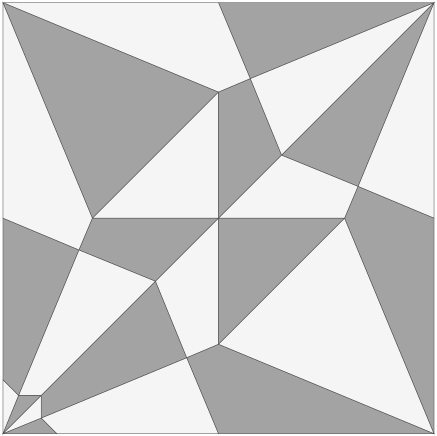
Amplituhedra and origami

Pavel Galashin (UCLA/Cornell)

Solvable Lattice Models & Interacting Particle Systems

August 26, 2025

Origami crease patterns (OCP)



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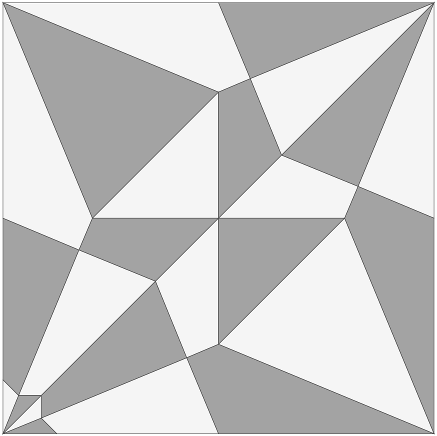

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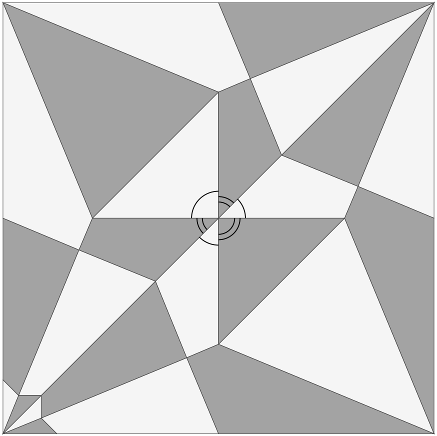

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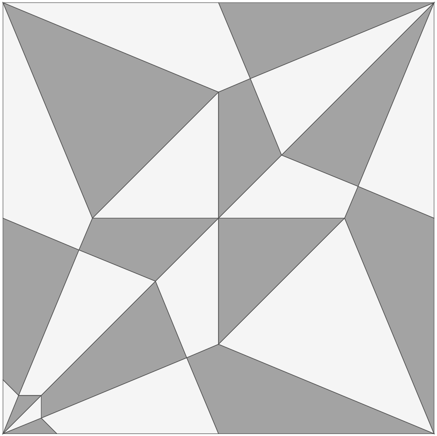

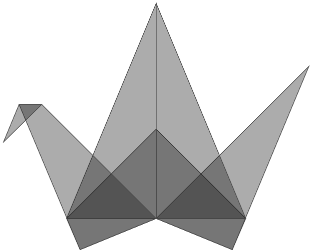

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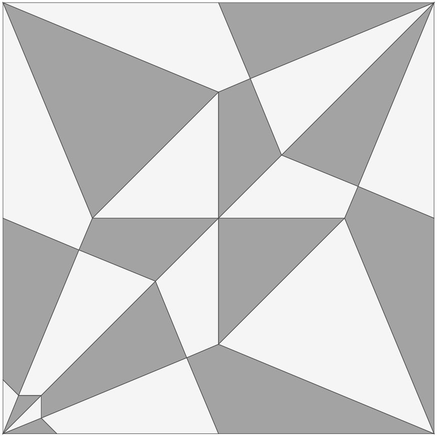

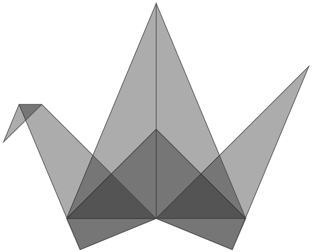

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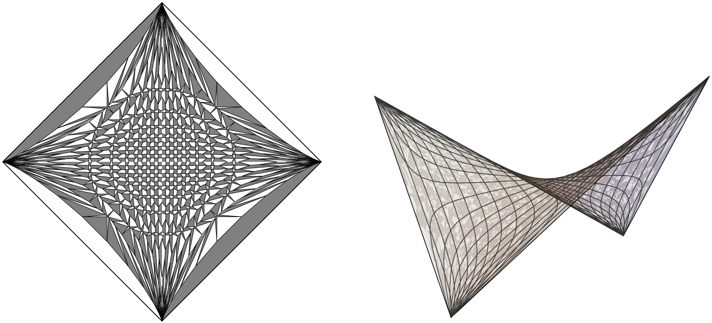
Convergence results for dimer model observables on OCPs:
[Kenyon–Lam–Ramassamy–Russkikh '18], [Chelkak–Laslier–Russkikh '21]



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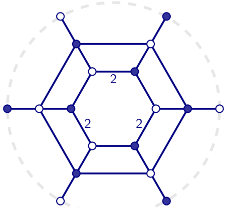

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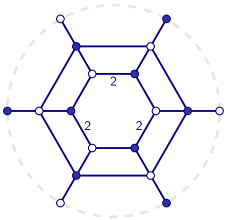

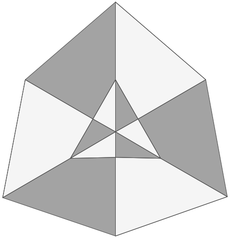

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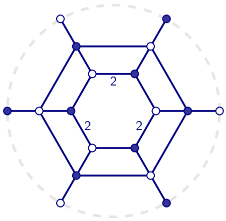

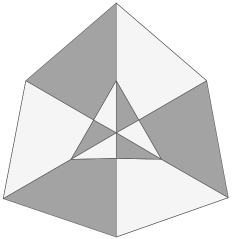

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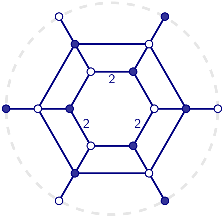

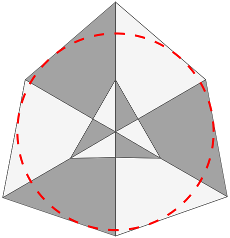

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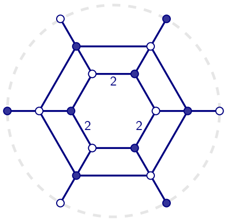
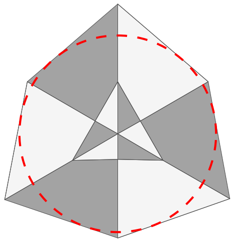
Theorem (G. '24)

Part 1 is true.





