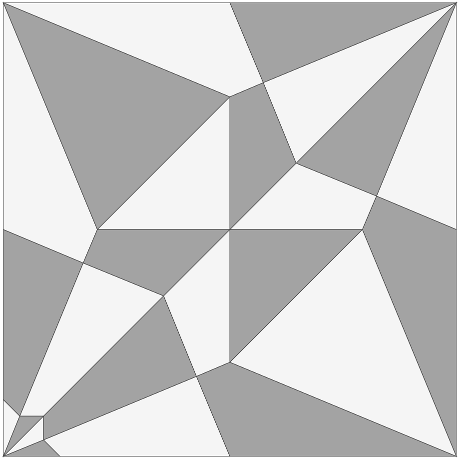
Amplituhedra and origami

Pavel Galashin (UCLA/Cornell)

The Amplituhedron: Structure, Combinatorics, and Positive Geometry

July 3, 2025

Origami crease patterns (OCP)



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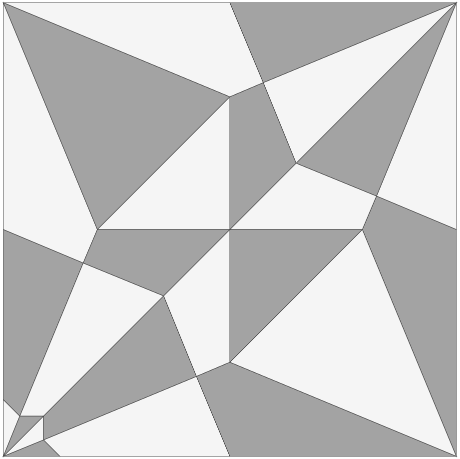

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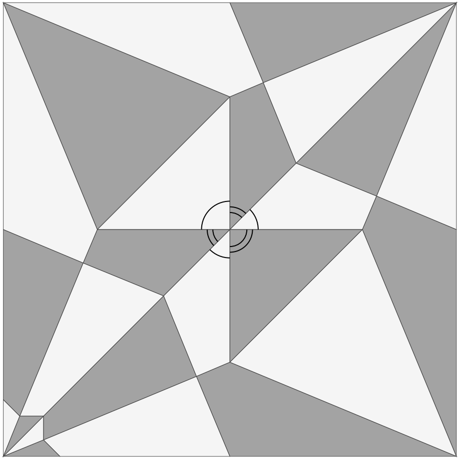

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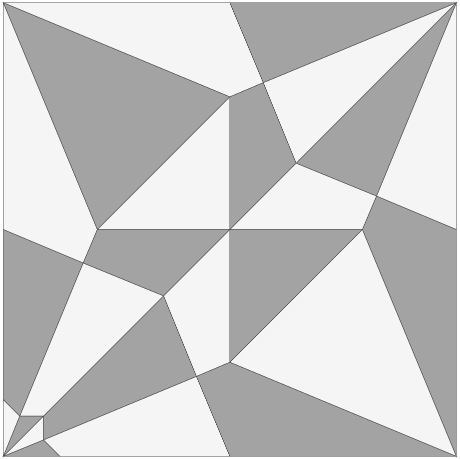

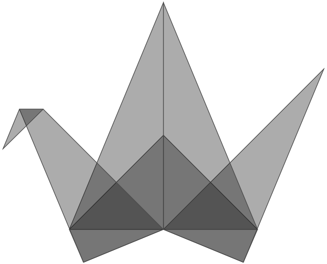

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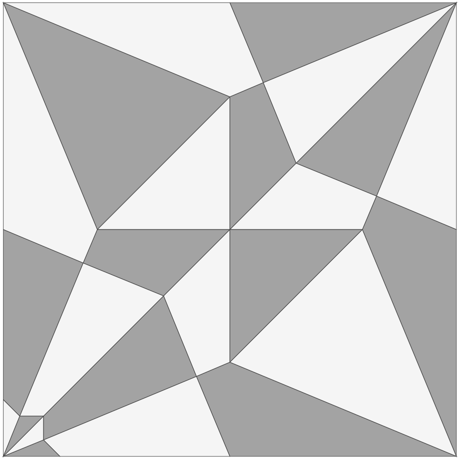

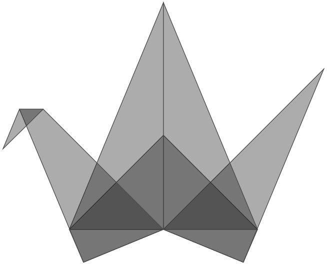

(restrict to flat-foldable OCPs.)

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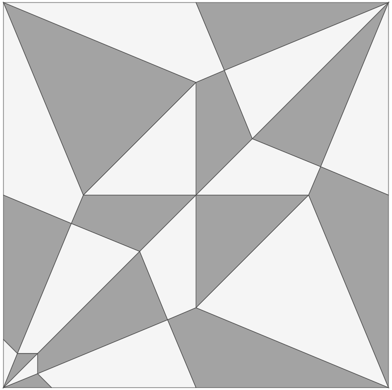
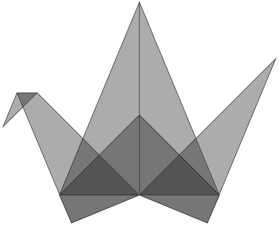
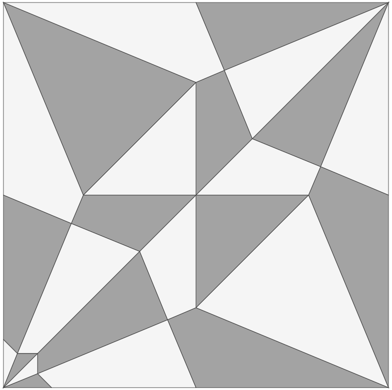
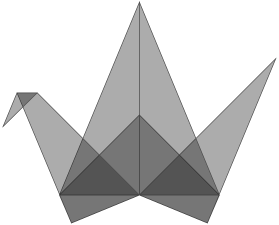




- OCP + its origami folding = 2-dimensional discrete PL surface in $\mathbb{R}^{2,2}$.





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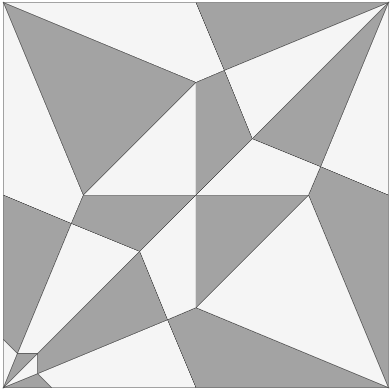

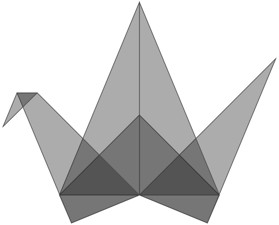

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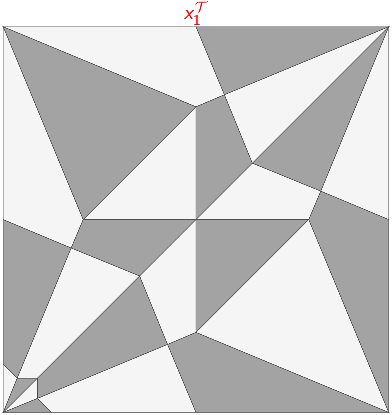

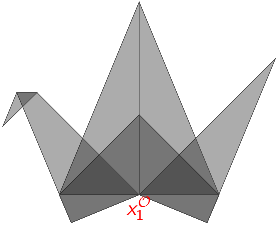

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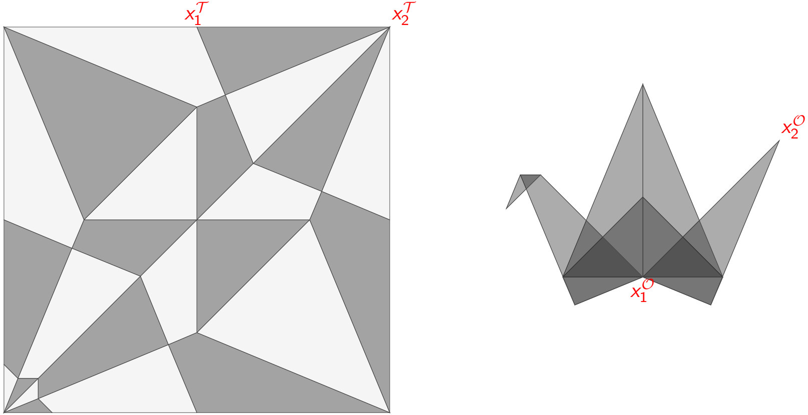

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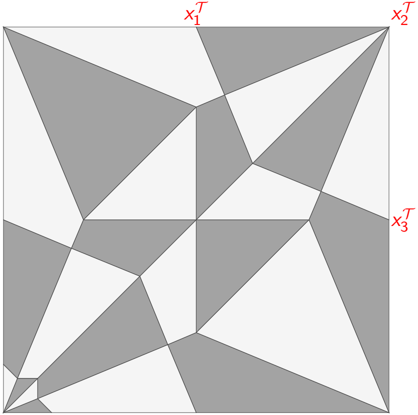

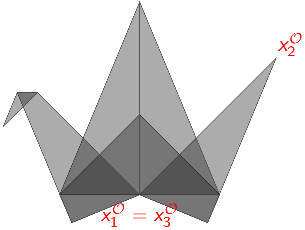

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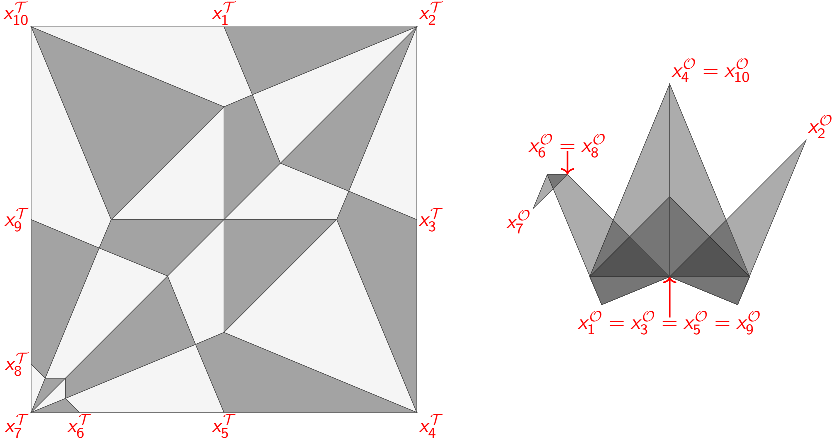

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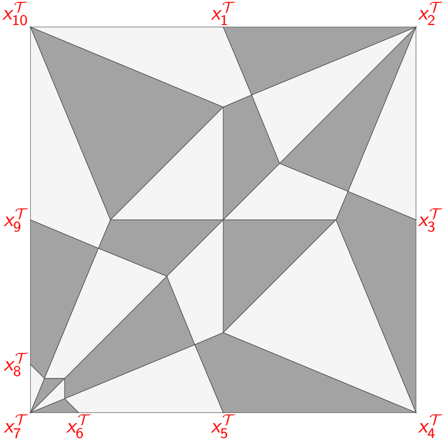

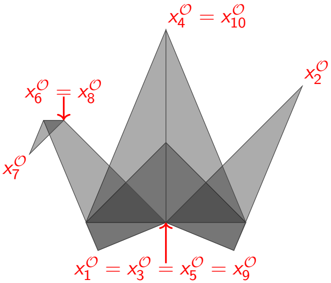

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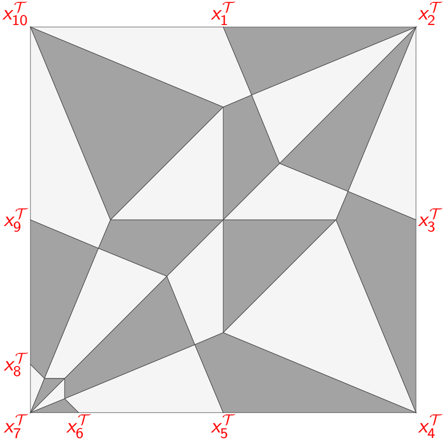
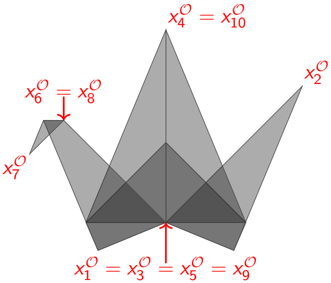
- Alternatively, $\mathbb{R}^{2,2} \cong \mathrm{Mat}_{2\times 2}(\mathbb{R})$. For $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$, set $x^2 = \det(x)$.





