Amplituhedra and origami

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Combinatorics of Fundamental Physics Workshop
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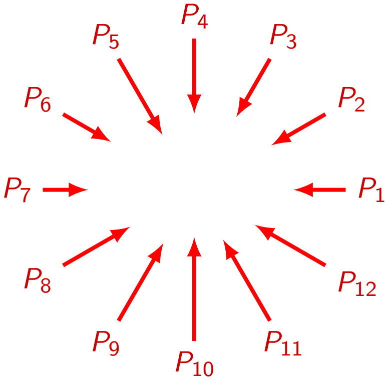
Particle momenta

- Consider incoming particles with momenta $P_1, P_2, \ldots, P_n \in \mathbb{R}^{3,1}$ which are null ($P_i^2 = 0$) and satisfy $P_1 + P_2 + \cdots + P_n = 0$.



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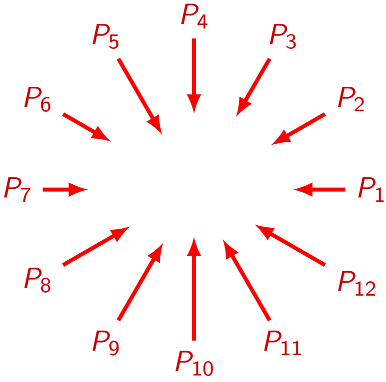

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- Consider incoming particles with momenta $P_1, P_2, \ldots, P_n \in \mathbb{R}^{2,2}$ which are null ($P_i^2 = 0$) and satisfy $P_1 + P_2 + \cdots + P_n = 0$.
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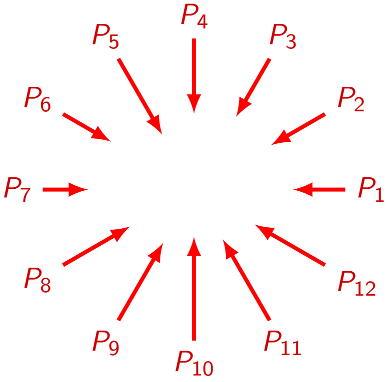

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On the contrary, for *complex-valued momenta* p^μ, the angle and square spinors are independent.[1] It may not seem physical to take p^μ complex, but it is a very very very useful strategy. We will see this repeatedly.

[1] One can keep p^μ real and change the spacetime signature to $(-, +, -, +)$; in that case, the angle and square spinors are real and independent.

[Elvang, Huang. *Scattering amplitudes in gauge theory and gravity.* Cambridge University Press, Cambridge, 2015.]

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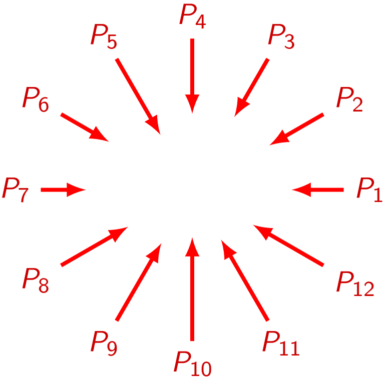

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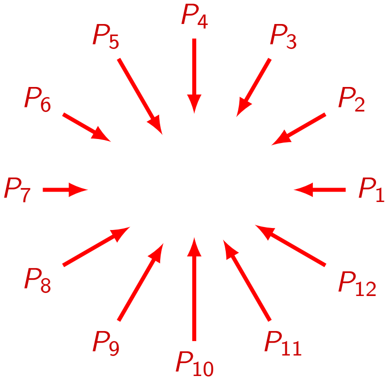

- Think of a null momentum vector $P \in \mathbb{R}^{2,2}$ as a pair $(P^{\mathcal{T}}, P^{\mathcal{O}}) \in \mathbb{C}^2$ of complex numbers satisfying $|P^{\mathcal{T}}| = |P^{\mathcal{O}}|$.

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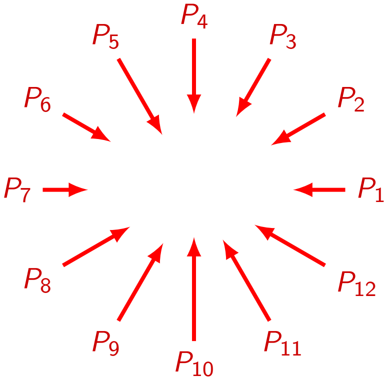

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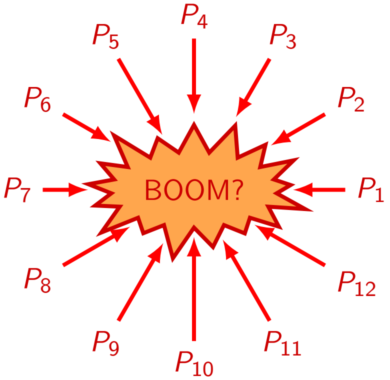

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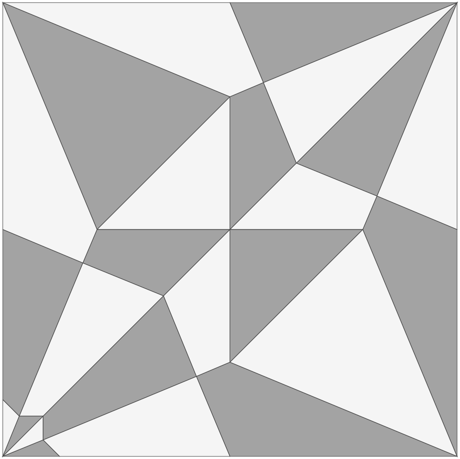
Origami crease patterns



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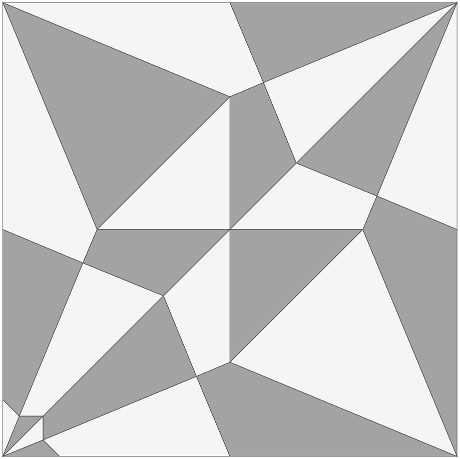

Faces: convex polygons colored black and white;

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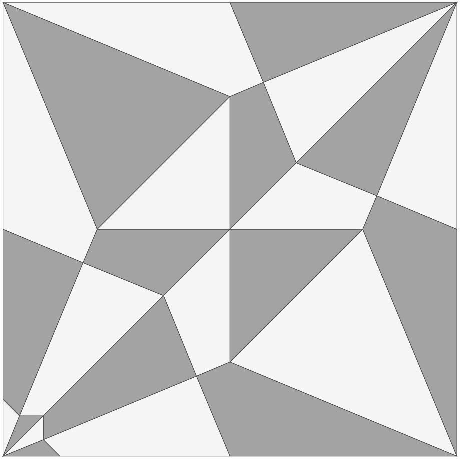

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sum(white angles) $= \pi$,
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around each interior vertex.

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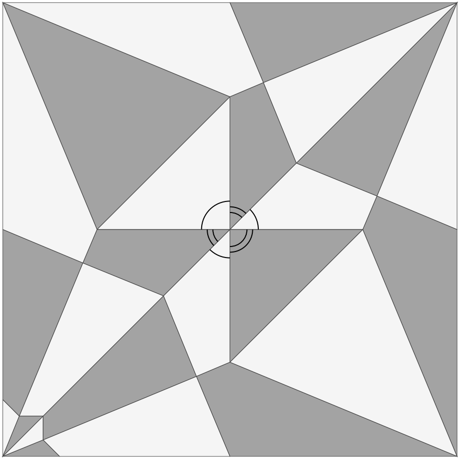

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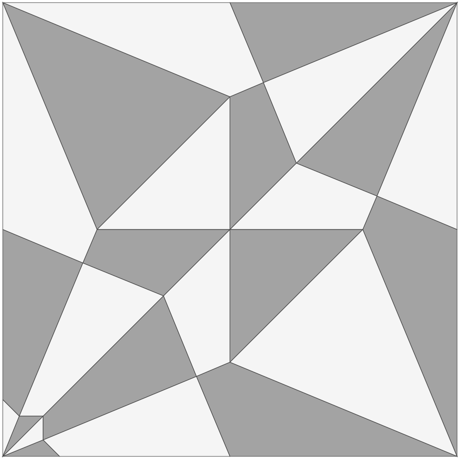

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isometry on each face preserving/reversing the orientations of white/black faces.