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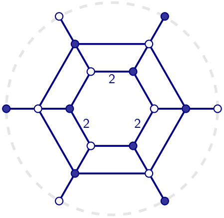

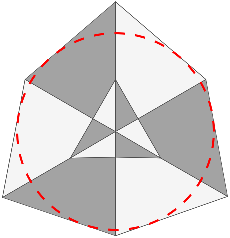

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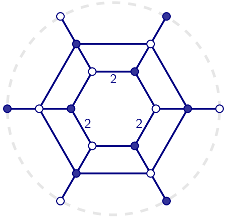

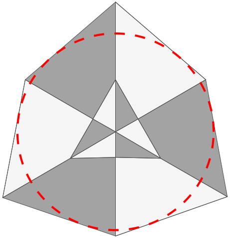

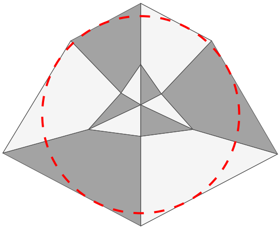

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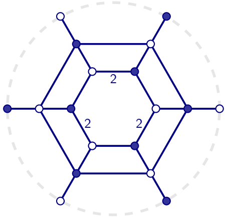

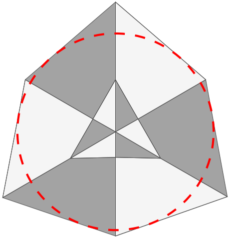

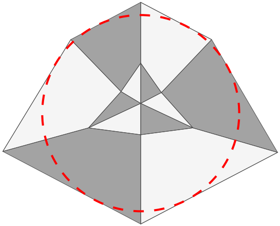

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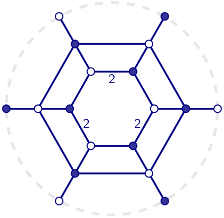
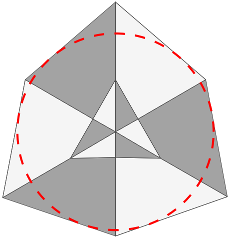
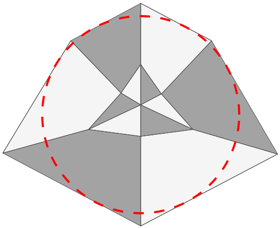
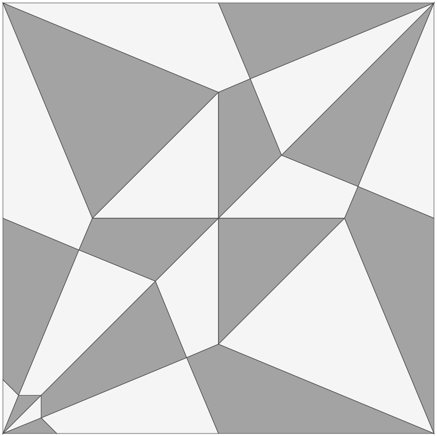
Proof idea: Total positivity + physics of scattering amplitudes.





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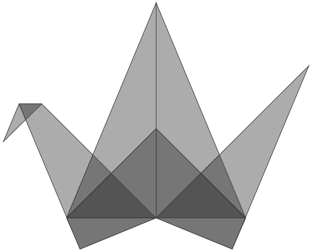

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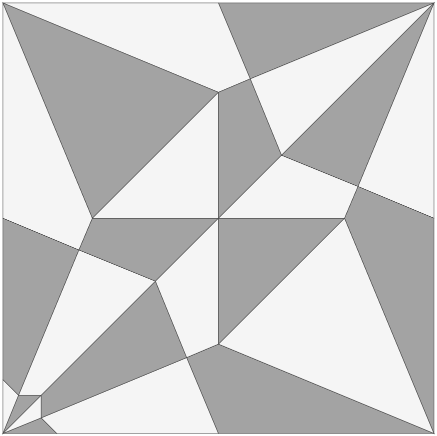

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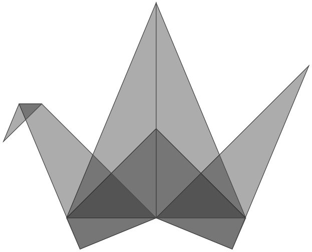

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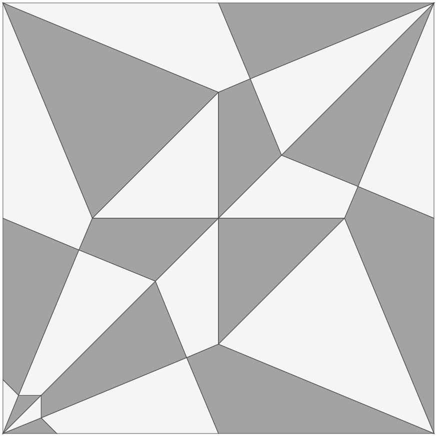

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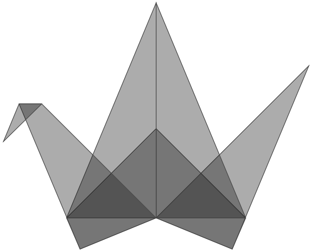

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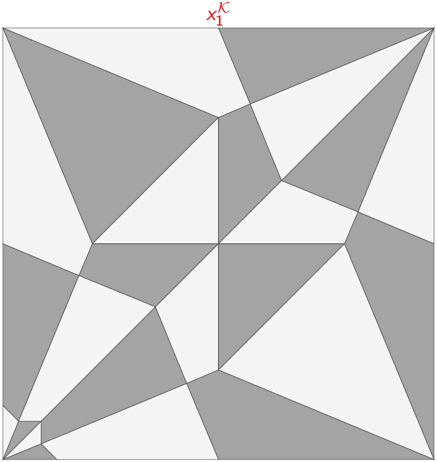

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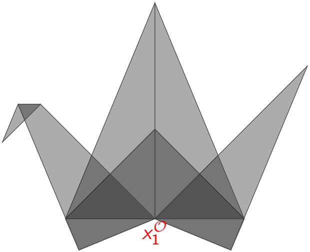

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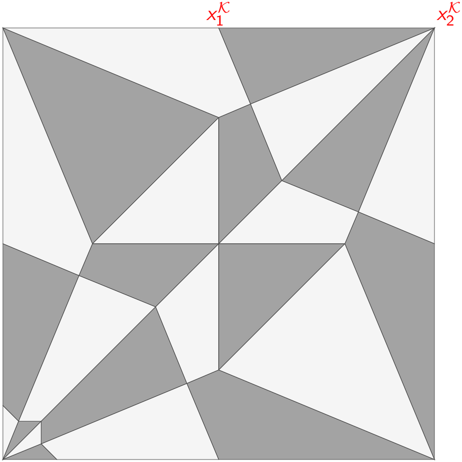

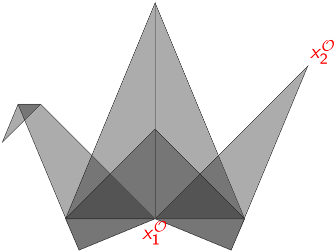

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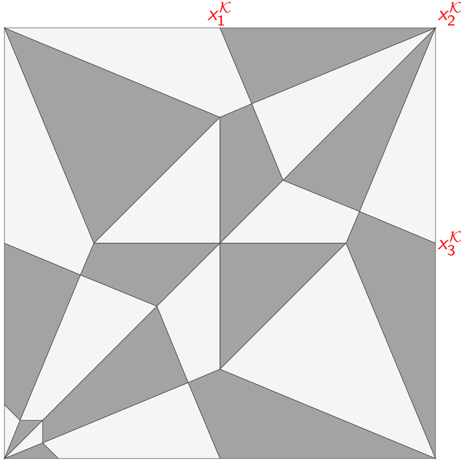