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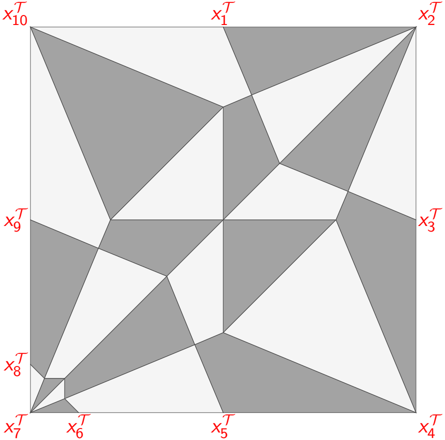

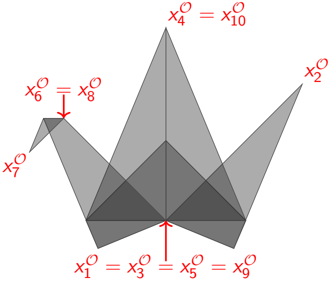

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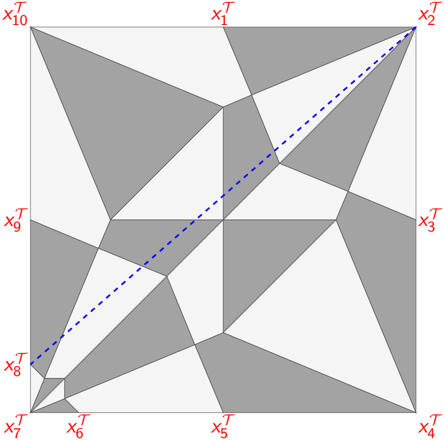
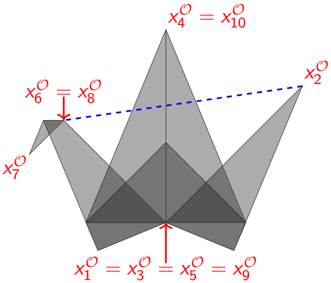
- Alternatively, $\mathcal{P} = \mathcal{P}_{\lambda,\tilde{\lambda}}$, where $\lambda \perp \tilde{\lambda} \in \mathrm{Gr}(2,n)$ and $P_i := x_i - x_{i-1} = \lambda_i \cdot \tilde{\lambda}_i^T$.





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- Mandelstam-positivity: $(x_i - x_j)^2 > 0 \iff |x_i^{\mathcal{T}} - x_j^{\mathcal{T}}| > |x_i^{\mathcal{O}} - x_j^{\mathcal{O}}|$.

Definition

Let $L \geqslant 0$. An *L-punctured polygon* is a pair $(\mathcal{P}, \vec{\mathbf{y}})$, where $\mathcal{P} = (x_1, x_2, \ldots, x_n) \in (\mathbb{R}^{2,2})^n$ and $\vec{\mathbf{y}} = (y_1, y_2, \ldots, y_L) \in (\mathbb{R}^{2,2})^L$, such that

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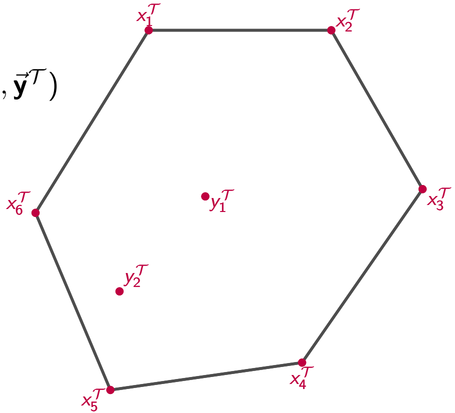
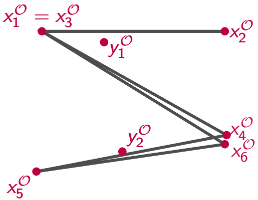
$$(x_i - x_{i-1})^2 = 0, \quad (x_i - x_j)^2 > 0, \quad (x_i - y_\rho)^2 > 0, \quad (y_\rho - y_\gamma)^2 > 0, \quad \text{and}$$

$(\mathcal{P}^{\mathcal{T}}, \vec{\mathbf{y}}^{\mathcal{T}})$





$(\mathcal{P}^{\mathcal{O}}, \vec{\mathbf{y}}^{\mathcal{O}})$

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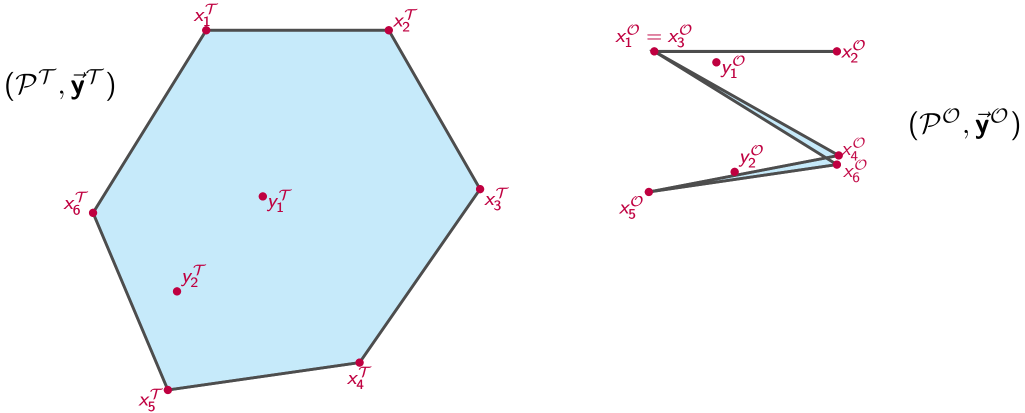
each $y_\rho^{\mathcal{T}}$ is located inside the polygon $\mathcal{P}^{\mathcal{T}} = (x_1^{\mathcal{T}}, x_2^{\mathcal{T}}, \ldots, x_n^{\mathcal{T}})$.



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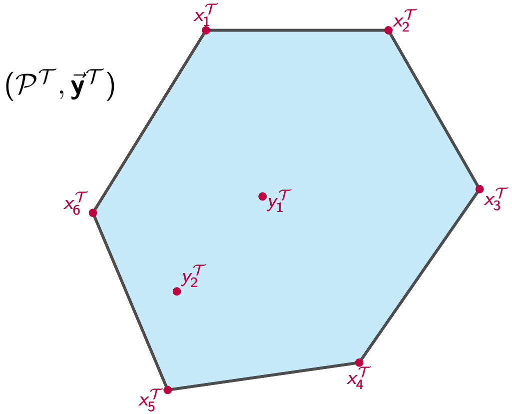
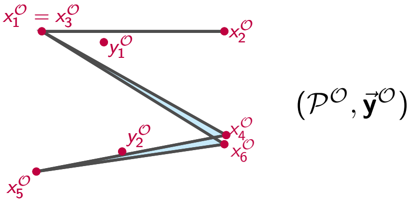
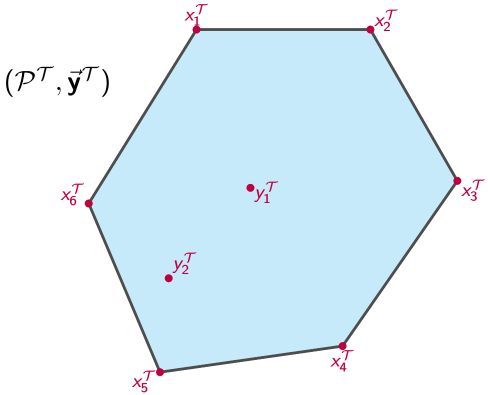
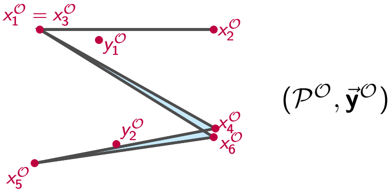




Problem: Find an OCP with boundary $\mathcal{P}^{\mathcal{T}}$ such that the origami map sends $x_i^{\mathcal{T}} \mapsto x_i^{\mathcal{O}}$ and $y_\rho^{\mathcal{T}} \mapsto y_\rho^{\mathcal{O}}$ for all i, ρ.





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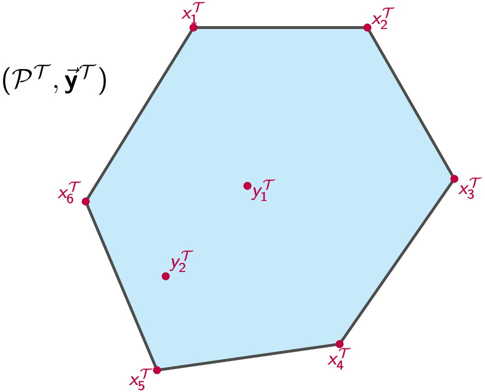

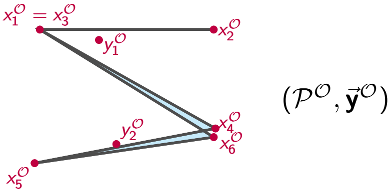

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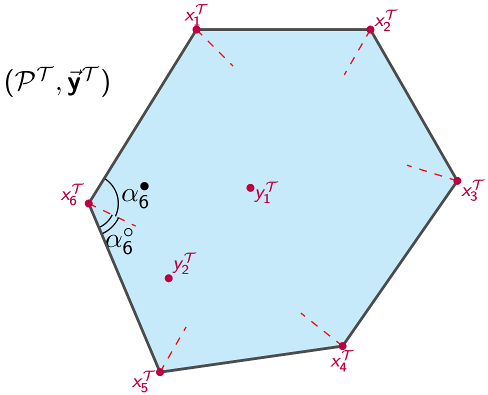
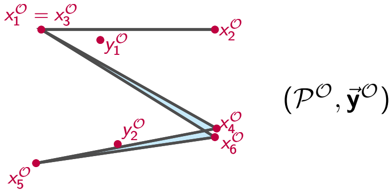
- Recover white/black angle sums $(\alpha_i^\circ, \alpha_i^\bullet)$ from the geometry of $\mathcal{P}$:

$$\alpha_i^\circ + \alpha_i^\bullet = \alpha_i^{\mathcal{T}} \quad \text{and} \quad \alpha_i^\circ - \alpha_i^\bullet \equiv \alpha_i^{\mathcal{O}} \text{ mod } 2\pi.$$





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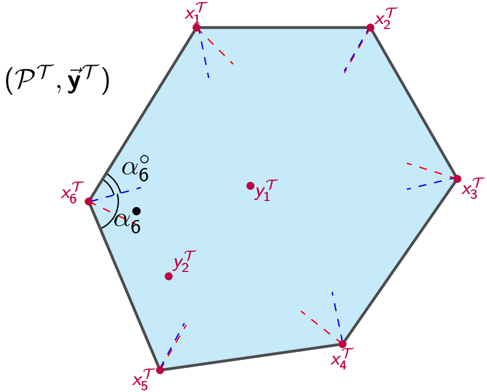
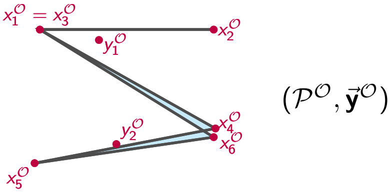
- Red folding ray splits the angle $\alpha_i^{\mathcal{T}}$ into angles $(\alpha_i^{\bullet}, \alpha_i^{\circ})$.





