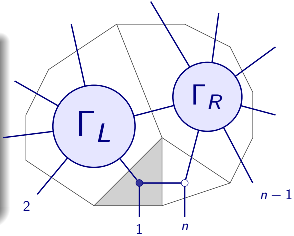
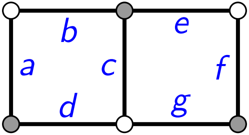
Theorem (G. (2024))

1. $\mathcal{K}^{\mathrm{M}^+}_{k,n} = \bigsqcup_{\Gamma \in \mathbf{\Gamma}^{\mathrm{BCFW}}_{k,n}} \mathcal{K}^{\mathrm{M}^+}_{\Gamma}$.
2. *Each $(\lambda, \tilde{\lambda}) \in \mathcal{K}^{\mathrm{M}^+}_{\Gamma}$ is the boundary of a unique origami crease pattern planar dual to Γ.*



Dimer model

- Dimer model: study of perfect matchings on weighted bipartite graphs



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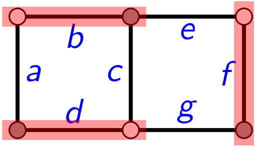

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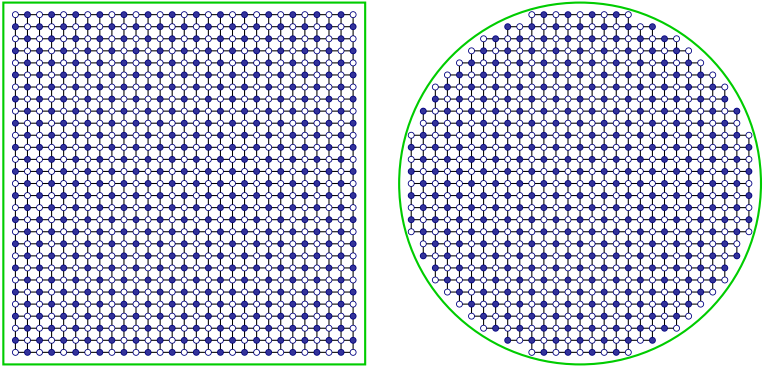

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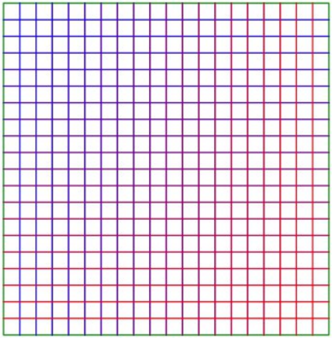

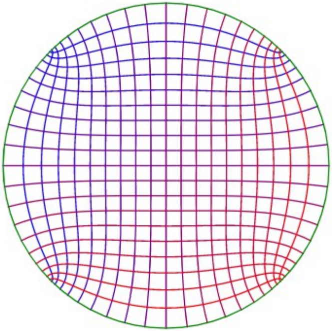

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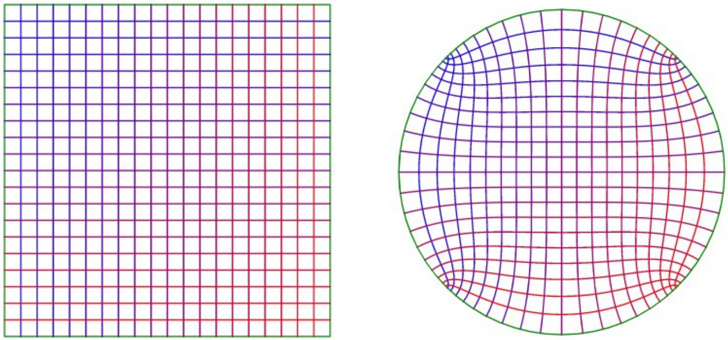

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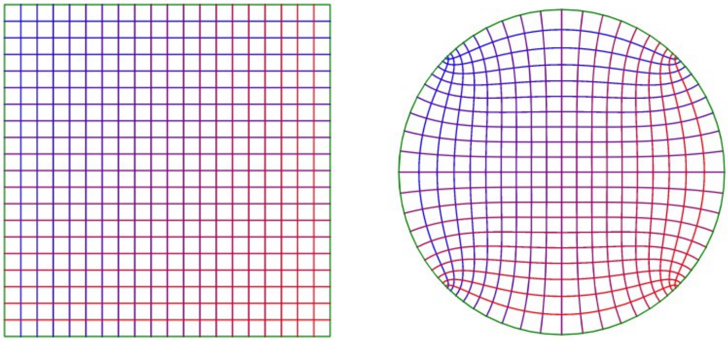