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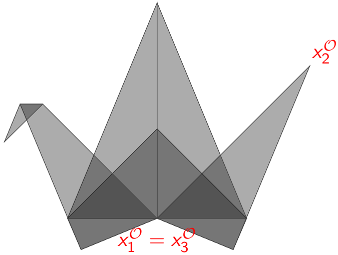

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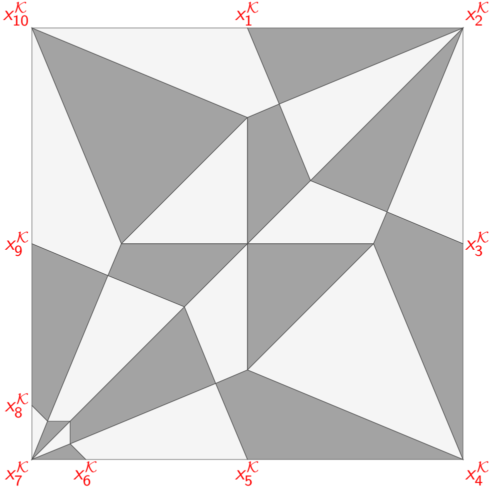

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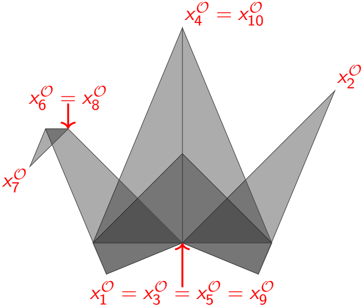

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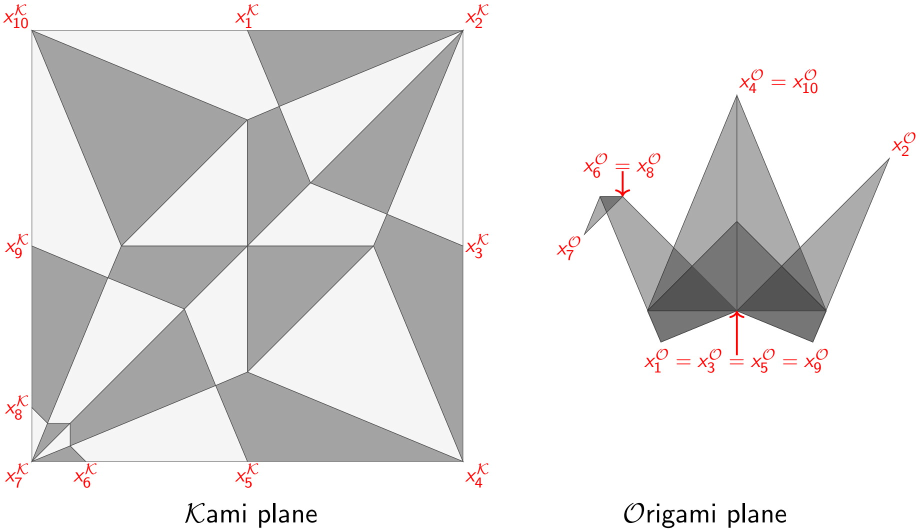

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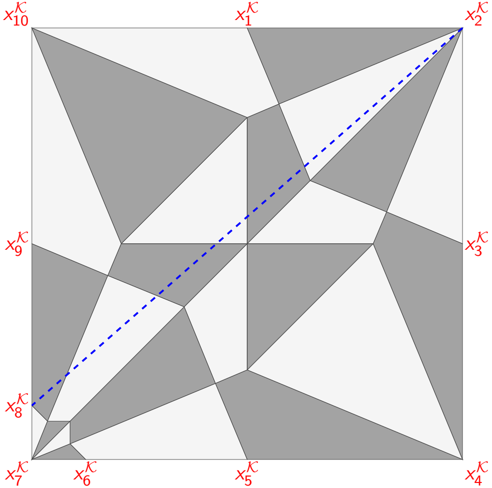

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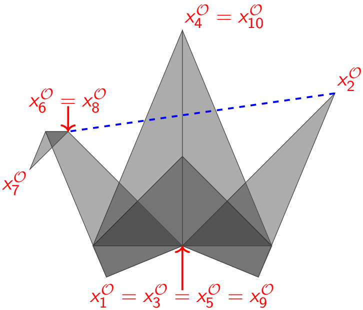

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- Minkowski distance vanishes on each white/black face.
- Origami map does not increase distances: $(x_i - x_j)^2 \geqslant 0$, i.e., $|x_i^{\mathcal{K}} - x_j^{\mathcal{K}}| \geqslant |x_i^{\mathcal{O}} - x_j^{\mathcal{O}}|$.

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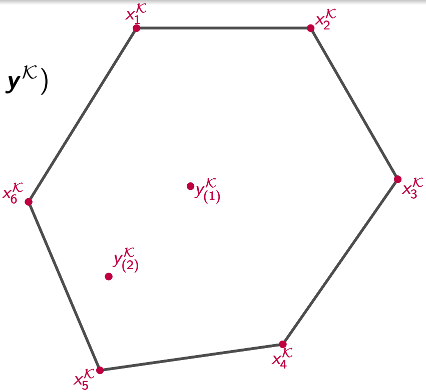
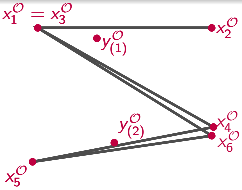
Definition

Let $L \geqslant 0$. An *L-punctured polygon* is a pair $(\mathcal{P}, \boldsymbol{y})$, where $\mathcal{P} = (x_1, x_2, \ldots, x_n) \in (\mathbb{R}^{2,2})^n$ and $\boldsymbol{y} = (y_{(1)}, y_{(2)}, \ldots, y_{(L)}) \in (\mathbb{R}^{2,2})^L$, such that

$(\mathcal{P}^{\mathcal{K}}, \boldsymbol{y}^{\mathcal{K}})$





$(\mathcal{P}^{\mathcal{O}}, \boldsymbol{y}^{\mathcal{O}})$

- Origami map does not increase distances: $(x_i - x_j)^2 \geqslant 0$, i.e., $|x_i^{\mathcal{K}} - x_j^{\mathcal{K}}| \geqslant |x_i^{\mathcal{O}} - x_j^{\mathcal{O}}|$.