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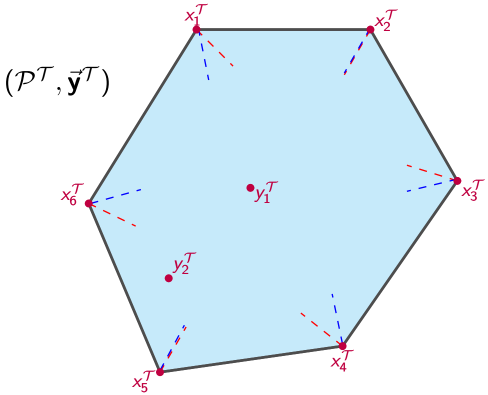
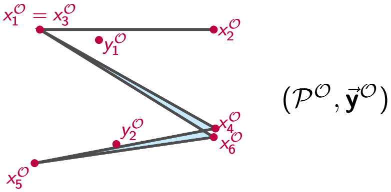
- Red folding ray splits the angle $\alpha_i^{\mathcal{T}}$ into angles $(\alpha_i^\bullet, \alpha_i^\circ)$.
- Blue folding ray splits the angle $\alpha_i^{\mathcal{T}}$ into angles $(\alpha_i^\circ, \alpha_i^\bullet)$.





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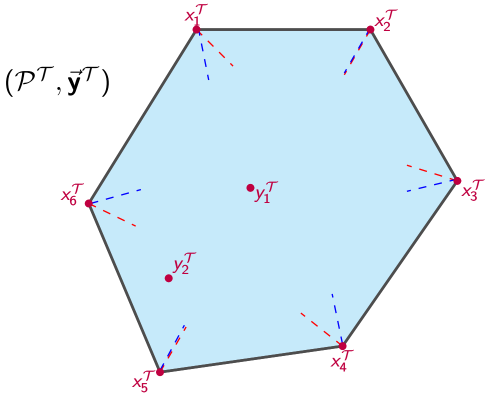

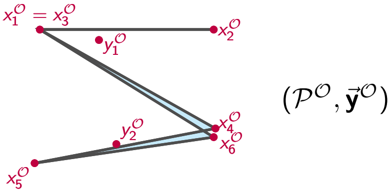

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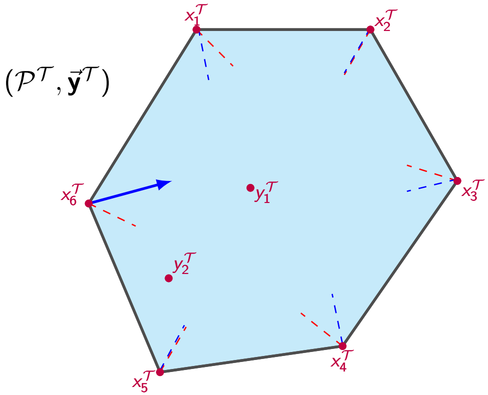

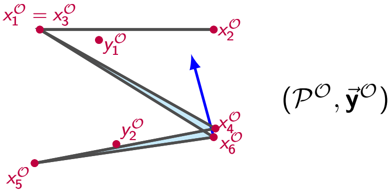

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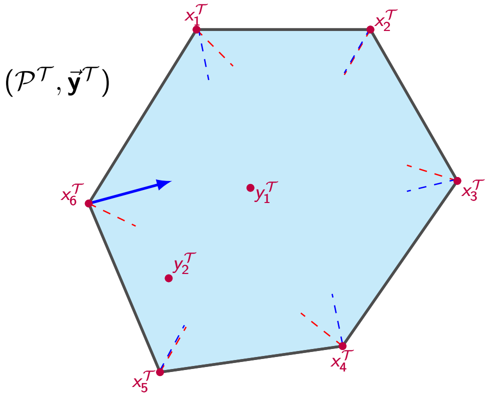

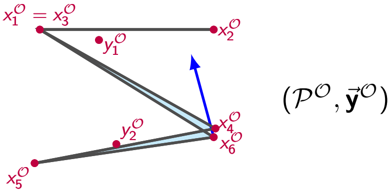

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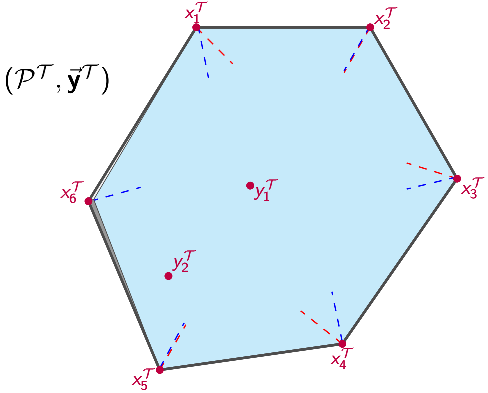

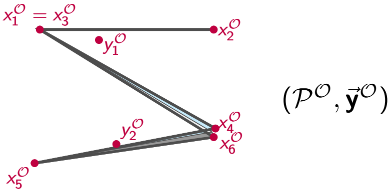

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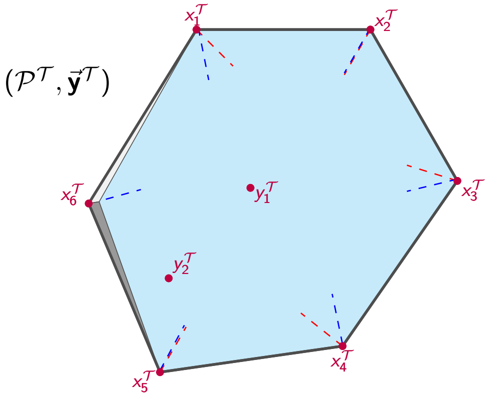

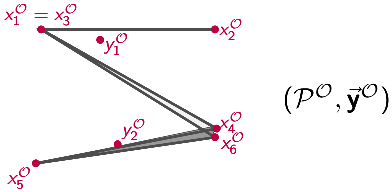

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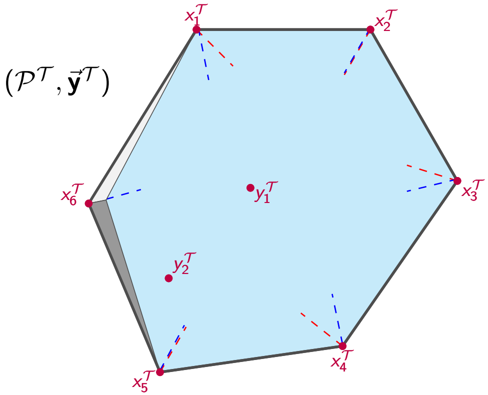

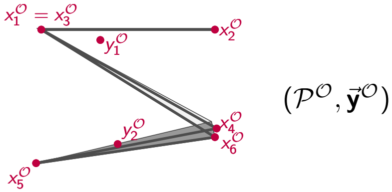

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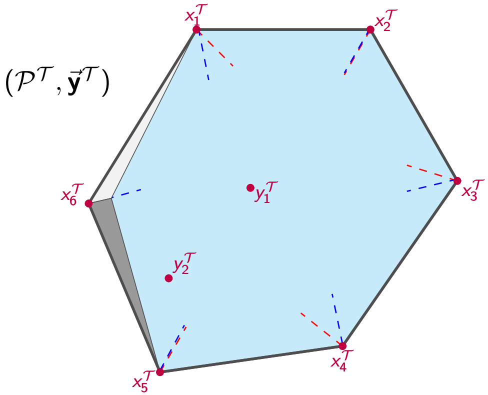

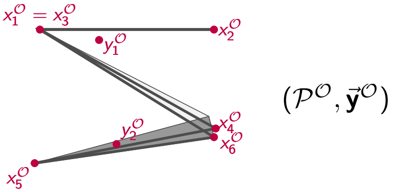

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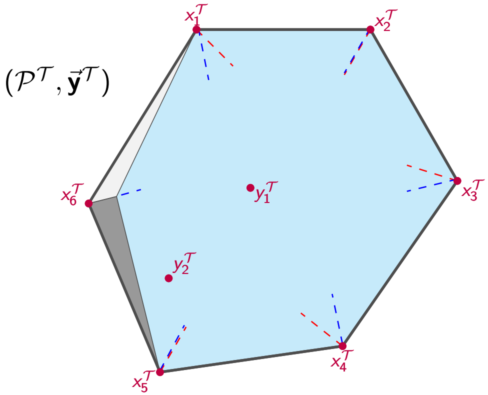

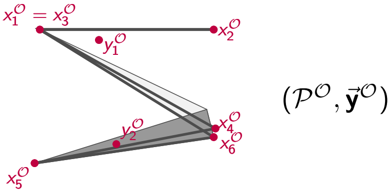

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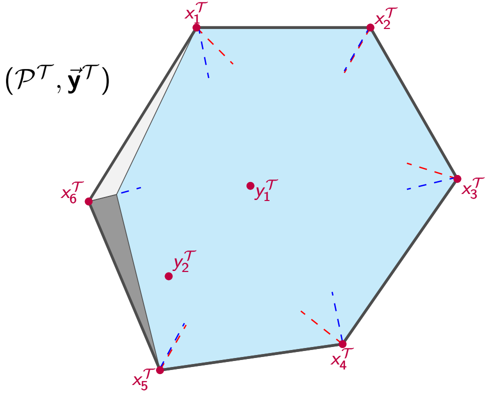

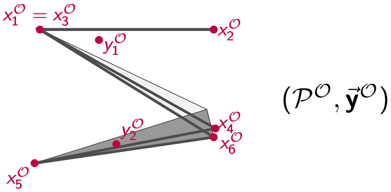

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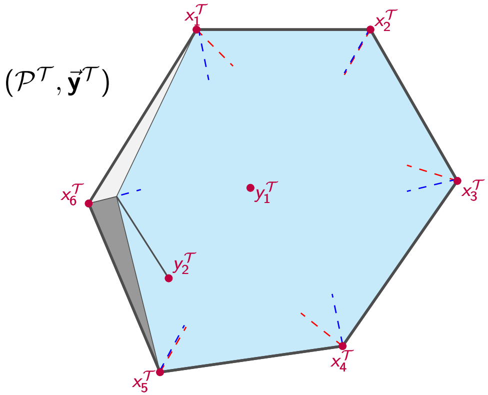