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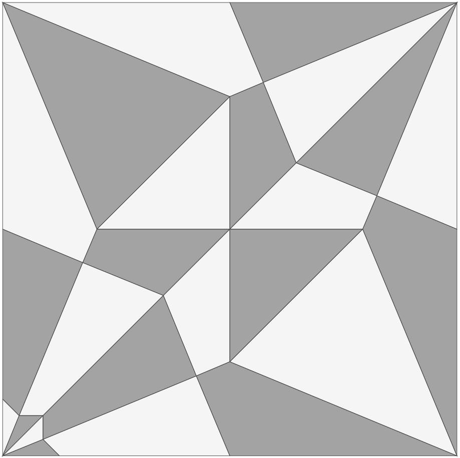

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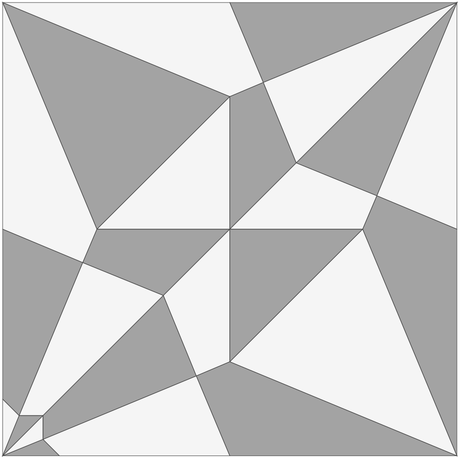

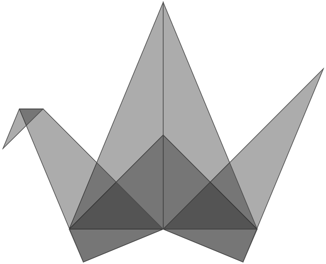

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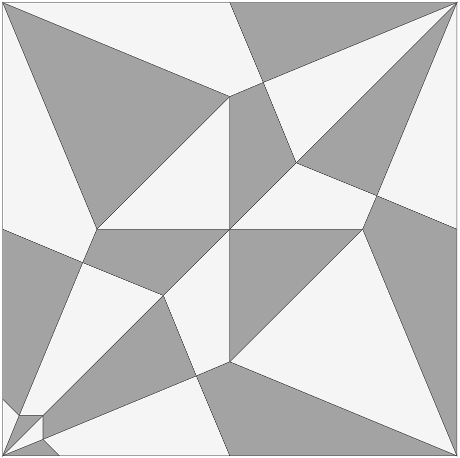

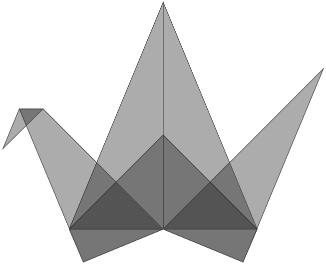

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Boundary vectors $P_i^{\mathcal{T}}$ and their images $P_i^{\mathcal{O}}$ under $\mathcal{O}$ satisfy $|P_i^{\mathcal{T}}| = |P_i^{\mathcal{O}}|$!

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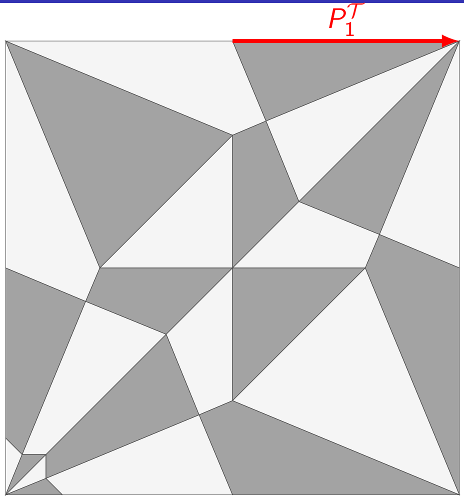

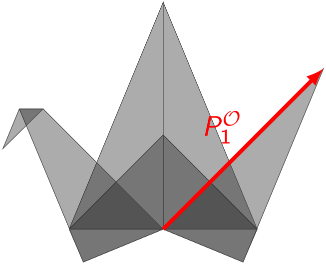

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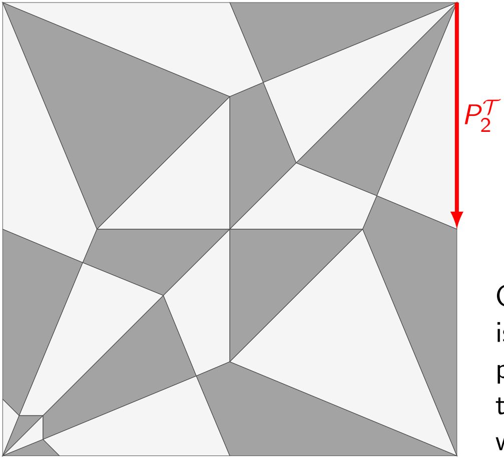

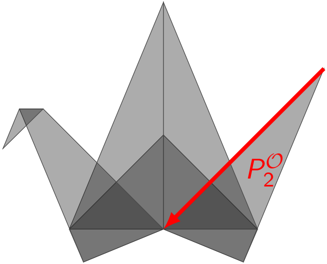

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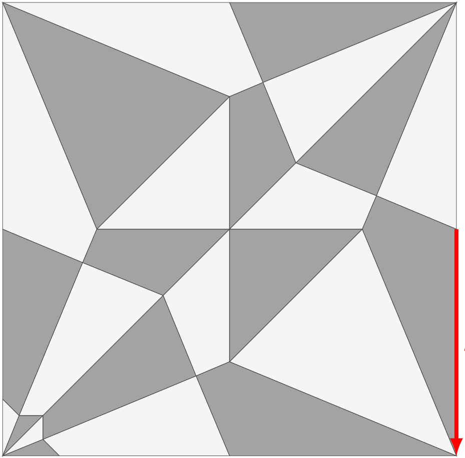

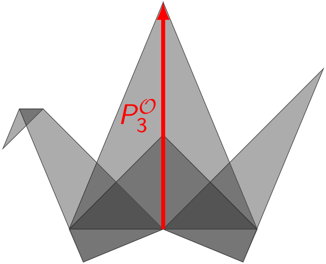

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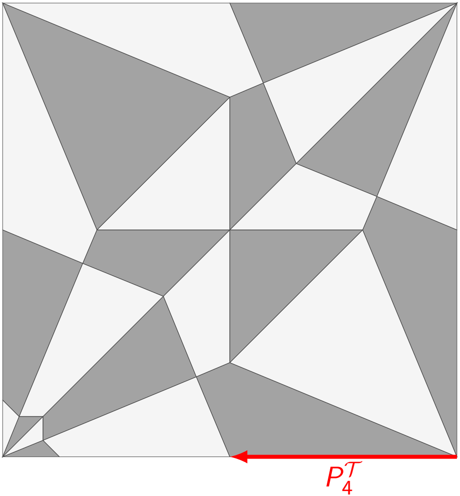

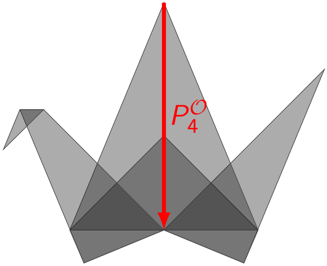

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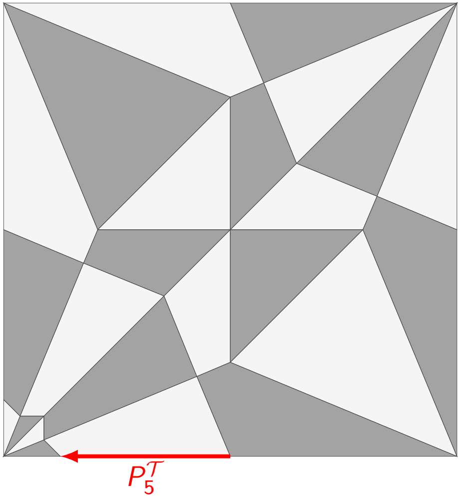