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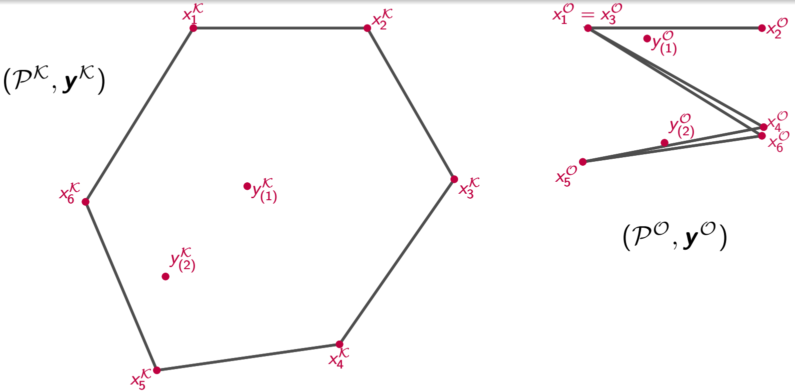
$$(x_i - x_{i-1})^2 = 0, \quad (x_i - x_j)^2 > 0, \quad (x_i - y_{(\rho)})^2 > 0, \quad (y_{(\rho)} - y_{(\gamma)})^2 > 0$$

for all non-adjacent $1 \leqslant i, j \leqslant n$ and all $1 \leqslant \rho \neq \gamma \leqslant L$,



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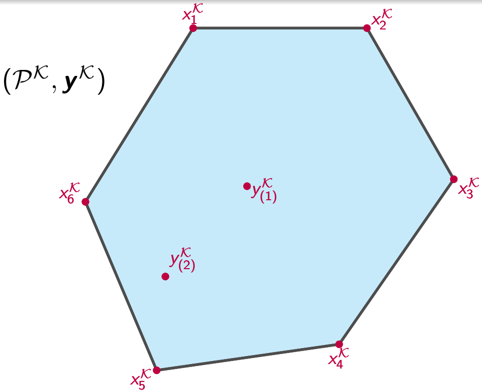
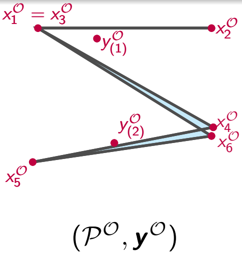
for all non-adjacent $1 \leqslant i, j \leqslant n$ and all $1 \leqslant \rho \neq \gamma \leqslant L$, and such that

each $y_{(\rho)}^{\mathcal{K}}$ is located inside the polygon $\mathcal{P}^{\mathcal{K}} = (x_1^{\mathcal{K}}, x_2^{\mathcal{K}}, \ldots, x_n^{\mathcal{K}})$.





- Origami map does not increase distances: $(x_i - x_j)^2 \geqslant 0$, i.e., $|x_i^{\mathcal{K}} - x_j^{\mathcal{K}}| \geqslant |x_i^{\mathcal{O}} - x_j^{\mathcal{O}}|$.

Definition

Let $L \geqslant 0$. An L-punctured polygon is a pair $(\mathcal{P}, \boldsymbol{y})$, where $\mathcal{P} = (x_1, x_2, \dots, x_n) \in (\mathbb{R}^{2,2})^n$ and $\boldsymbol{y} = (y_{(1)}, y_{(2)}, \dots, y_{(L)}) \in (\mathbb{R}^{2,2})^L$, such that

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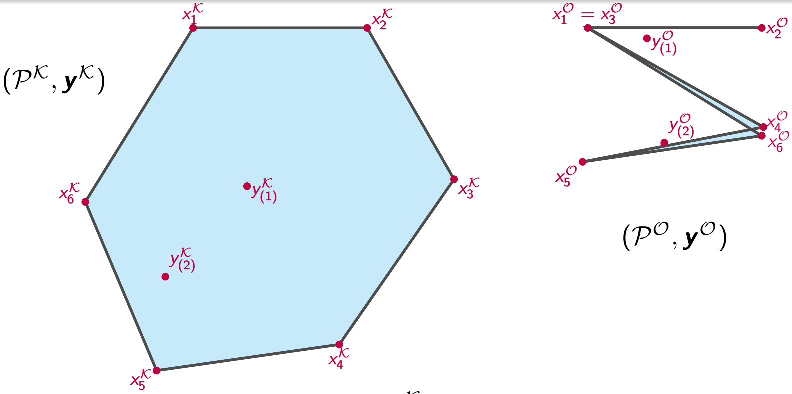

Problem: Find an OCP with boundary $\mathcal{P}^{\mathcal{K}}$ such that the origami map sends

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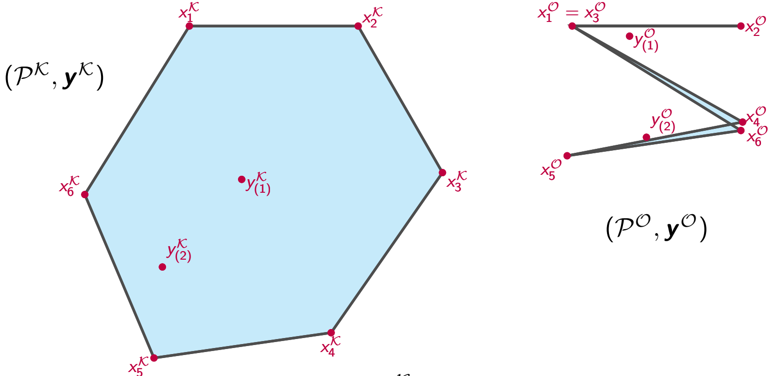

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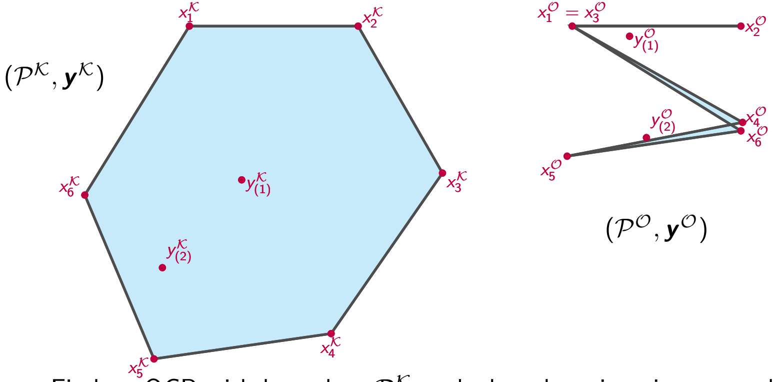

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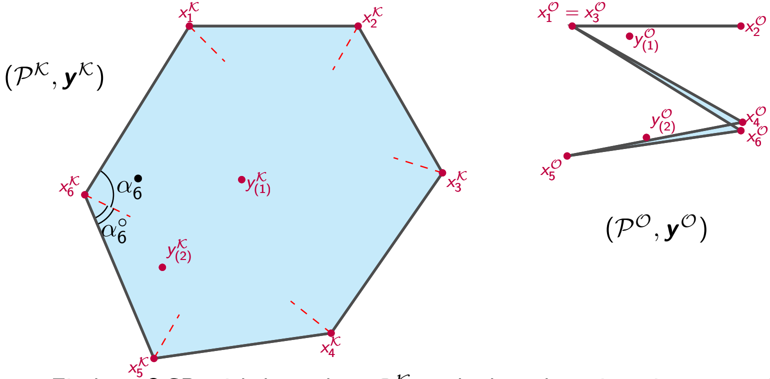
- Recover white/black angle sums $(\alpha_i^\circ, \alpha_i^\bullet)$ from the geometry of $\mathcal{P}$:

$$\alpha_i^\circ + \alpha_i^\bullet = \alpha_i^{\mathcal{K}} \quad \text{and} \quad \alpha_i^\circ - \alpha_i^\bullet \equiv \alpha_i^{\mathcal{O}} \bmod 2\pi.$$