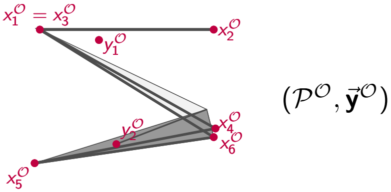

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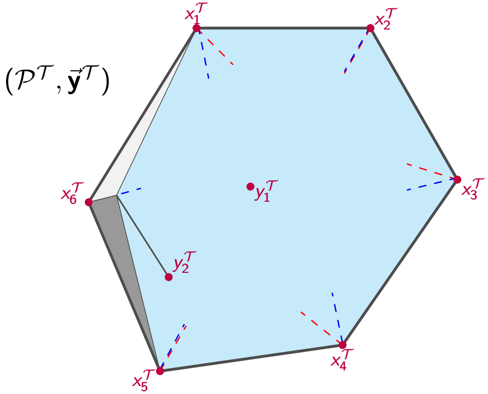

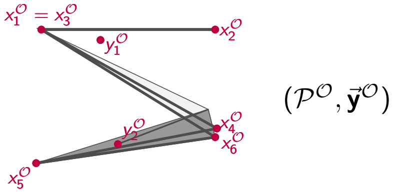

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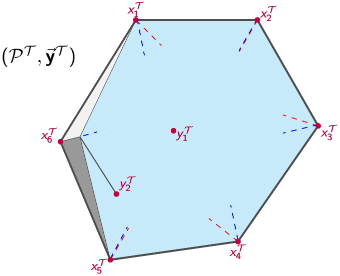

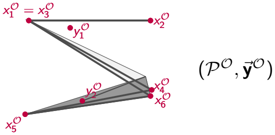

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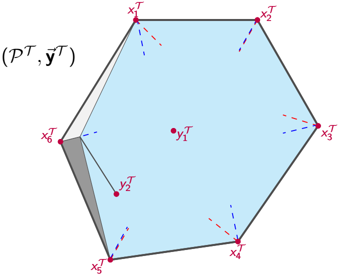

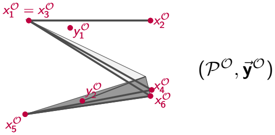

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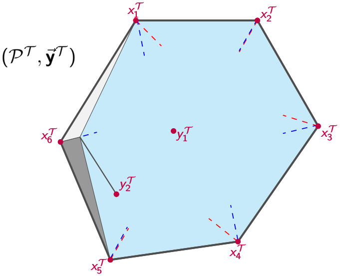
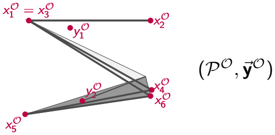
Theorem (G.'25+)

The crystallization algorithm works.





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The crystallization algorithm works.
For generic L-punctured polygons $(\mathcal{P}, \vec{\mathbf{y}})$, it outputs a valid, embedded OCP.

Conjecture (BCFW triangulation conjecture [AHT'14])

The BCFW cells triangulate the amplituhedron.

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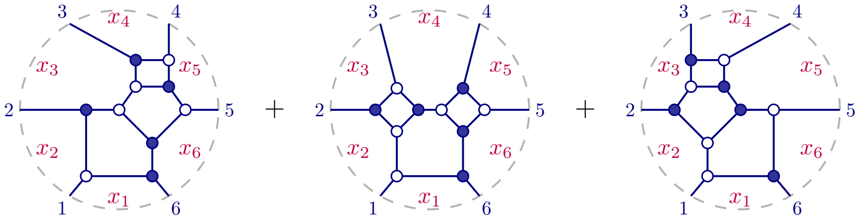

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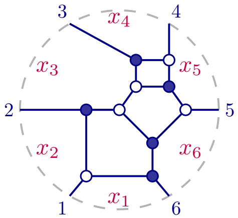

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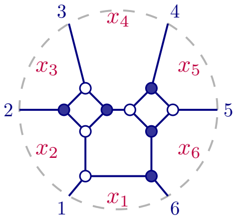

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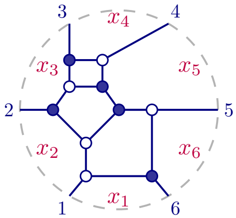

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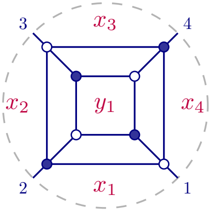

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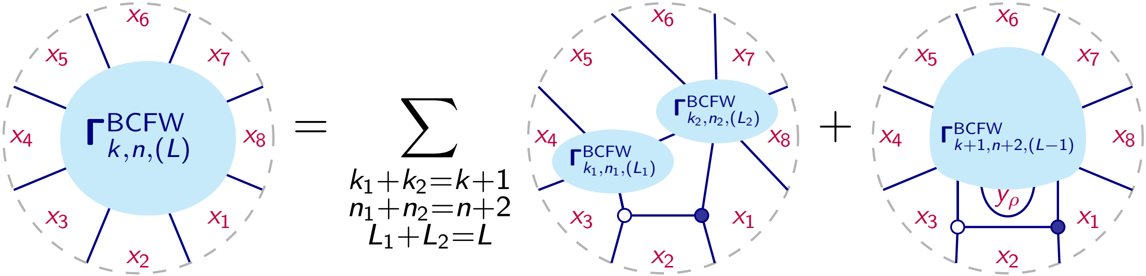

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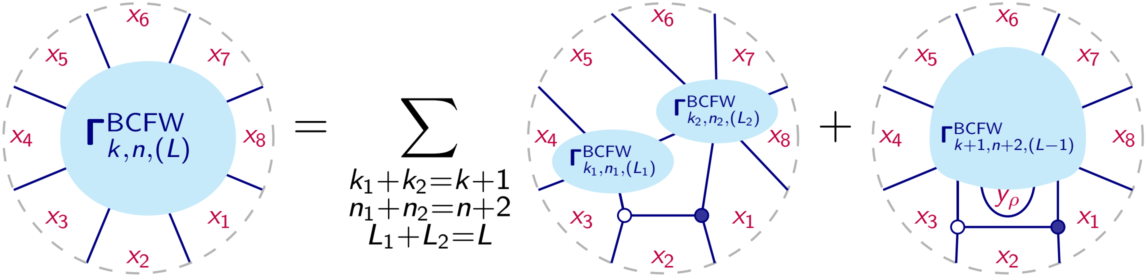

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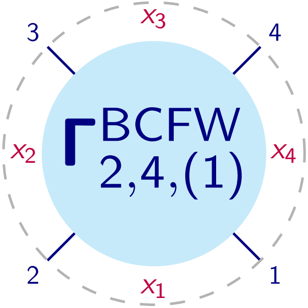

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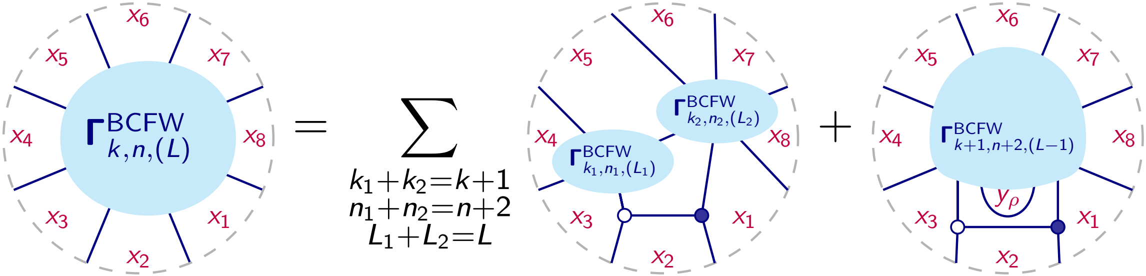

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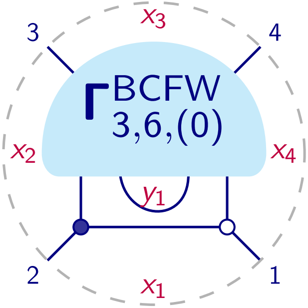

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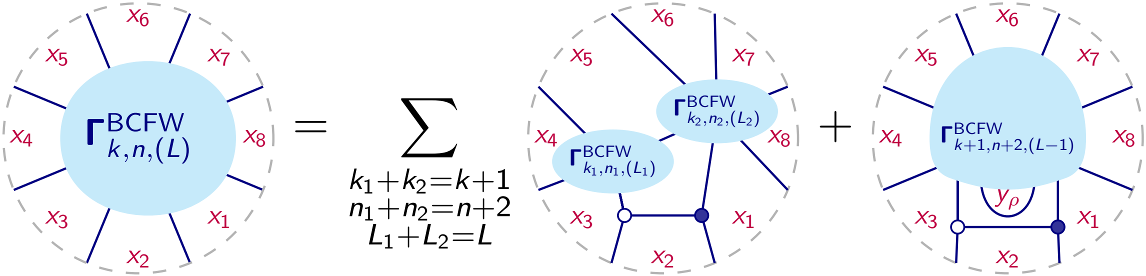

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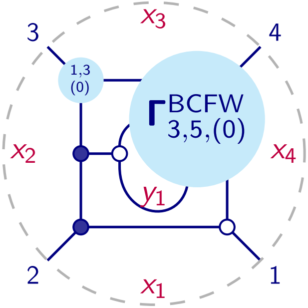

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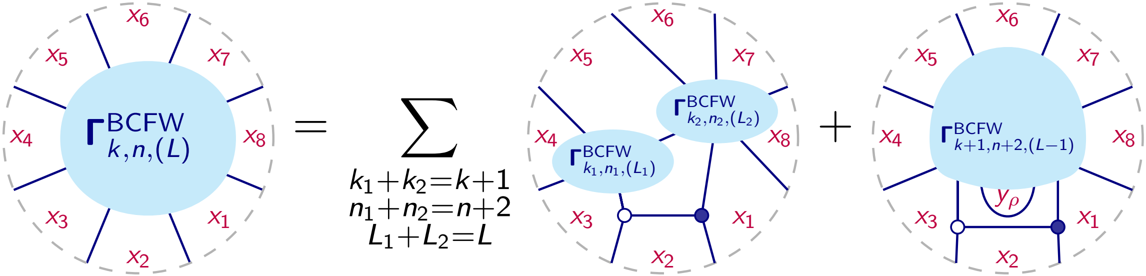

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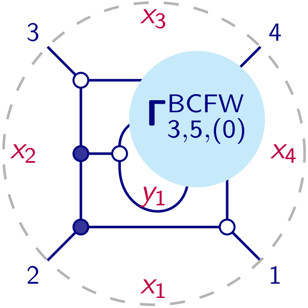

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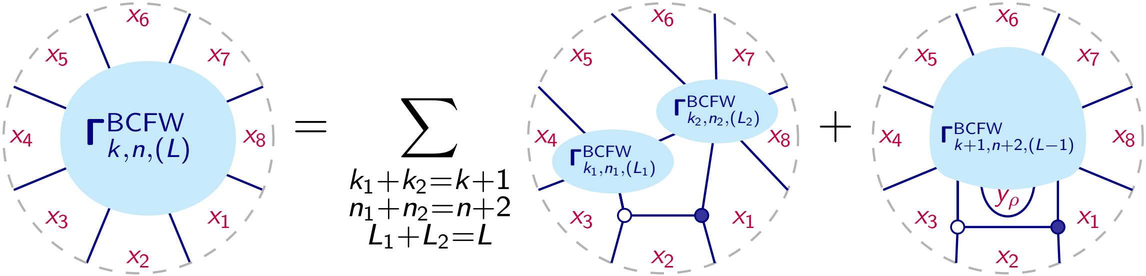