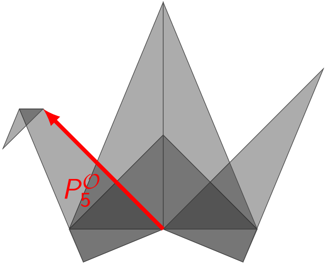

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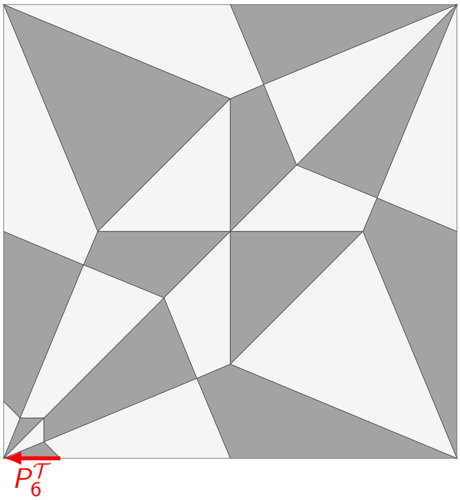

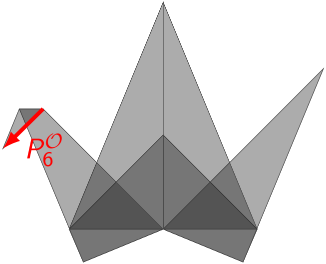

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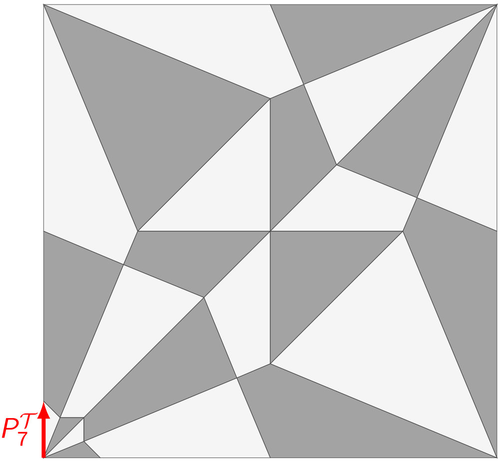

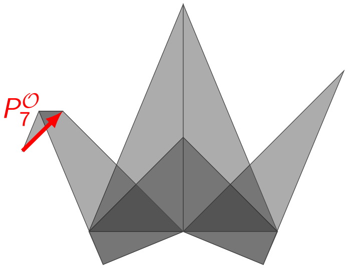

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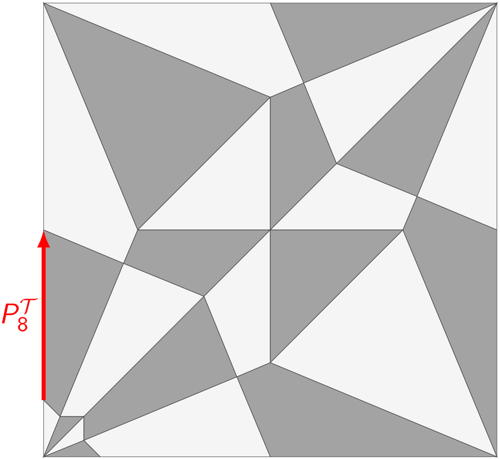

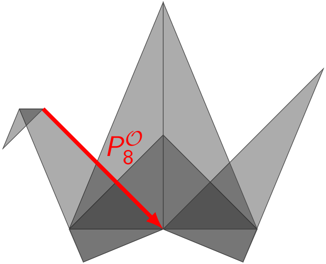

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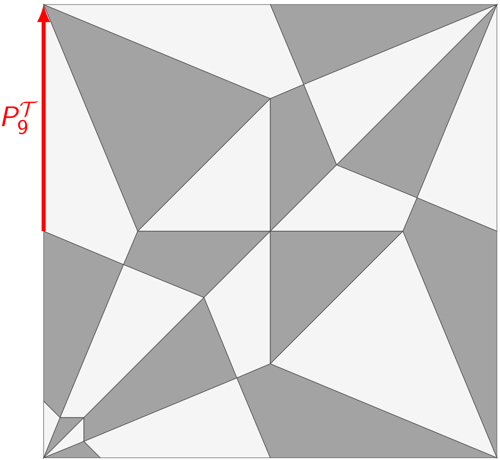

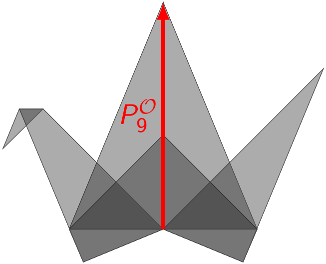

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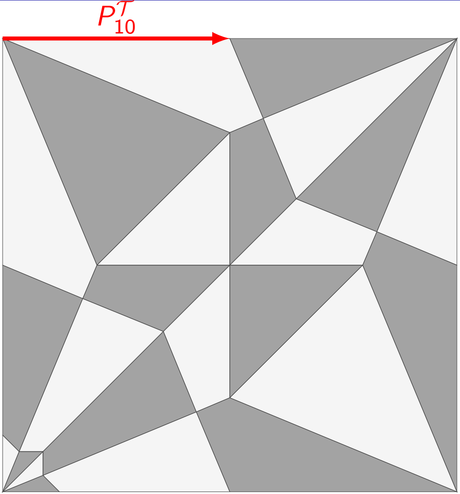

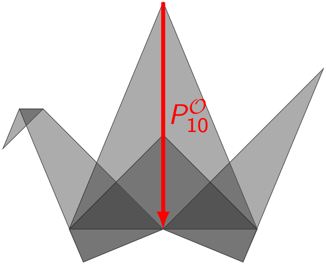

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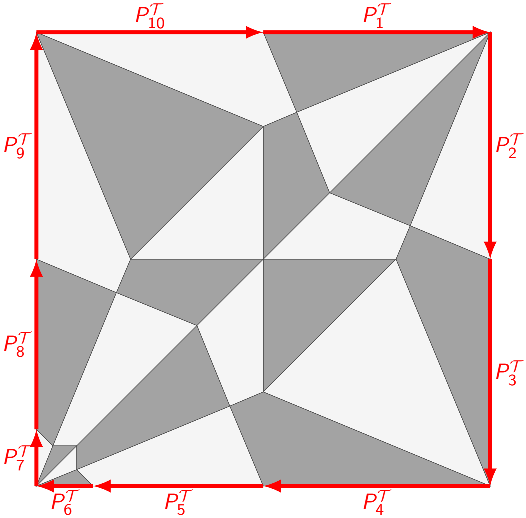

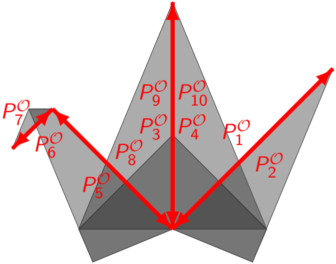

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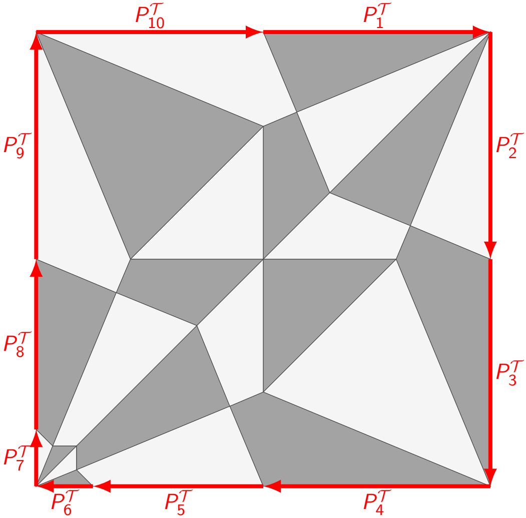

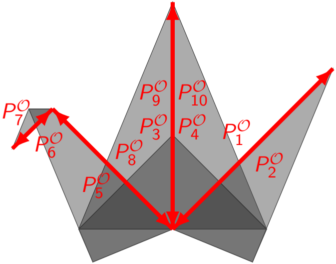

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Main result (preview):

$\mathcal{A}(P_1, \ldots, P_n) =$ integral over origami crease patterns with boundary $P_1, \ldots, P_n$.