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- Recover white/black angle sums $(\alpha_i^{\circ}, \alpha_i^{\bullet})$ from the geometry of $\mathcal{P}$:

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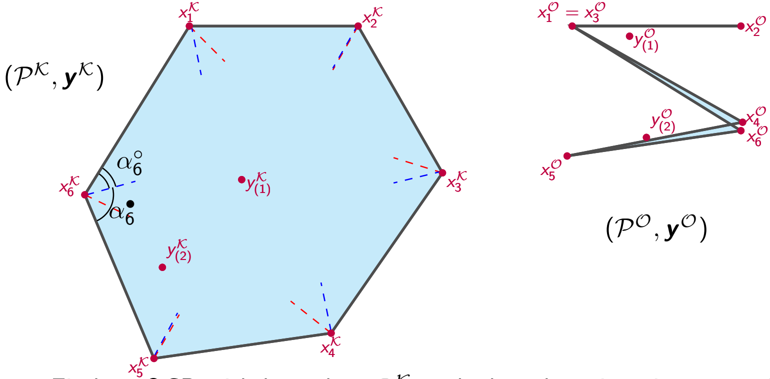
- Red folding ray splits the angle $\alpha_i^{\mathcal{K}}$ into angles $(\alpha_i^{\bullet}, \alpha_i^{\circ})$.



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- Recover white/black angle sums $(\alpha_i^{\circ}, \alpha_i^{\bullet})$ from the geometry of $\mathcal{P}$:

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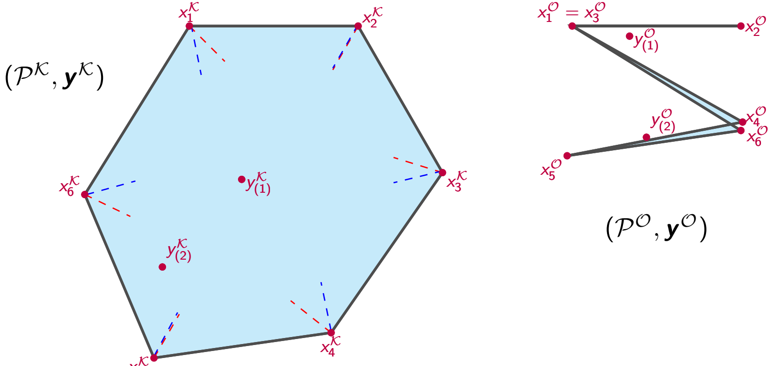
- Red folding ray splits the angle $\alpha_i^{\mathcal{K}}$ into angles $(\alpha_i^{\bullet}, \alpha_i^{\circ})$.
- Blue folding ray splits the angle $\alpha_i^{\mathcal{K}}$ into angles $(\alpha_i^{\circ}, \alpha_i^{\bullet})$.



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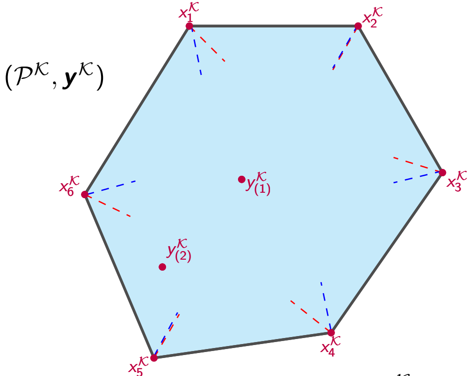
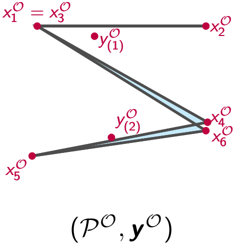
Algorithm (Crystallization algorithm)





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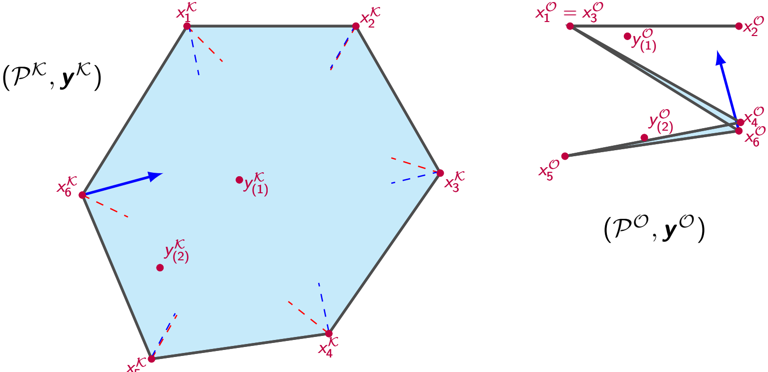

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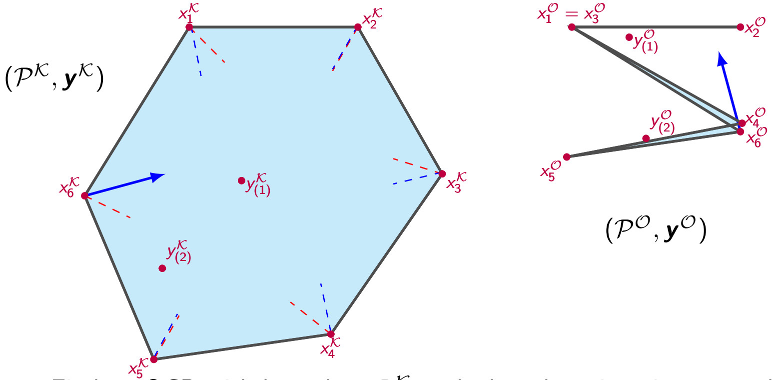

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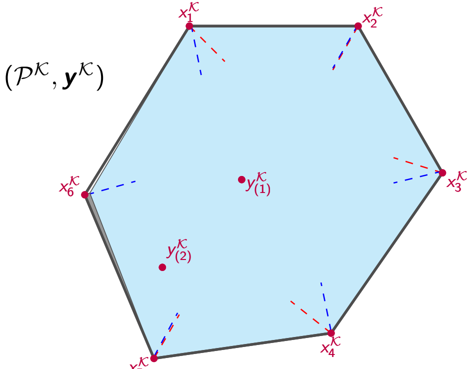