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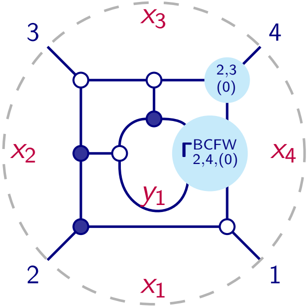

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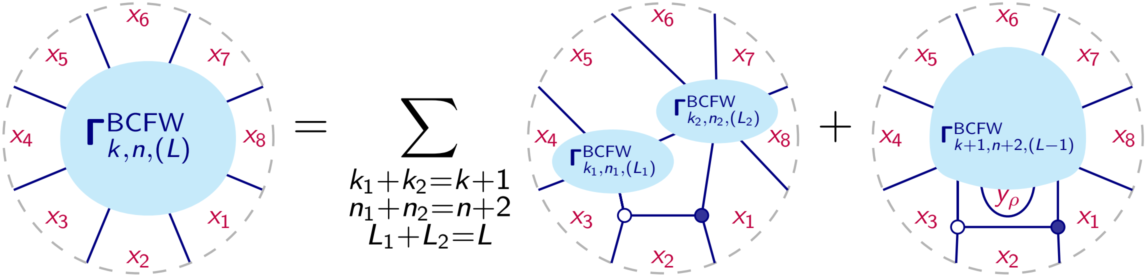

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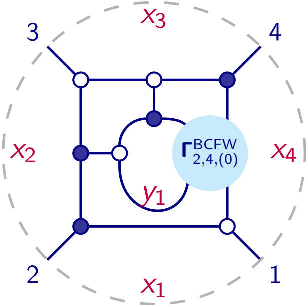

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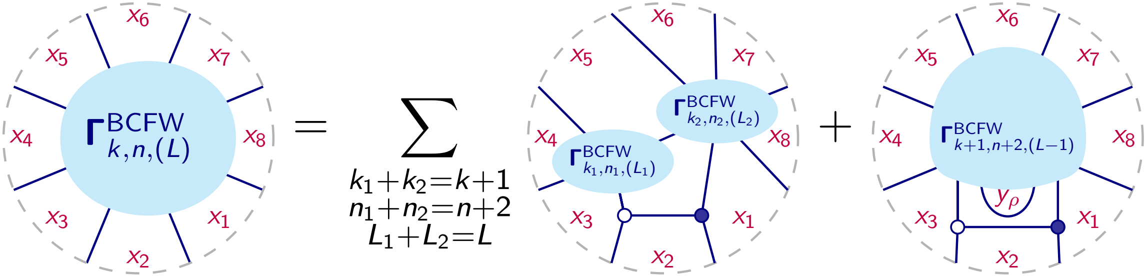

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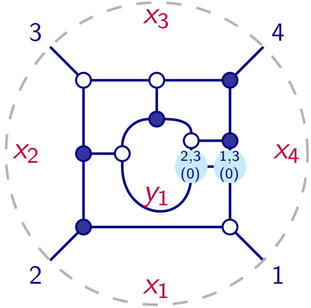

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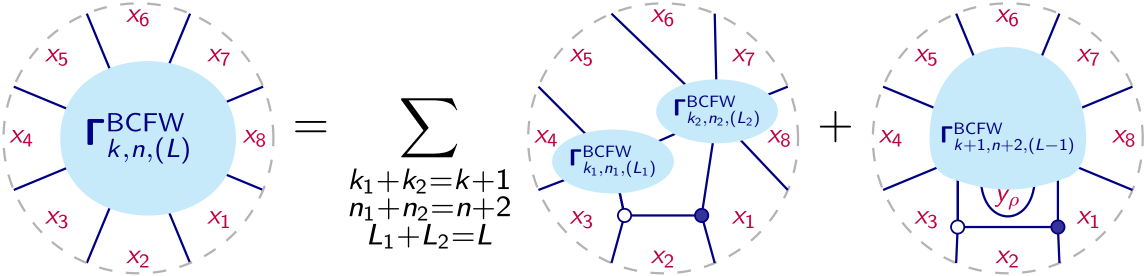

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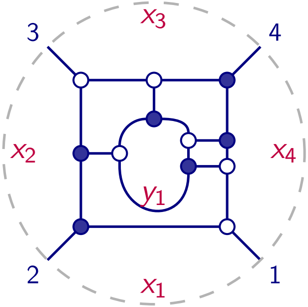

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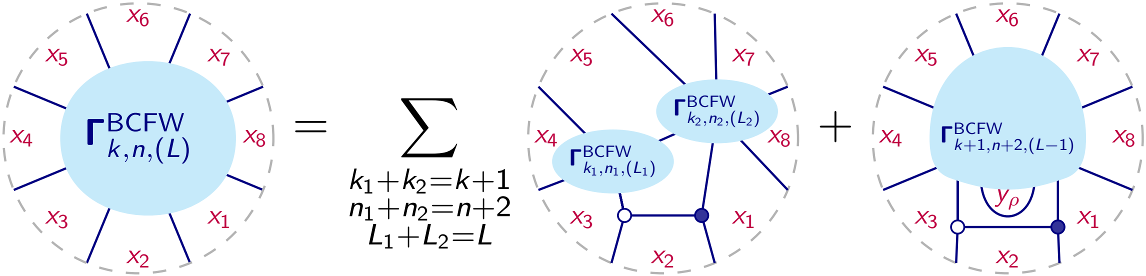

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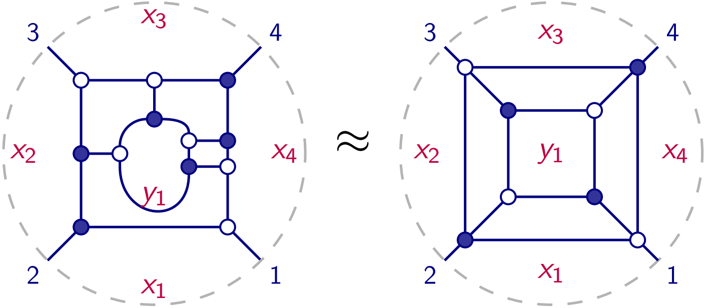

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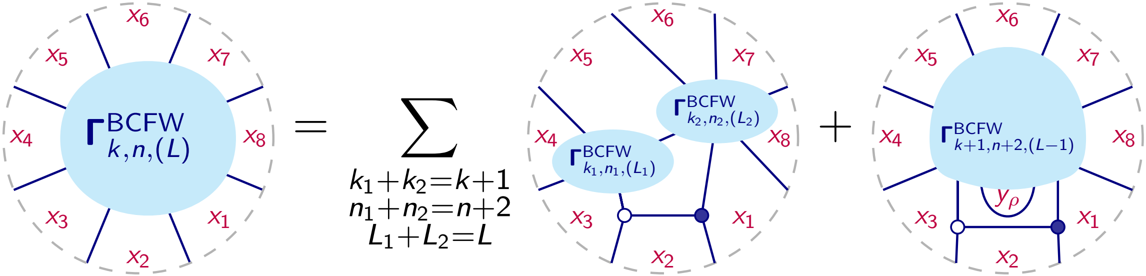

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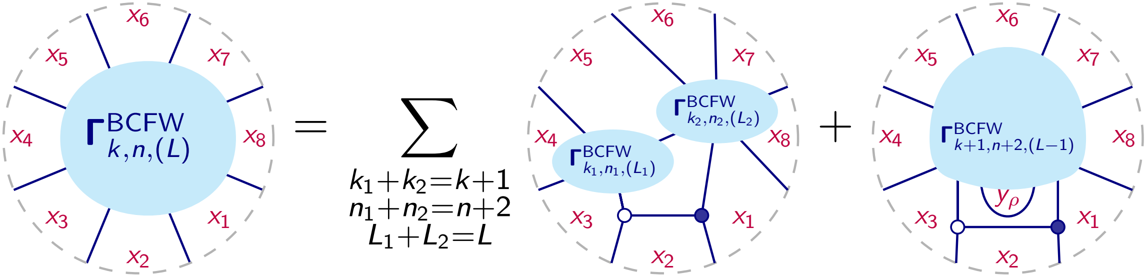

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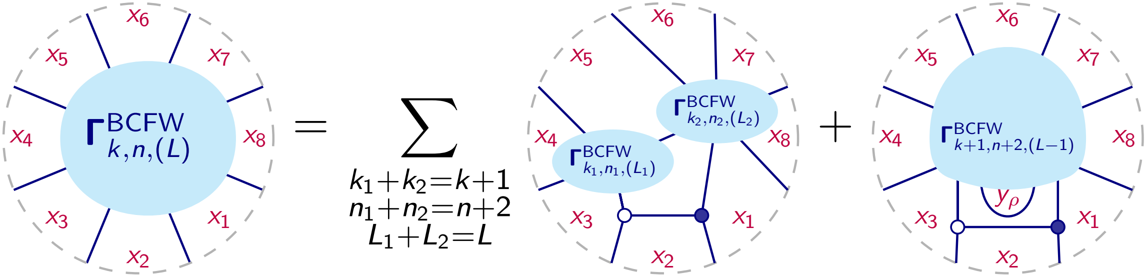

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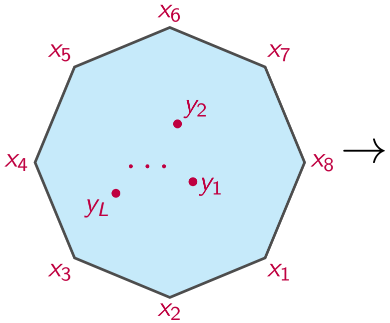

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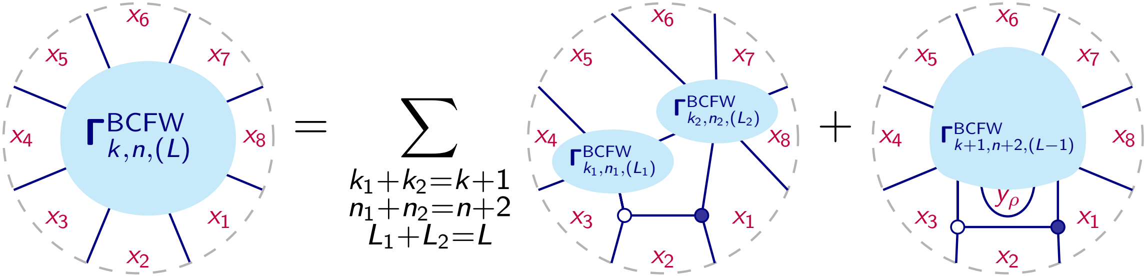