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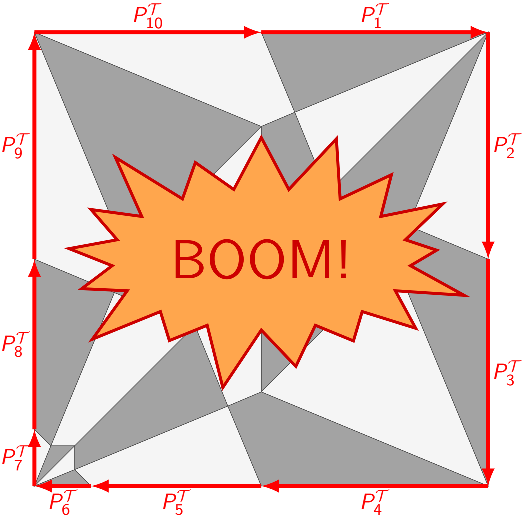

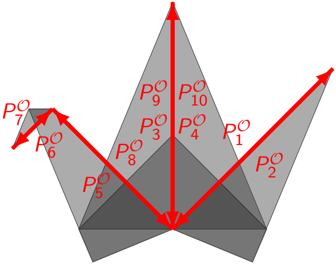

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Positive kinematic space

- Spinor-helicity formalism: Since $P_i = (P_i^{\mathcal{T}}, P_i^{\mathcal{O}}) \in \mathbb{C}^2$ with $|P_i^{\mathcal{T}}| = |P_i^{\mathcal{O}}|$, can choose $\lambda_i, \tilde{\lambda}_i \in \mathbb{C}$ such that $P_i^{\mathcal{T}} = \lambda_i \tilde{\lambda}_i$ and $P_i^{\mathcal{O}} = \bar{\lambda}_i \tilde{\lambda}_i$.

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- 2-planes $\lambda, \tilde{\lambda} \in \mathrm{Gr}(2, n)$ with columns $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^2$, $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n \in \mathbb{R}^2$.

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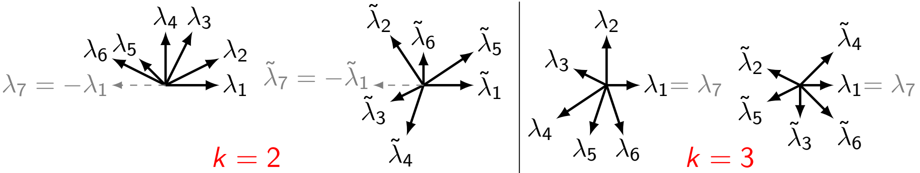

$\mathrm{wind}(\lambda) = \pi$ $\qquad$ $\mathrm{wind}(\tilde{\lambda}) = 3\pi$ $\qquad$ $\mathrm{wind}(\lambda) = 2\pi$ $\qquad$ $\mathrm{wind}(\tilde{\lambda}) = 4\pi$

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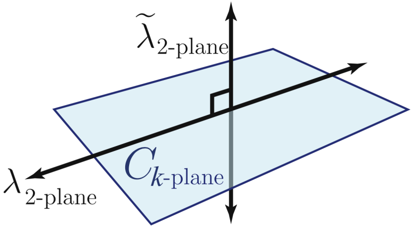
- $\mathcal{A}(P_1,\dots,P_n)$ = integral over $\{C \in \mathrm{Gr}_{\geqslant 0}(k,n) \mid \lambda \subset C \subset \tilde\lambda^\perp\}$ [ABCGPT '16].

As we saw in section 7, this can also be written as a residue of the top-form,

$$f_{\sigma}^{(k)} = \oint_{C\subset\Gamma_{\sigma}} \frac{d^{k\times n}C}{\mathrm{vol}(GL(k))}\,\frac{\delta^{k\times 4}(C\cdot\widetilde{\eta})}{(1\cdots k)\cdots(n\cdots k{-}1)}\,\delta^{k\times 2}(C\cdot\widetilde{\lambda})\,\delta^{2\times(n-k)}(\lambda\cdot C^{\perp}). \quad (8.2)$$

Recall from section 4, the (ordinary) δ-functions in (8.2) have the geometric interpretation of constraining the k-plane C to be *orthogonal to* the 2-plane $\widetilde{\lambda}$ and to *contain* the 2-plane λ, [14]:



(8.3)

[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka. *Grassmannian Geometry of Scattering Amplitudes*. Cambridge University Press, Cambridge, 2016.]

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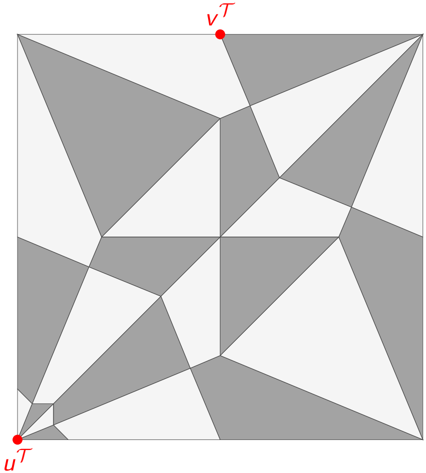
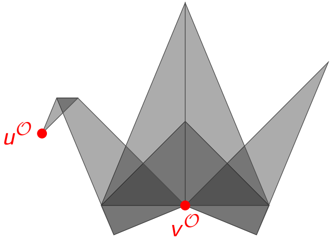
- Corollary: BCFW cells triangulate (Mandelstam-positive region of) $\mathcal{K}^+_{k,n}$.

- Let $u^{\mathcal{T}}, v^{\mathcal{T}}$ be two vertices of an origami crease pattern and $u^{\mathcal{O}}, v^{\mathcal{O}}$ be their images under the origami map.





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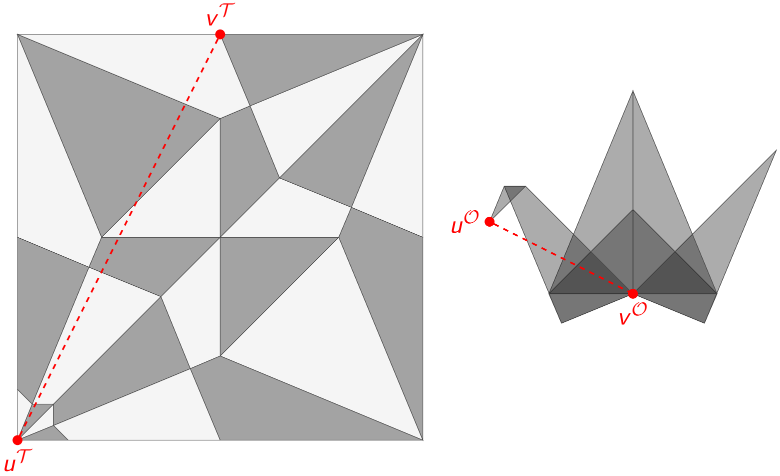

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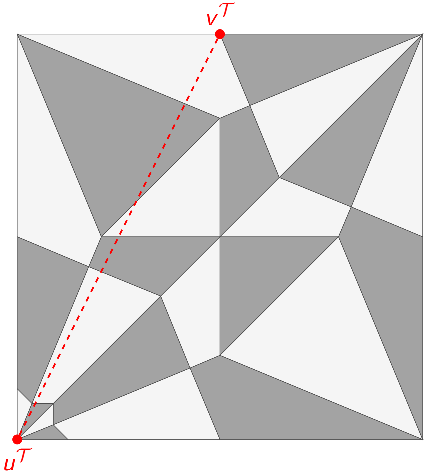

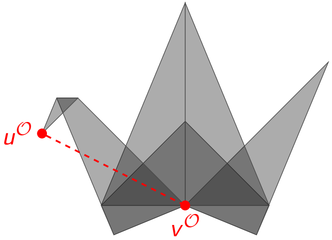

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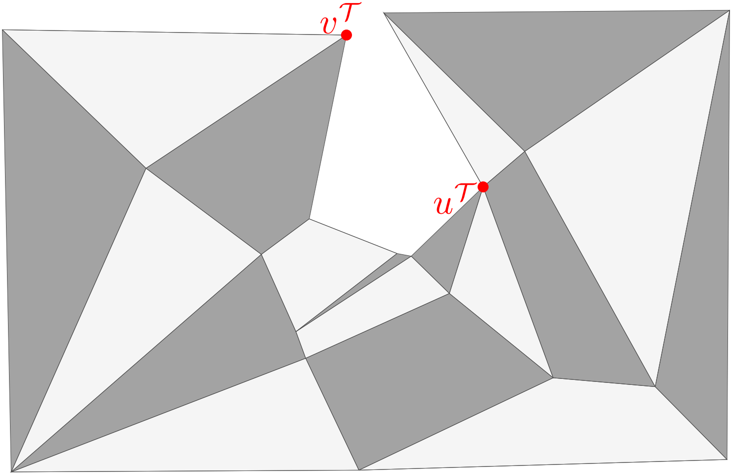