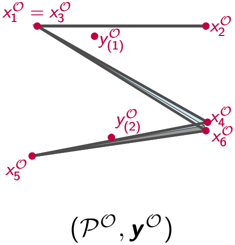

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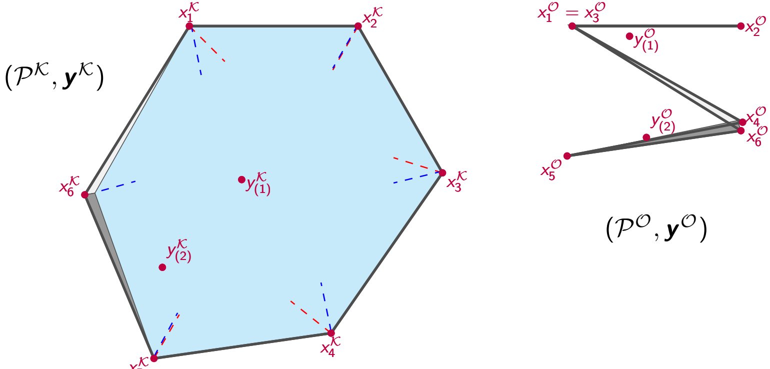

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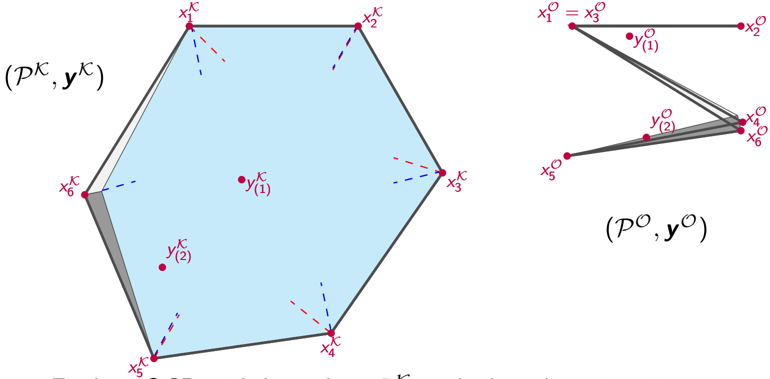

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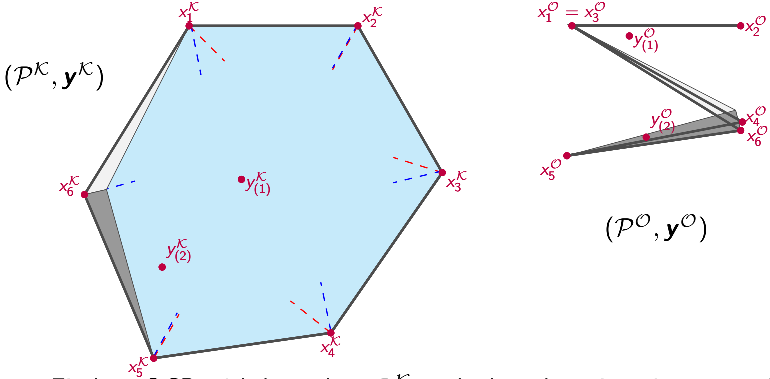

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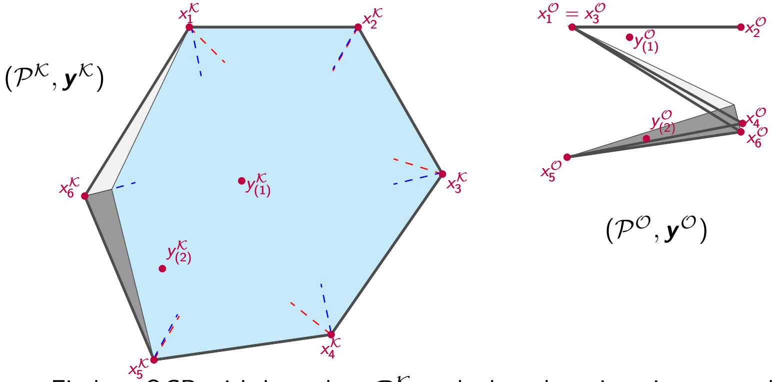

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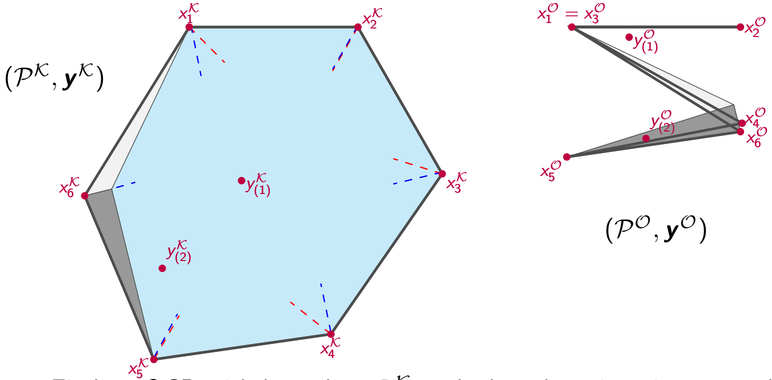

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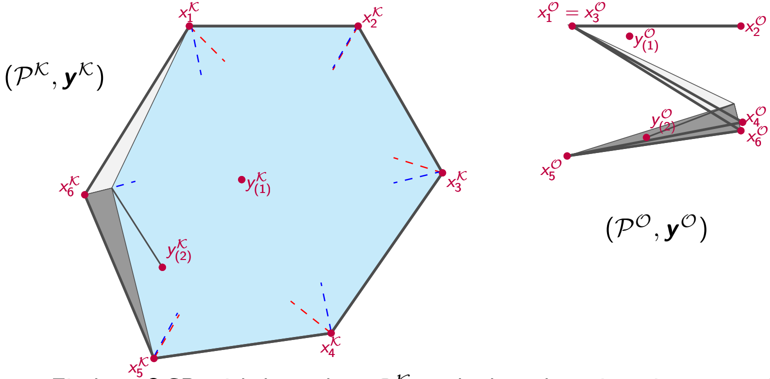

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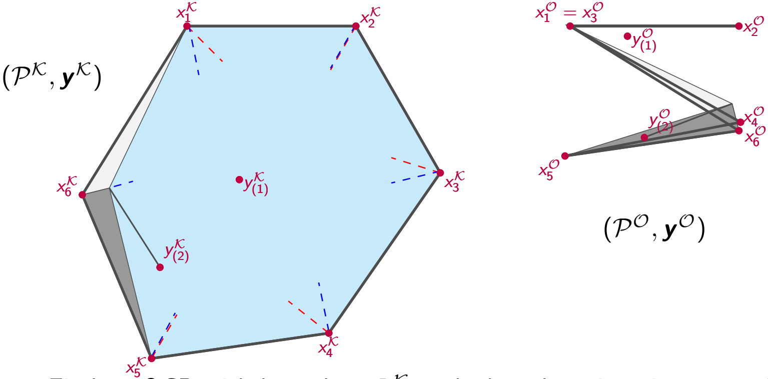

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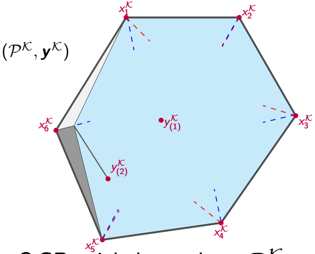