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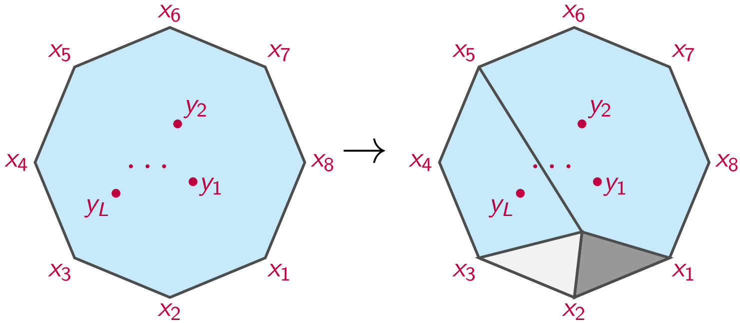

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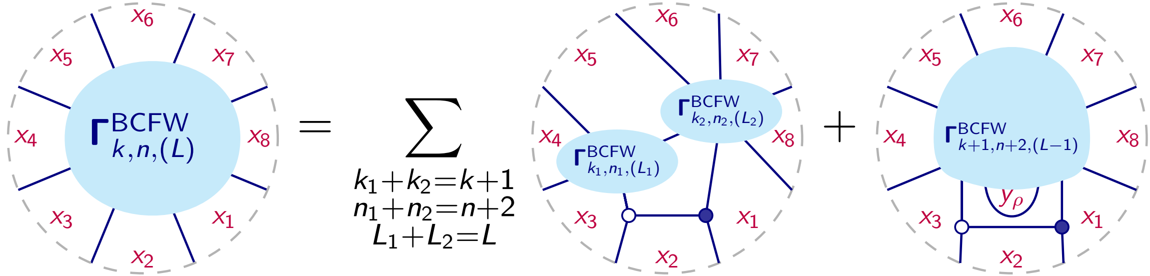

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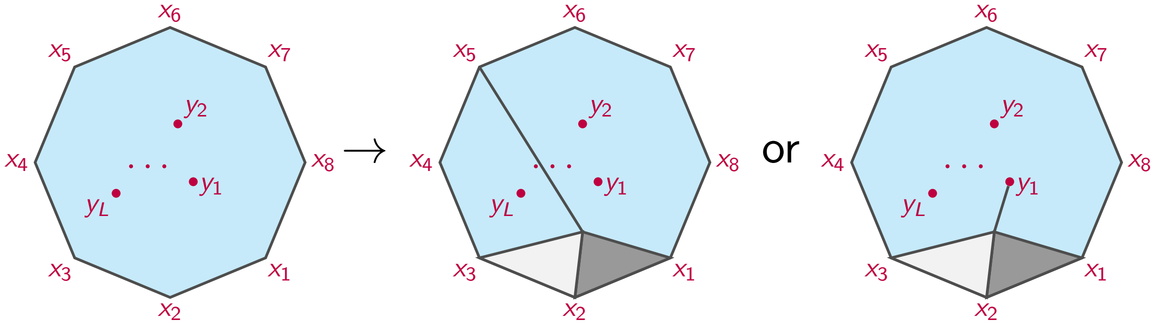

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Standard / All BCFW cells triangulate the *tree / loop* *momentum-twistor / momentum* amplituhedron.

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$$= \text{linear projection of } \begin{cases} \mathrm{Gr}_{\geqslant 0}(k,n) & (\text{tree level}); \\ ??? & (\text{loop level}). \end{cases}$$

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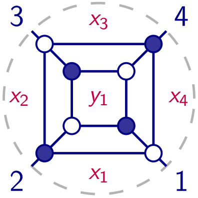

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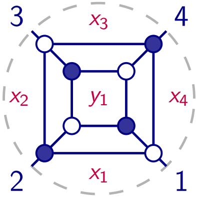

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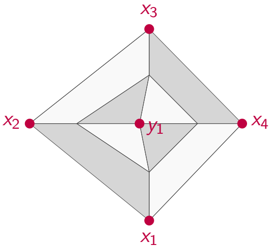

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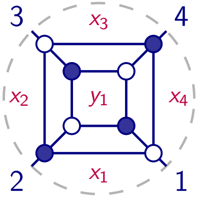

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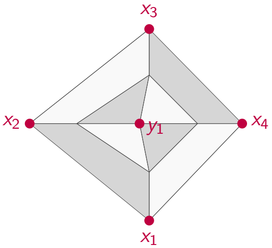

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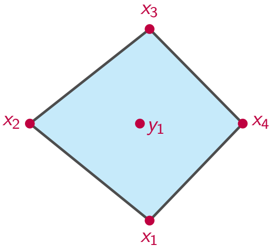

L-punctured polygon

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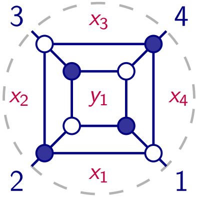

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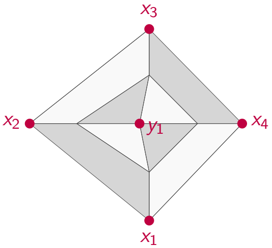

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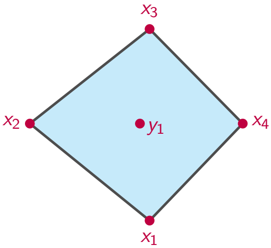

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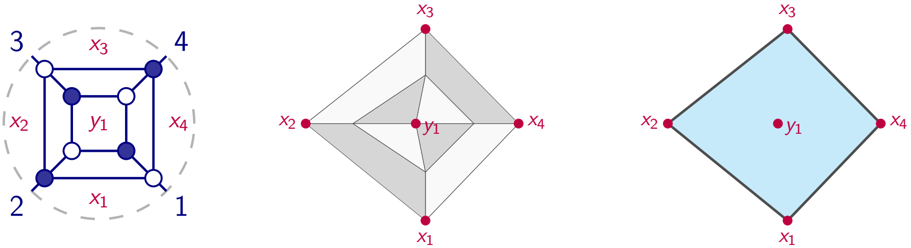

Γ | $H : \text{Faces}(\Gamma) \to \mathbb{R}^{2,2}$ | L-punctured polygon

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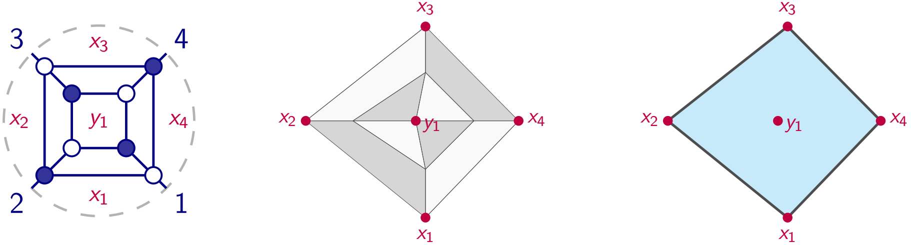
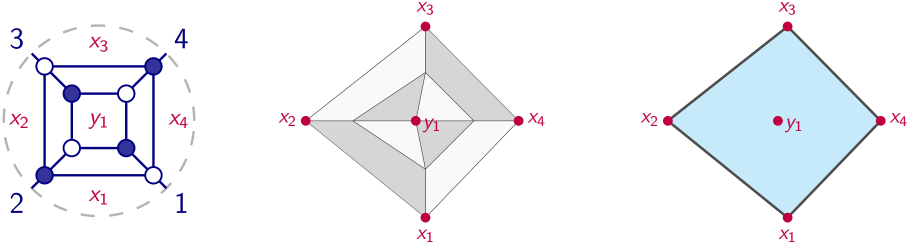


Γ | $H : \mathrm{Faces}(\Gamma) \to \mathbb{R}^{2,2}$ | L-punctured polygon

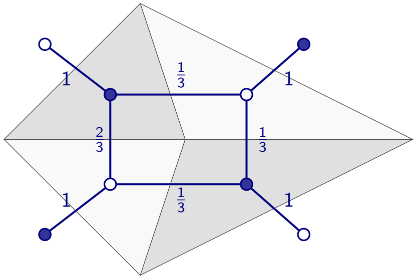
- Ambient momentum amplituhedron via sign flips [G.'25]:

$$\mathcal{M}_{k,n}^{(L)} := \left\{ (\mathcal{P}, \vec{\mathbf{y}}) \in (\mathbb{R}^{2,2})^{n+L} \;\middle|\; \begin{array}{c} (x_i - x_j)^2 > 0,\ (x_i - y_\rho)^2 > 0,\ (y_\rho - y_\gamma)^2 > 0, \\ \text{flips}(\lambda) = k-2,\ \text{flips}(\tilde{\lambda}) = k \\ y_\rho^{\mathcal{T}} \text{ is inside } \mathcal{P}^{\mathcal{T}} \end{array} \right\}.$$

- Ambient momentum amplituhedron as a "linear projection" of $\mathrm{Gr}_{\geqslant 0}^{(L)}(k,n)$:
  - Pick an L-punctured plabic graph Γ.
  - Pick an OCP $H : \text{Faces}(\Gamma) \to \mathbb{R}^{2,2}$ (embedded dual to Γ) with Mandelstam-positive boundary.
  - Restrict H to boundary and punctured faces of $\Gamma \implies$ get L-punctured polygon $(\mathcal{P}, \vec{\mathbf{y}}) \in \mathcal{M}_{k,n}^{(L)}$.
- **Corollary:** these two definitions of $\mathcal{M}_{k,n}^{(L)}$ coincide.
- **Question:** How to pick $H : \text{Faces}(\Gamma) \to \mathbb{R}^{2,2}$? How is this a linear projection?



Γ $\qquad$ $H : \text{Faces}(\Gamma) \to \mathbb{R}^{2,2}$ $\qquad$ L-punctured polygon



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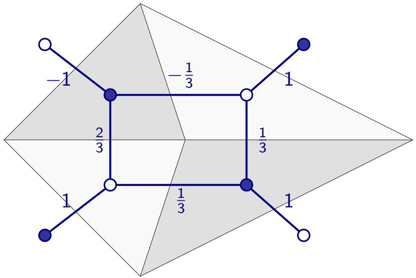
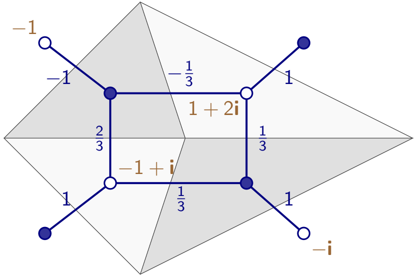
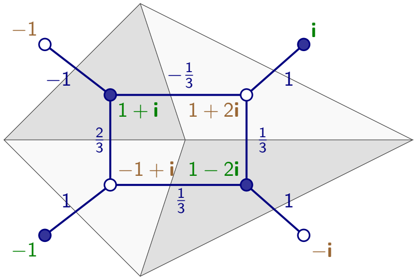


- Pick Kasteleyn edge weights $K(w, b)$ on Γ.