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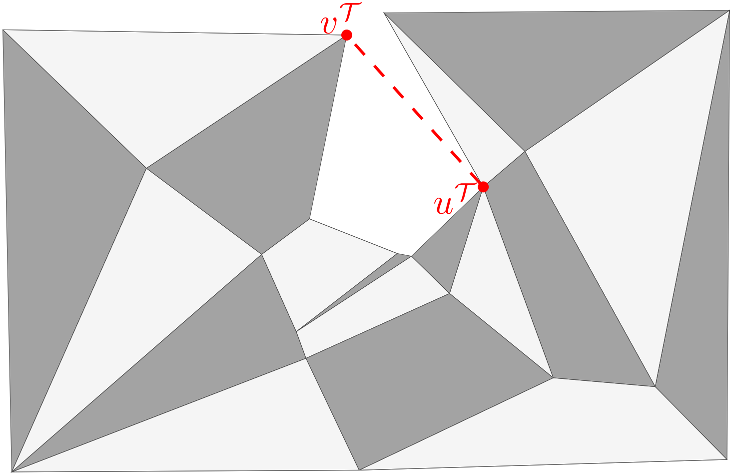

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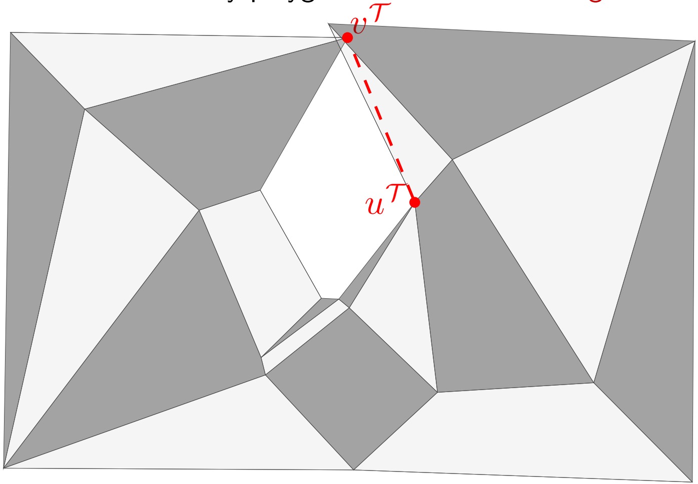

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$\mathcal{K}_{k,n}^{\mathrm{M}^+} := \{(\lambda, \tilde{\lambda}) \in \mathcal{K}_{k,n}^+ \mid S_{\lambda,\tilde{\lambda}}(i,j) > 0 \text{ for all } i,j\}$.

Theorem (G. (2024), "BCFW triangulation")

BCFW cells triangulate $\mathcal{K}_{k,n}^{\mathrm{M}^+}$.

See also: [Arkani-Hamed–Trnka '14], [Even-Zohar–Lakrec–Tessler '21], [Even-Zohar–Lakrec–Parisi–Tessler–Sherman-Bennett–Williams '23].

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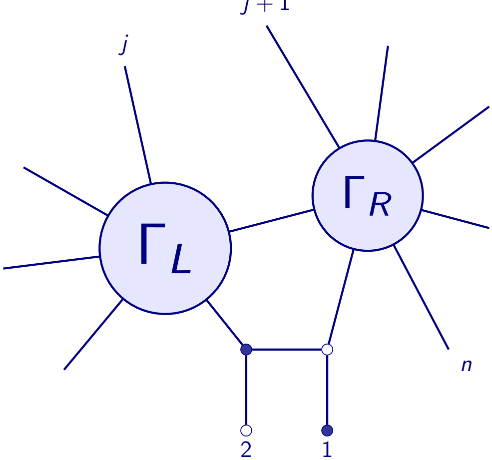

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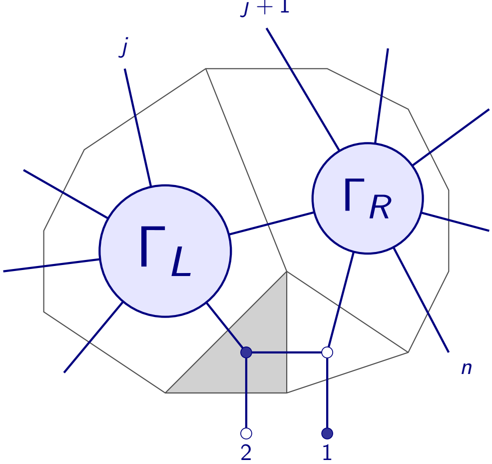

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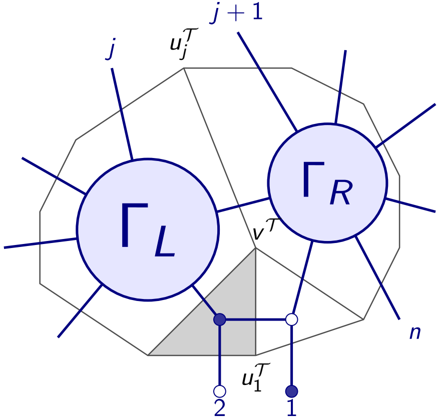

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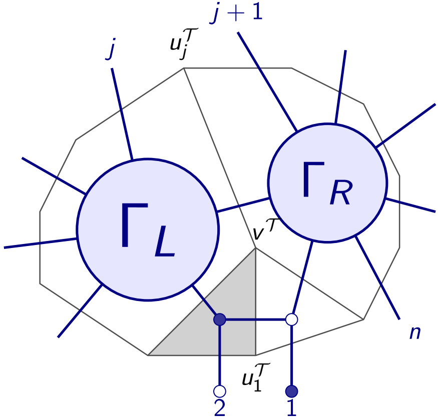

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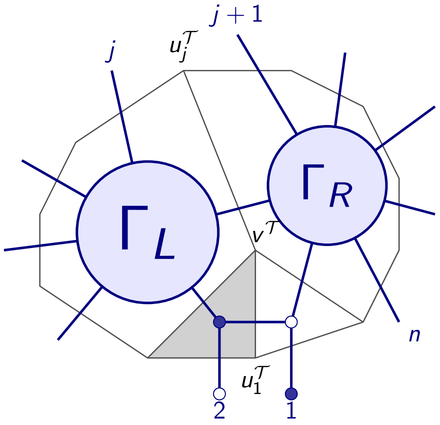

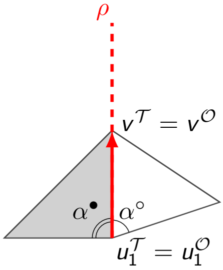

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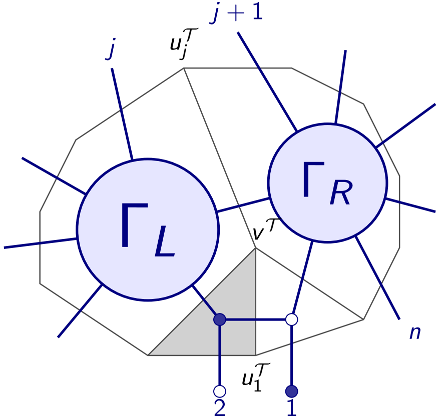

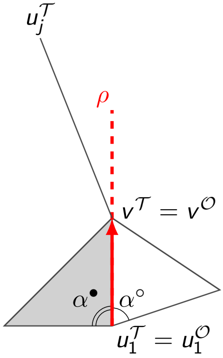

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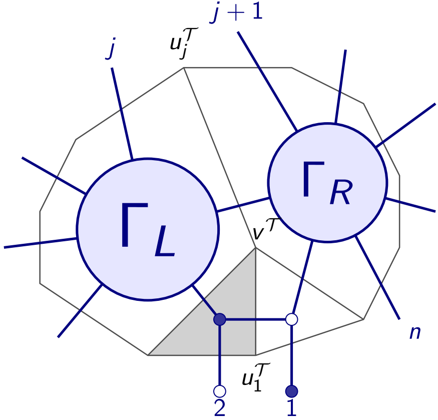