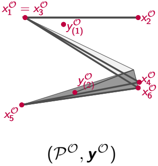

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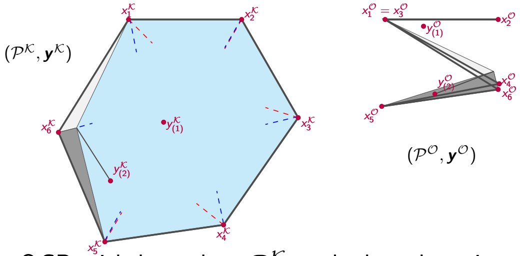

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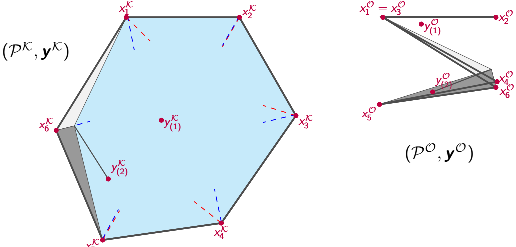
Theorem (G.'25+)

The crystallization algorithm works.



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Theorem (G.'25+)

The crystallization algorithm works.
For generic L-punctured polygons $(\mathcal{P}, \mathbf{y})$, it outputs a valid, embedded OCP.

Scattering amplitudes

Input: n particles with incoming momenta $P_1, P_2, \ldots, P_n \in \mathbb{R}^{2,2}$ satisfying

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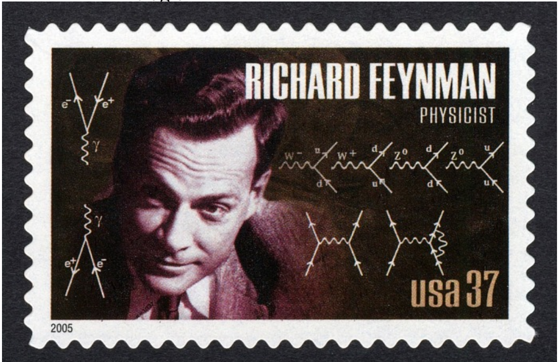

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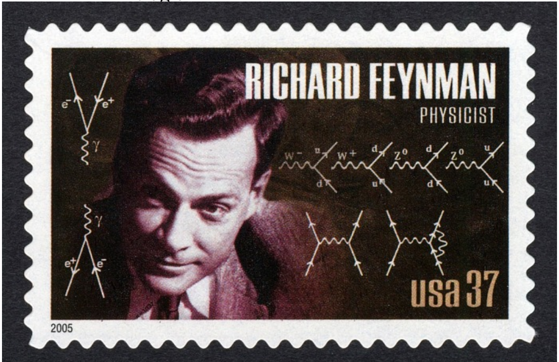

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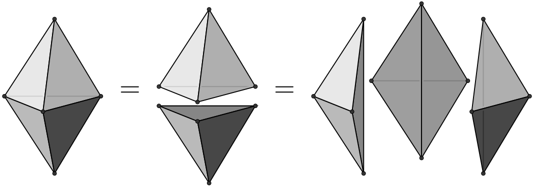

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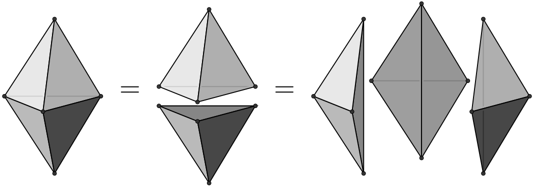

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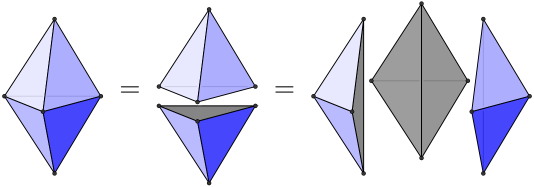

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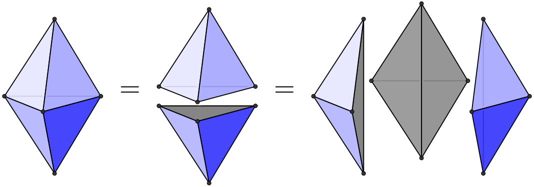

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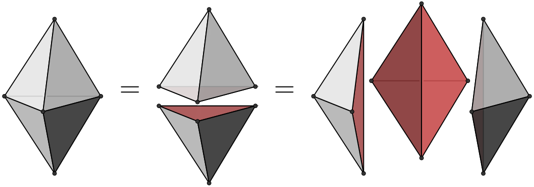

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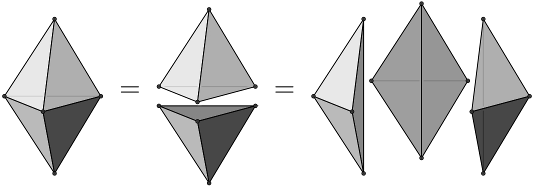

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Conjecture (BCFW triangulation conjecture [AHT'14])

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- The tree amplituhedron corresponds to $A^{(\text{tree})}$, and the more general loop amplituhedron corresponds to $A^{(L)}$ with arbitrary number $L \geqslant 0$ of loops.

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All 4 conjectures are true.

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- Loop momentum amplituhedron $\mathcal{M}_{k,n}^{(L)}$ is a linear slice of the space $\left\{(\mathcal{P}, \boldsymbol{y}) \mid (x_i - x_j)^2, (x_i - y_{(\rho)})^2, (y_{(\rho)} - y_{(\gamma)})^2 > 0, \text{ and } y_{(\rho)}^{\mathcal{K}} \text{ is inside } \mathcal{P}^{\mathcal{K}}\right\}$ of L-punctured polygons.

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- Loop momentum amplituhedron $\mathcal{M}^{(L)}_{k,n}$ is a linear slice of the space
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- With this formulation, (4) becomes equivalent to the previous result that the crystallization algorithm always outputs a valid, embedded OCP.

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- With this formulation, (4) becomes equivalent to the previous result that the crystallization algorithm always outputs a valid, embedded OCP.
- Our approach also implies that (2) is equivalent to (4). □

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$$C = \begin{pmatrix} 1 & b & 0 & -a \\ 0 & c & 1 & d \end{pmatrix} \quad \in \text{Gr}(2,4)$$

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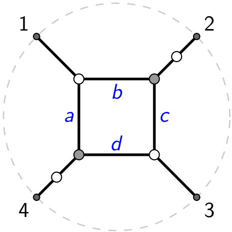

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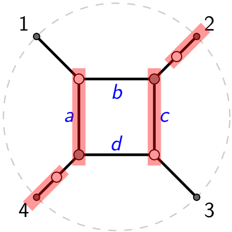

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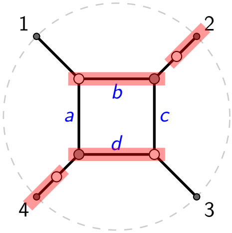

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Theorem (G. '24, "Main bijection")