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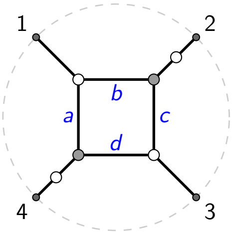

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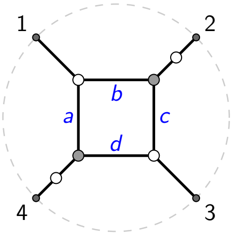

$$C = \begin{pmatrix} 1 & b & 0 & -a \\ 0 & c & 1 & d \end{pmatrix}$$

$$\Delta_{12}(C) = c \quad \Delta_{23}(C) = b \quad \Delta_{34}(C) = a \quad \Delta_{14}(C) = d$$

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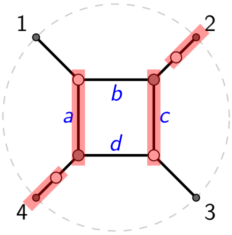

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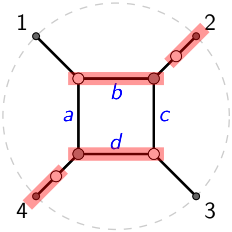

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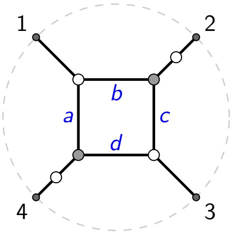

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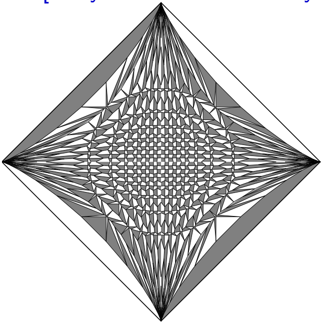

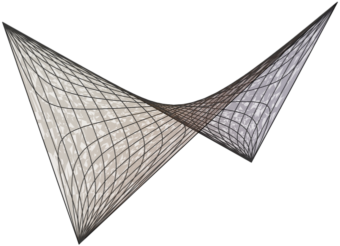

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Conjecture ([KLRR '18], [CLR '21])

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Theorem (G. (2024))

Part 1 is true, part 2 is false.

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Theorem (G. (2024), "Main bijection")

Origami crease patterns are in natural bijection with triples $\lambda \subset C \subset \tilde{\lambda}^{\perp}$ such that $(\lambda, \tilde{\lambda}) \in \mathcal{K}^{+}_{k,n}$ and $C \in \mathrm{Gr}_{\geqslant 0}(k, n)$.*

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Theorem (G. (2024), "Main bijection")

Origami crease patterns are in natural bijection with triples $\lambda \subset C \subset \tilde{\lambda}^{\perp}$ such that $(\lambda, \tilde{\lambda}) \in \mathcal{K}^{+}_{k,n}$ and $C \in \mathrm{Gr}_{\geqslant 0}(k, n)$.*

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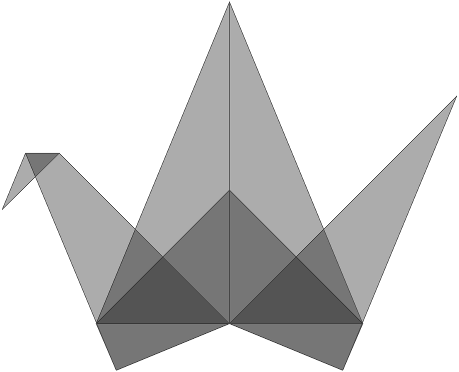
    - Hard direction: if $(\lambda, \tilde{\lambda}) \in \mathcal{K}^{+}_{k,n}$ then $\int \lambda^{\circ}\tilde{\lambda}^{\bullet}\, dz$ gives a valid (embedded) origami crease pattern.



Thanks!