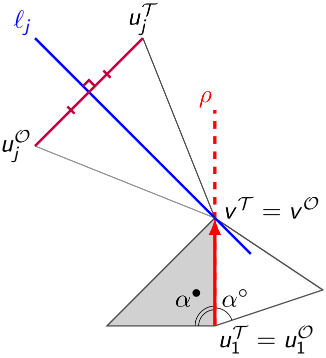

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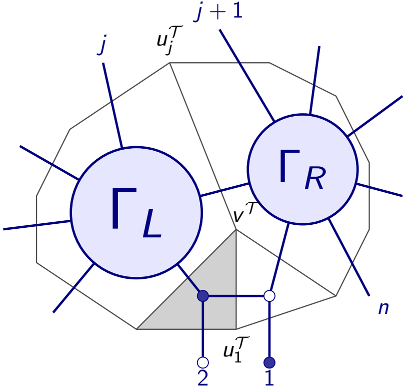

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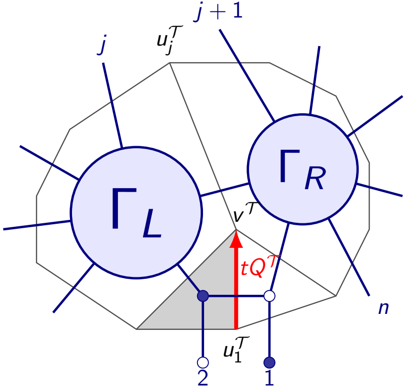

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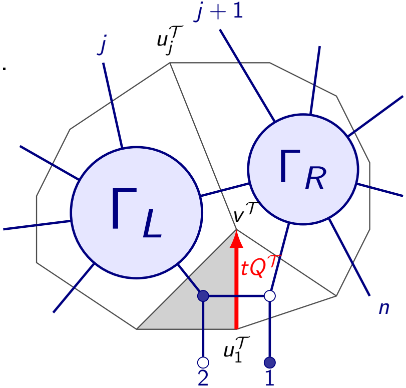

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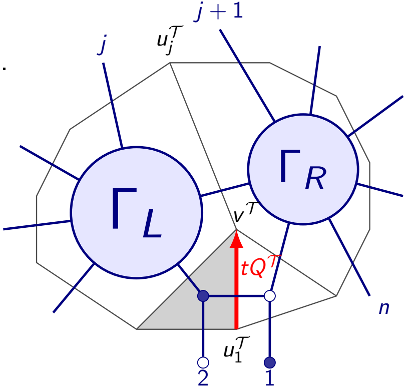

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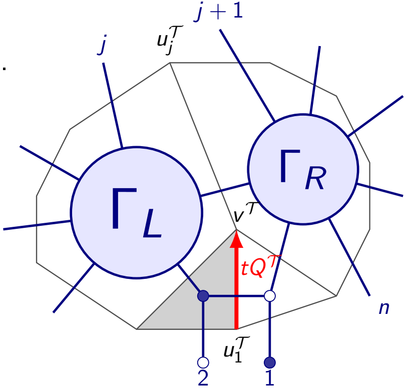

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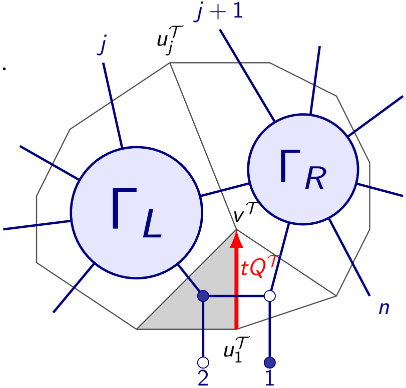
Step 3 (surjectivity). Show that the above algorithm always outputs a valid origami crease pattern.



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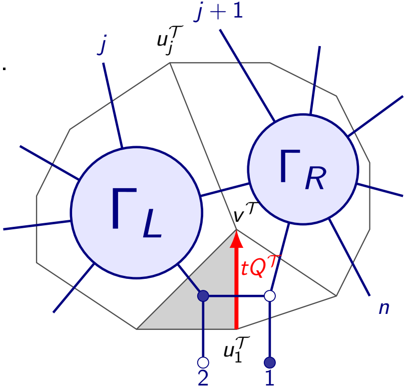

Theorem (G. (2024), "Main bijection")

Origami crease patterns are in natural bijection with triples* $\lambda \subset C \subset \tilde{\lambda}^{\perp}$ *such that* $(\lambda, \tilde{\lambda}) \in \mathcal{K}^{+}_{k,n}$ *and* $C \in \mathrm{Gr}_{\geqslant 0}(k, n)$.

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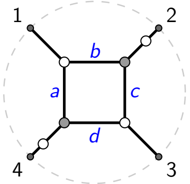
- Boundary measurement map: $C \in \mathrm{Gr}_{\geqslant 0}(k,n) \longleftrightarrow (\Gamma, \mathrm{wt})$ = planar bipartite graph in a disk with edge weights $\mathrm{wt} : E(\Gamma) \to \mathbb{R}_{>0}$ [Postnikov '06].



(Γ, wt)

$$C = \mathrm{Meas}(\Gamma, \mathrm{wt}) = \begin{pmatrix} 1 & b & 0 & -a \\ 0 & c & 1 & d \end{pmatrix}$$

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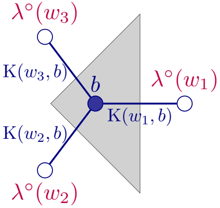

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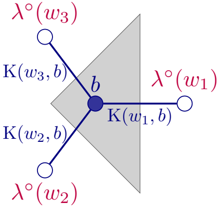

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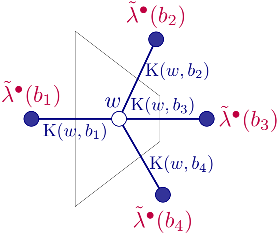

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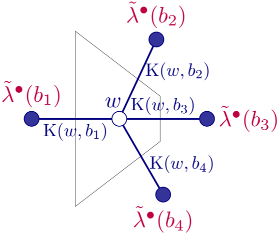

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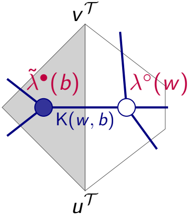

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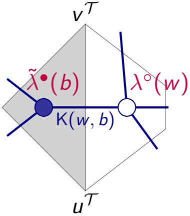

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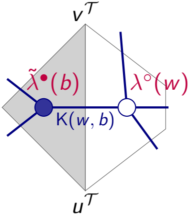

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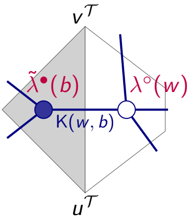

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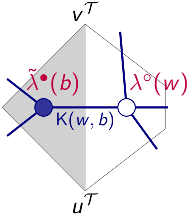

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If $(\lambda, \tilde{\lambda}) \in \mathcal{K}^{+}_{k,n}$*,* $\lambda \subset C$*, and* $C \in \mathrm{Gr}_{\geqslant 0}(k, n)$*, then* $C \cdot Q_\lambda \in \mathrm{Gr}_{\geqslant 0}(k-2, n)$*.*

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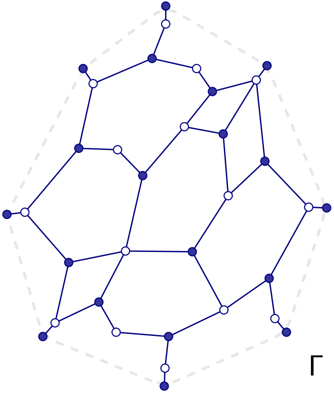
Shift by 2 for planar bipartite graphs ("T-duality")



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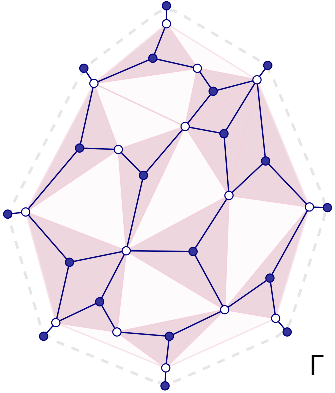