- Pick Kasteleyn edge weights $K(w, b)$ on Γ.
- Discrete holomorphic functions on Γ:
  $\lambda^\circ : \mathbf{V}^\circ \to \mathbb{C}$

$$\forall b \in \mathbf{V}^\bullet_{\text{int}} \quad \sum_{w \sim b} K(w, b)\lambda^\circ(w) = 0,$$



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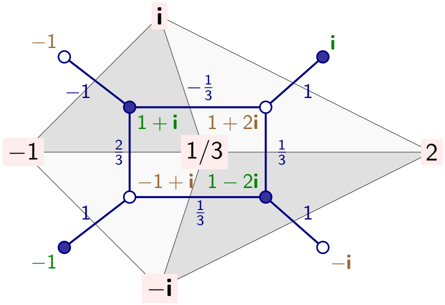

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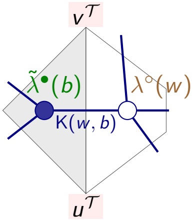

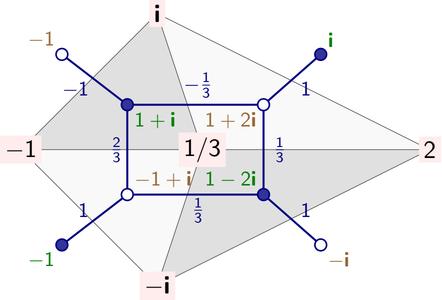

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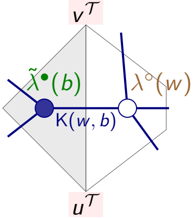

[Kenyon '02], [Smirnov '10], [Chelkak–Smirnov '12],
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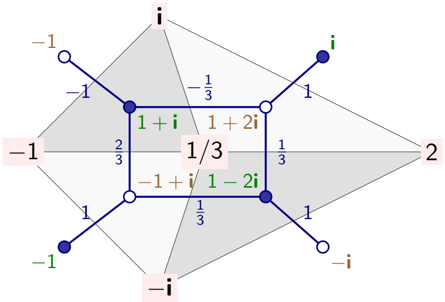

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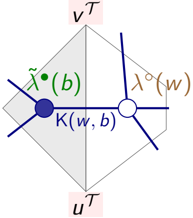

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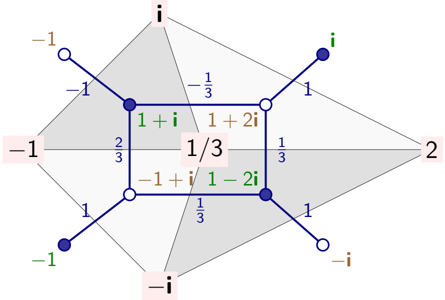

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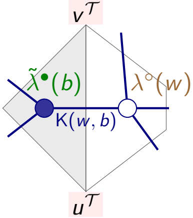
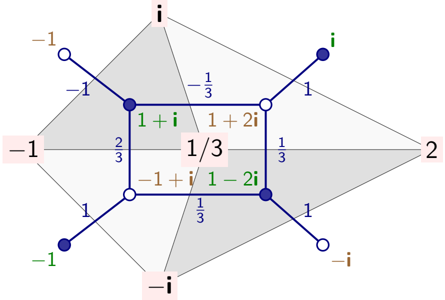


[Kenyon '02], [Smirnov '10], [Chelkak–Smirnov '12], [Kenyon–Lam–Ramassamy–Russkikh '18], [Chelkak–Laslier–Russkikh '21]

Easy: angle condition is satisfied mod 2π



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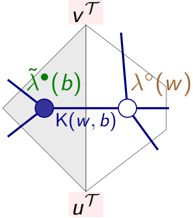

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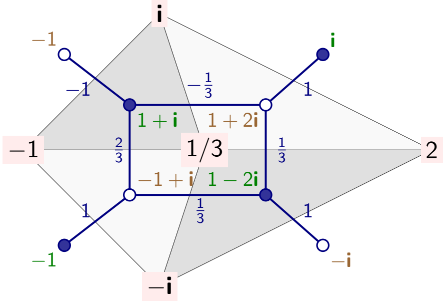
Hard: how to construct $(\lambda^\circ, \tilde{\lambda}^\bullet)$ such that H is a valid, embedded OCP?



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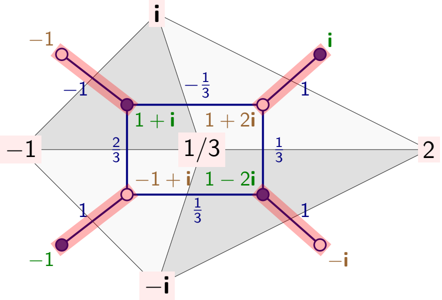

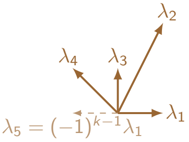

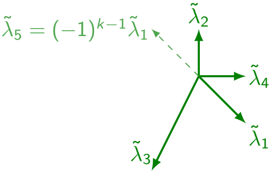

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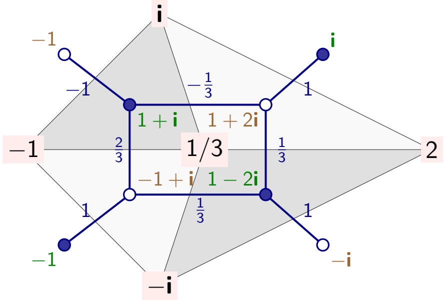

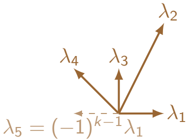

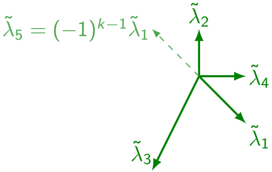

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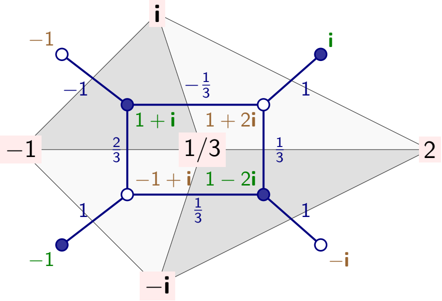

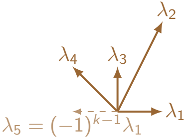

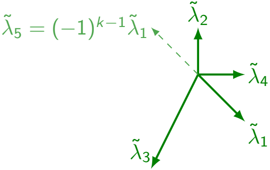

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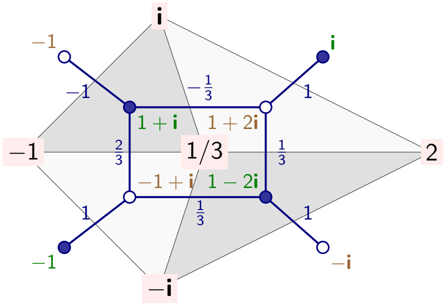

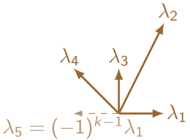

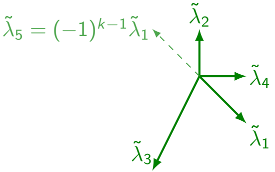

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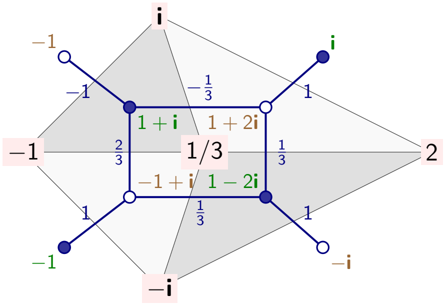

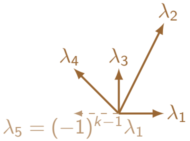

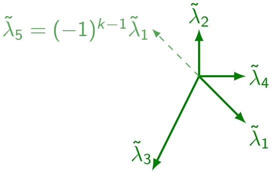

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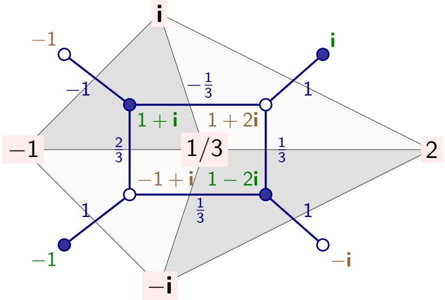
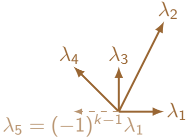
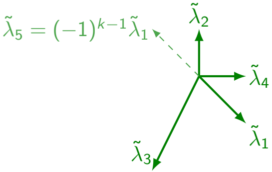
$$\langle a\,b\rangle := \det(\lambda_a|\lambda_b), \quad [a\,b] := \det(\tilde\lambda_a|\tilde\lambda_b), \quad \mathrm{flips}(\lambda) := \mathrm{flips}(\langle 1\,2\rangle, \langle 1\,3\rangle, \dots, \langle 1\,n\rangle)$$

Theorem (G. (2024))

H is a valid, embedded OCP $\iff (\lambda, \tilde\lambda)$ belongs to positive kinematic space:







$\langle 1\,2\rangle = 2,\ \langle 2\,3\rangle = 1,$
$\langle 3\,4\rangle = 1,\ \langle 4\,5\rangle = 1,$
$\mathrm{flips}(\lambda) = 0$

$[1\,2] = 1,\ [2\,3] = 1,$
$[3\,4] = 2,\ [4\,5] = 1,$
$\mathrm{flips}(\tilde\lambda) = 2$

- The OCP is obtained as the Kenyon–Smirnov primitive $H : \mathrm{Faces}(\Gamma) \to \mathbb{R}^{2,2}$:
  $H^{\mathcal{T}}(u) - H^{\mathcal{T}}(v) = \mathrm{K}(w,b)\lambda^\circ(w)\tilde\lambda^\bullet(b)$ and $H^{\mathcal{O}}(u) - H^{\mathcal{O}}(v) = \mathrm{K}(w,b)\overline{\lambda^\circ(w)}\tilde\lambda^\bullet(b)$.
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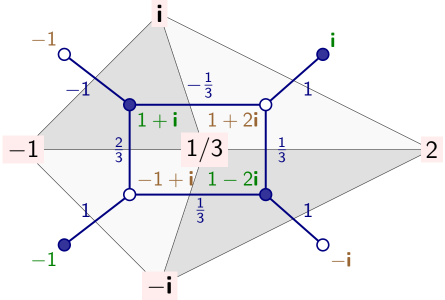
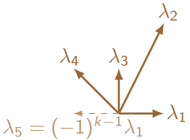
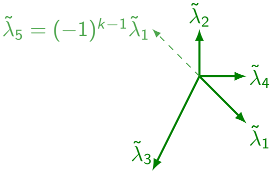
Theorem (G. (2024))

H is a valid, embedded OCP $\iff$ $(\lambda, \tilde\lambda)$ belongs to positive kinematic space:
$\lambda \cdot \tilde\lambda^T = 0_{2\times 2}$,







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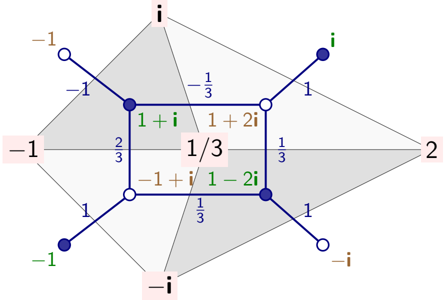
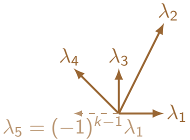
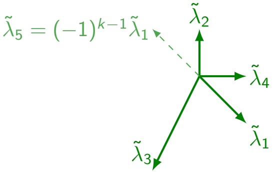
Theorem (G. (2024))

H is a valid, embedded OCP $\iff$ $(\lambda, \tilde\lambda)$ belongs to positive kinematic space:
$\lambda \cdot \tilde\lambda^T = 0_{2\times 2}, \quad \langle i\; i+1\rangle > 0, \quad [i\; i+1] > 0,$