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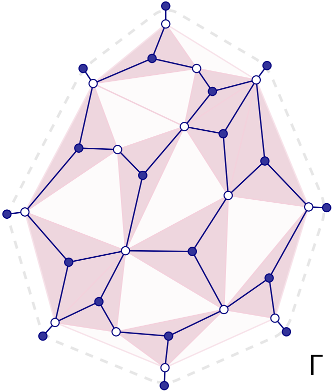
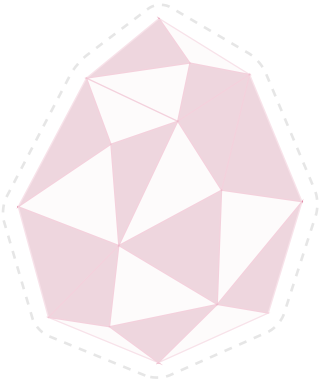


$\longrightarrow$



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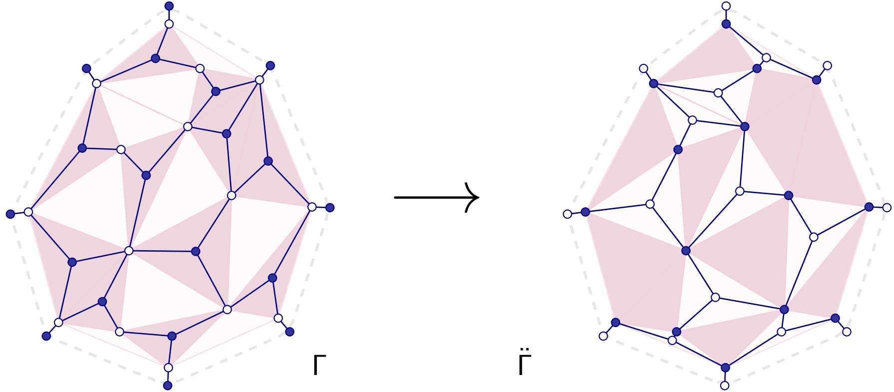

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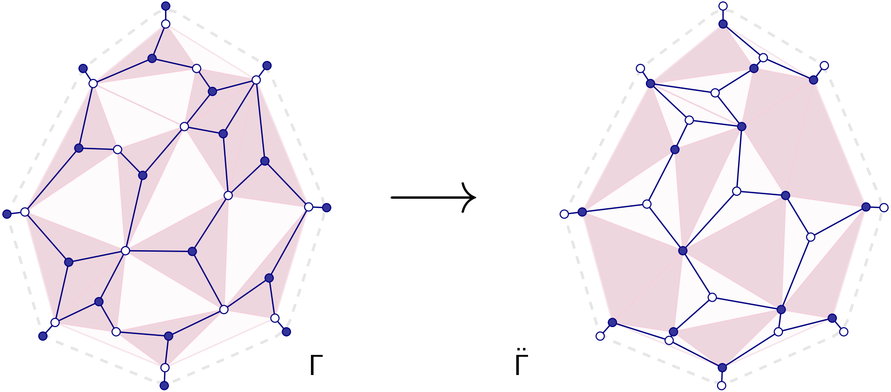

- [G. '17], [G.–Postnikov–Williams '19], [Balitskiy–Wellman '19], [Lukowski–Parisi–Williams '20], [Parisi–Sherman-Bennett–Williams '21].

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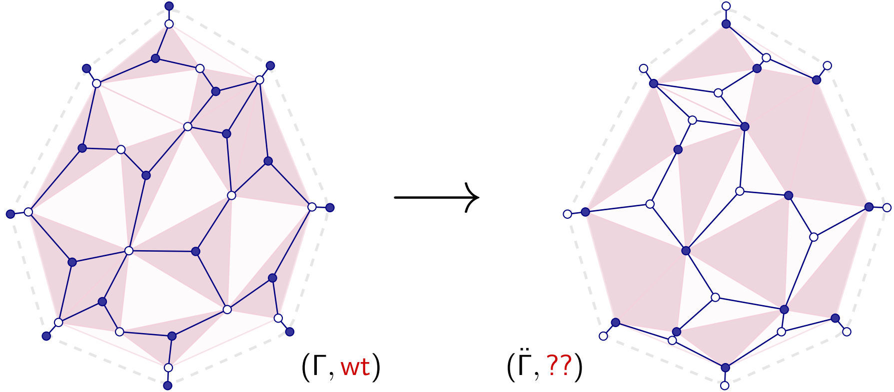

$(\Gamma, \mathrm{wt}) \longrightarrow (\ddot{\Gamma}, ??)$

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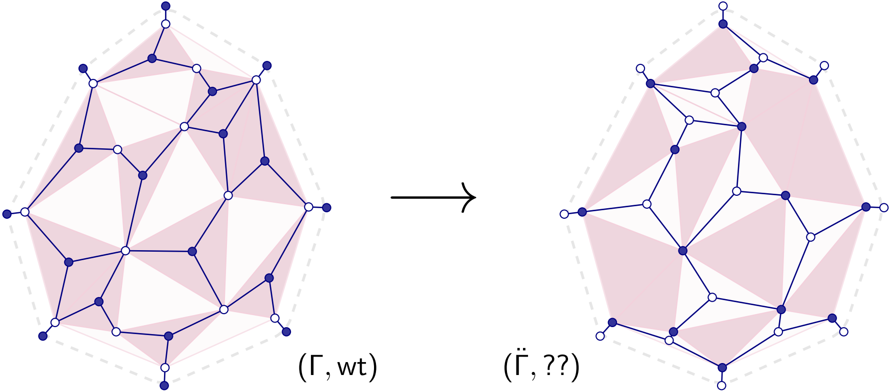

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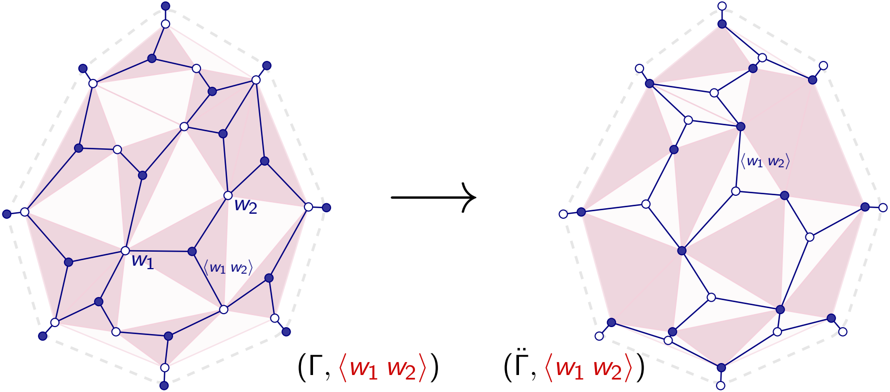

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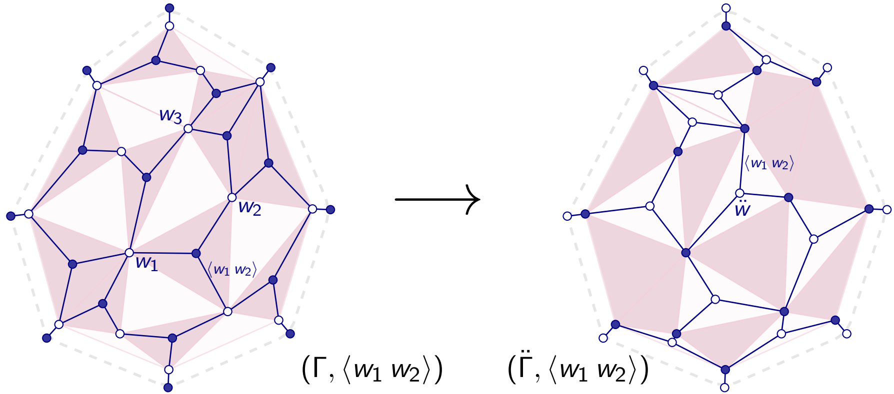

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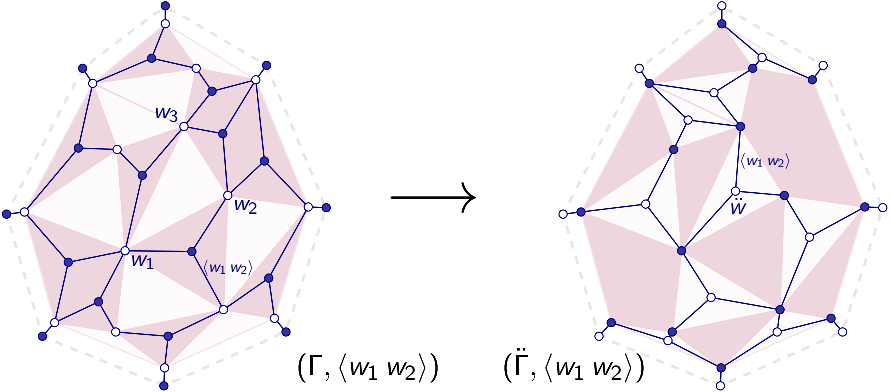

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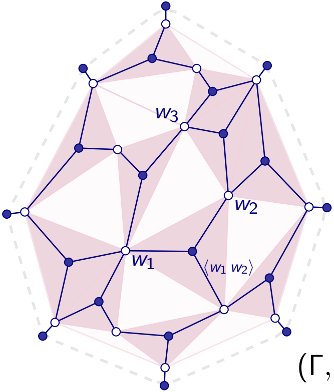

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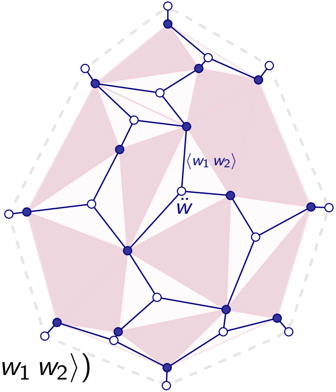

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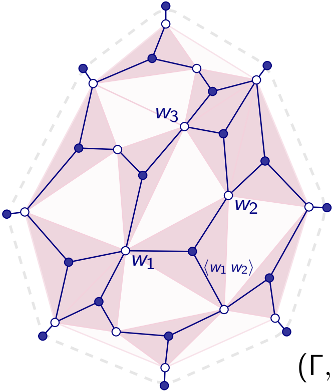

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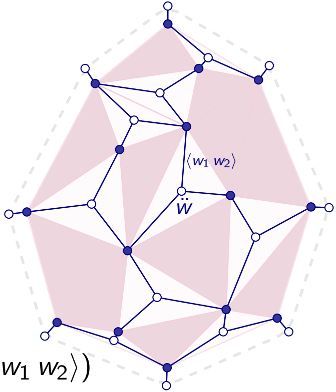

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- Originally conjectured by [Kenyon–Lam–Ramassamy–Russkikh '18], [Chelkak–Laslier–Russkikh '21].

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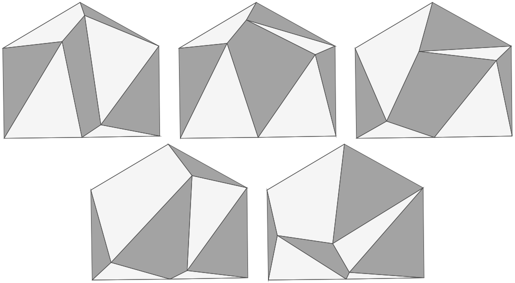

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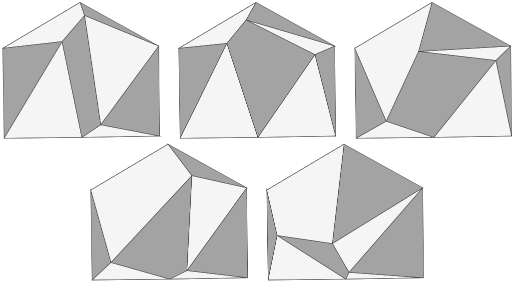

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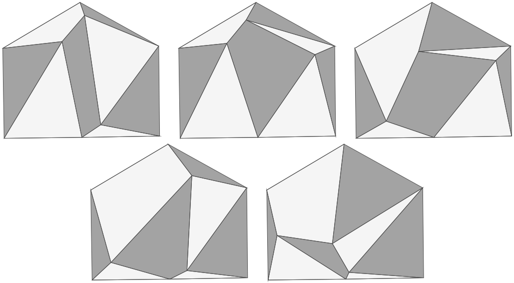

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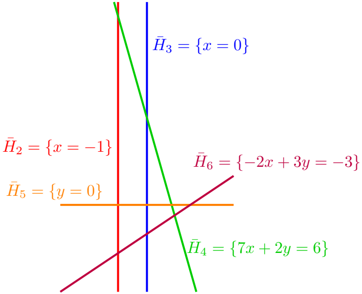

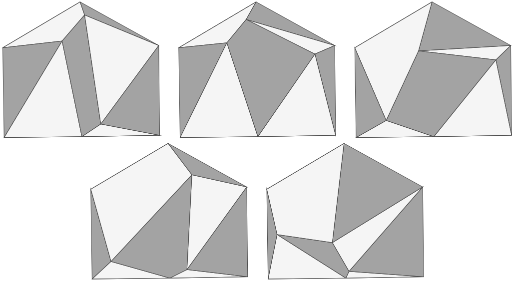

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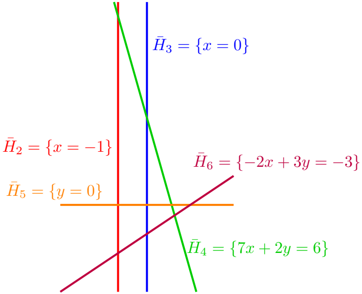

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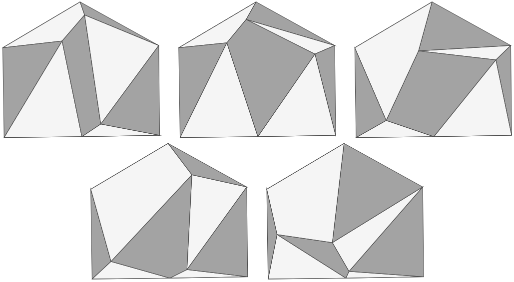

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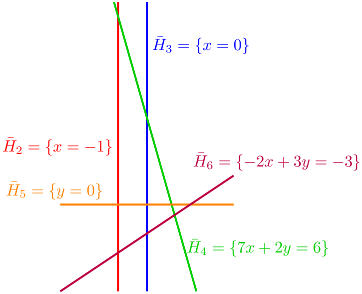

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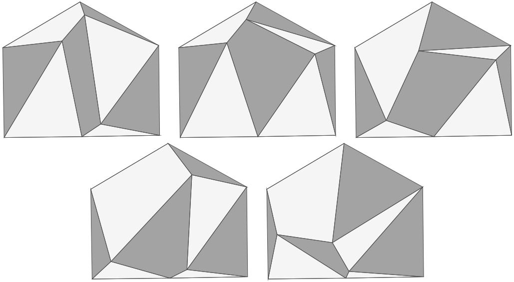

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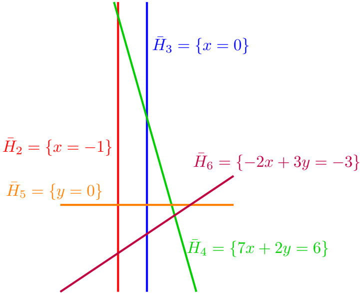

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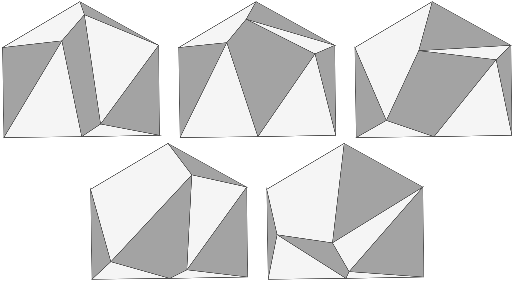

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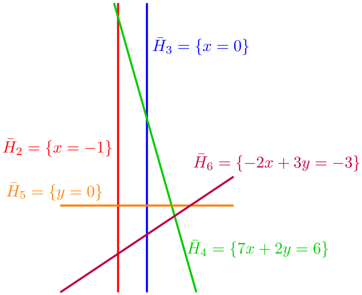

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