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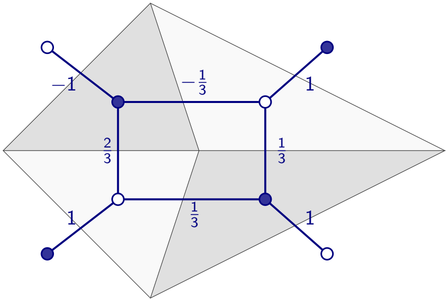
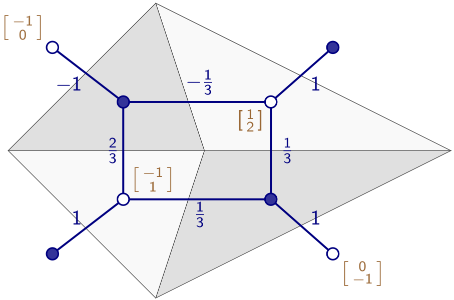
$$\lambda \cdot \tilde\lambda^T = 0_{2\times 2}, \quad \langle i\; i+1\rangle > 0, \quad [i\; i+1] > 0, \quad \mathrm{flips}(\lambda) = k-2, \quad \mathrm{flips}(\tilde\lambda) = k.$$



- Pick Kasteleyn edge weights $K(w,b)$ on Γ.
- Columns of $C \in \mathrm{Gr}_{\geqslant 0}(k,n)$ and $C^{\perp}$ extend to:
  [AGPR'19] $C^{\circ} : \mathbf{V}^{\circ} \to \mathbb{R}^{k}$, $\quad C^{\perp\bullet} : \mathbf{V}^{\bullet} \to \mathbb{R}^{n-k}$:

$$\forall b \in \mathbf{V}^{\bullet}_{\mathrm{int}} \quad \sum_{w \sim b} K(w,b) C^{\circ}(w) = 0,$$

$$\forall w \in \mathbf{V}^{\circ}_{\mathrm{int}} \quad \sum_{b \sim w} K(w,b) C^{\perp\bullet}(b) = 0.$$



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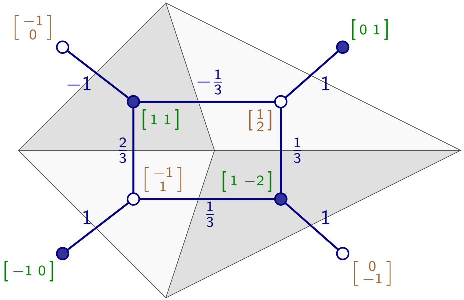

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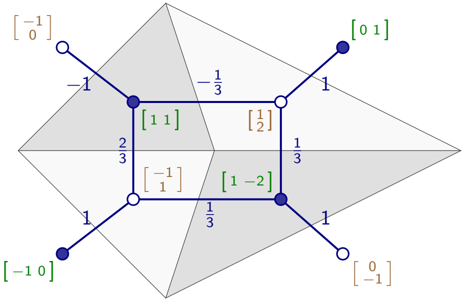

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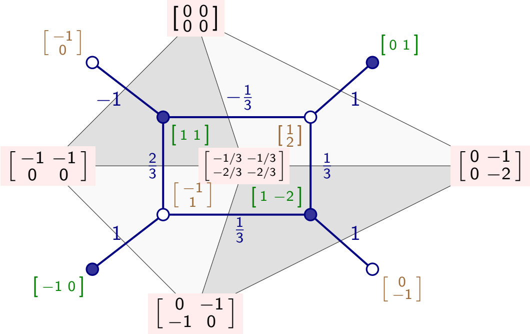

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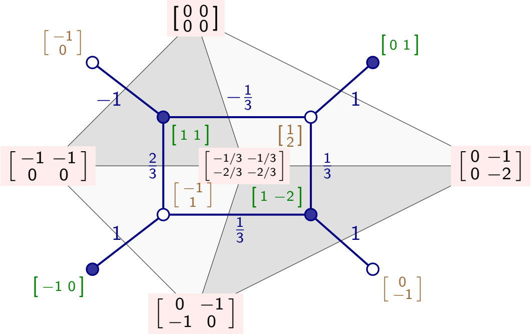

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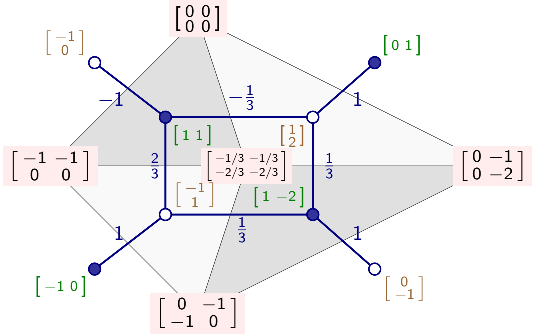
- Tangent space $T_C\,\mathrm{Gr}(k,n)$ is identified with the space $\mathbb{R}^{k\times(n-k)}$ of linear maps $C \to C^{\perp}$.



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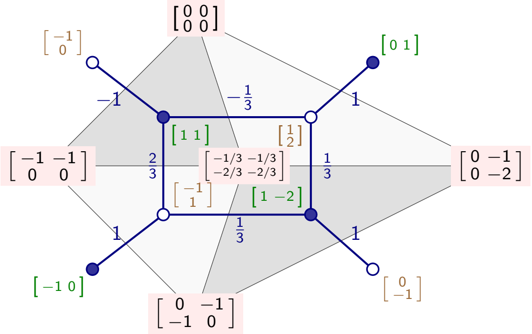
- Tangent space $T_C\,\mathrm{Gr}(k,n)$ is identified with the space $\mathbb{R}^{k\times(n-k)}$ of linear maps $C \to C^\perp$.
- So H is naturally viewed as a map $H : \mathrm{Faces}(\Gamma) \to T_C\,\mathrm{Gr}(k,n)$.



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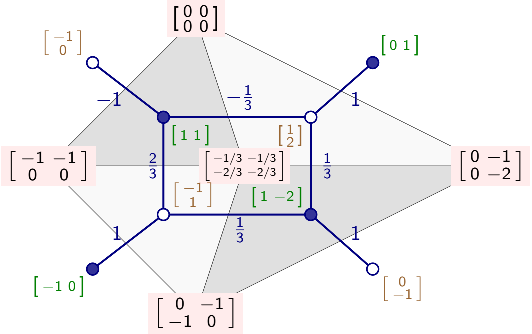
- **Conclusion:** L-punctured weighted plabic graph $\longrightarrow$ point $C \in \mathrm{Gr}_{\geqslant 0}(k,n)$ together with $n+L$ point configuration $(\mathcal{P},\vec{\mathbf{y}}) \in (T_C\,\mathrm{Gr}(k,n))^{n+L}$ defined up to shift.



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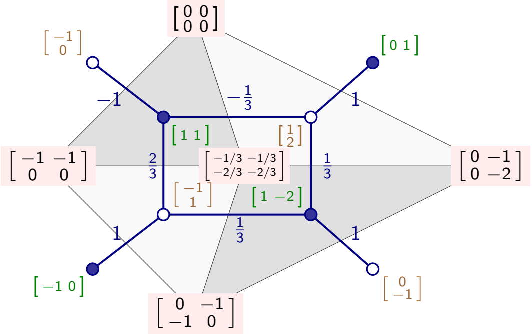

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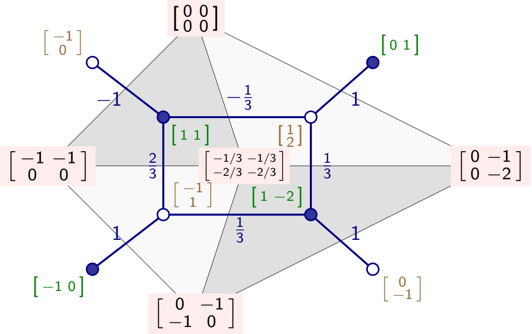
- Define the L-punctured positive Grassmannian $\mathrm{Gr}_{\geqslant 0}^{(L)}(k,n)$ as the image of the map
  $\{L\text{-punctured weighted plabic graphs}\} \longrightarrow T_*^{(L)}\,\mathrm{Gr}(k,n)$.



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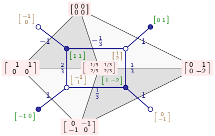
- Loop momentum amplituhedron $\mathcal{M}^{(L)}_{k,n}(\Lambda, \tilde{\Lambda})$ = linear slice of $\mathcal{M}^{(L)}_{k,n}$ = linear projection of $\mathrm{Gr}^{(L)}_{\geqslant 0}(k,n)$.



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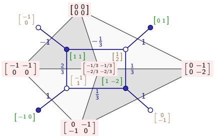

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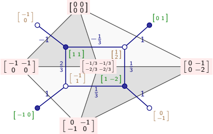

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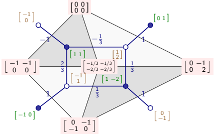

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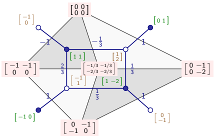

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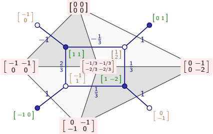

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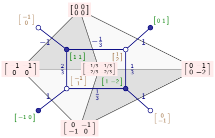

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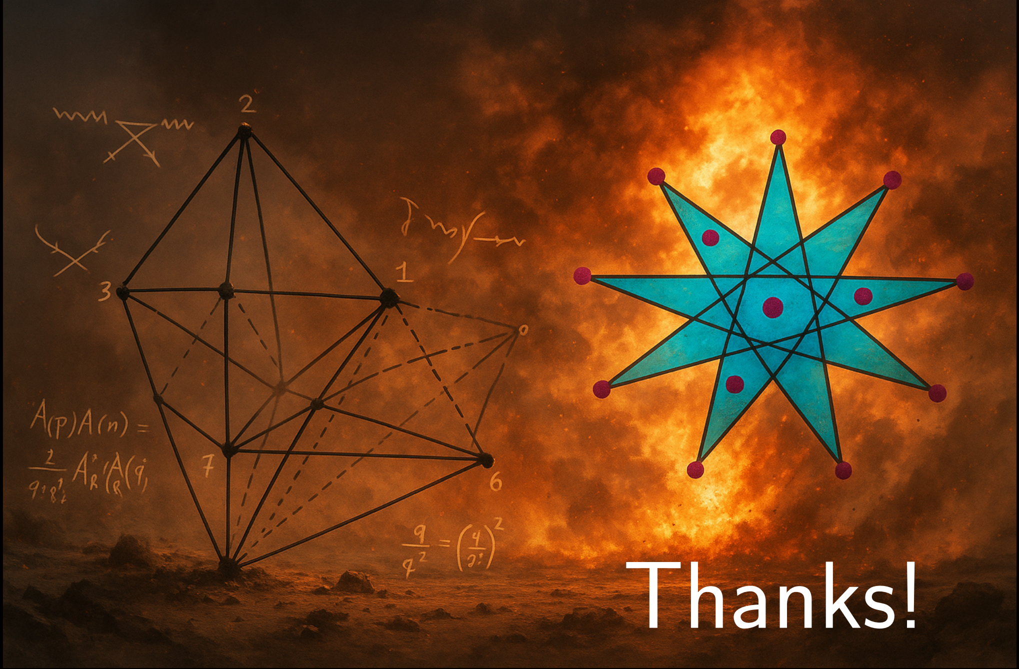

Thanks!