Embedded OCPs are in natural bijection with triples* $\lambda \subset C \subset \tilde{\lambda}^{\perp}$ *such that* $C \in \mathrm{Gr}_{\geqslant 0}(k,n)$ *and* $(\lambda, \tilde{\lambda})$ *belongs to the tree momentum amplituhedron:*

$$\mathcal{M}_{k,n}^{(tree)} := \left\{ \quad \middle| \quad \right\}.$$

$$C = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{pmatrix}$$

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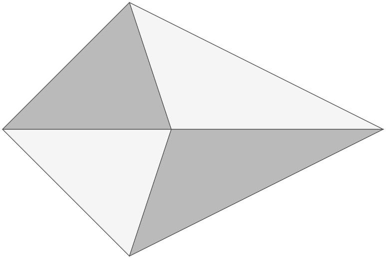

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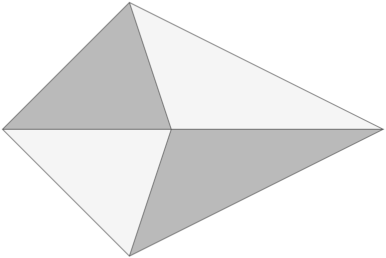

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- $C \longleftrightarrow$ weighted planar bipartite graph Γ in a disk

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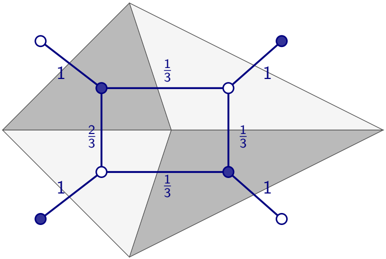

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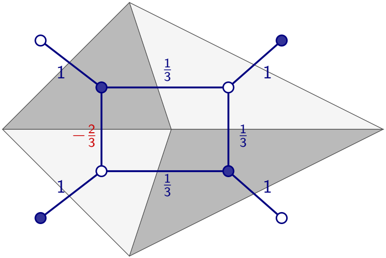

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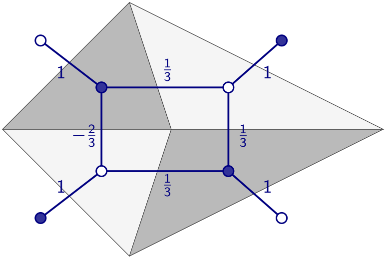

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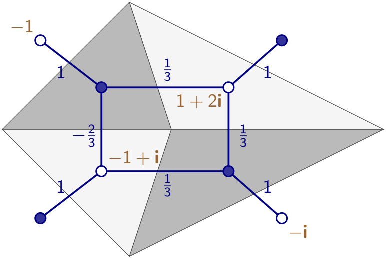

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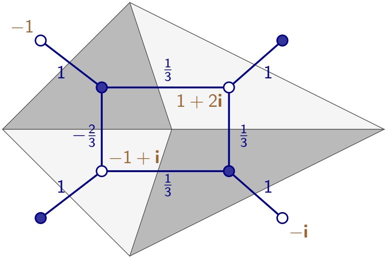

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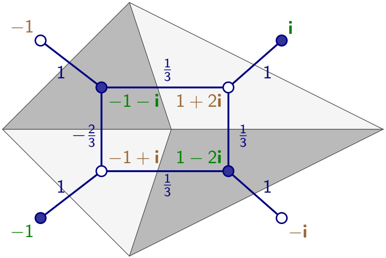

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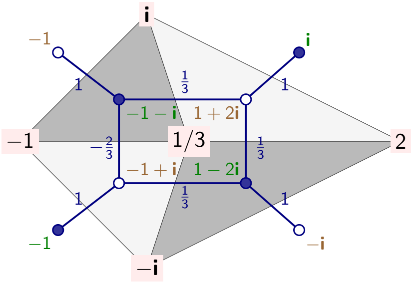

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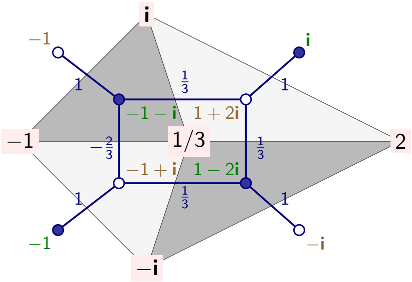

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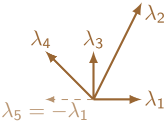

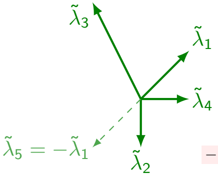

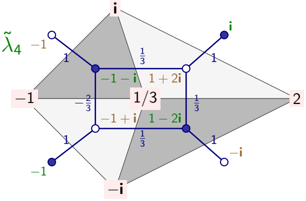

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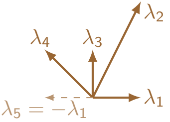

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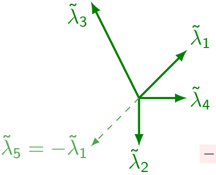

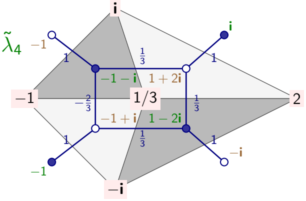

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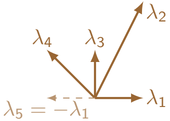

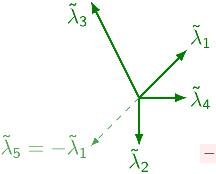

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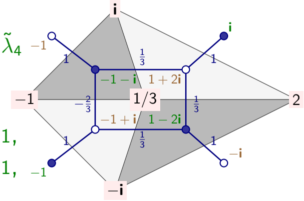

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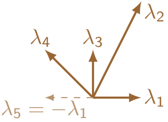

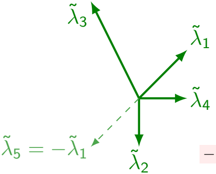

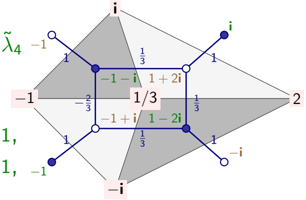

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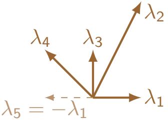

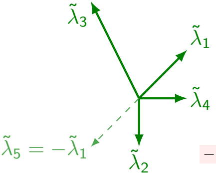

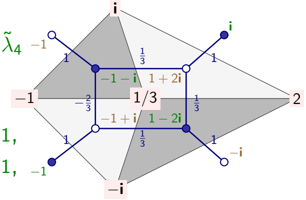

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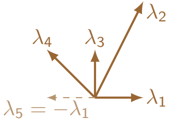

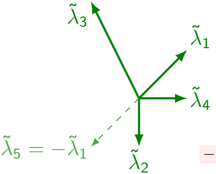

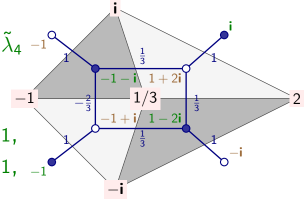

$\langle i\, i+1 \rangle := \det(\lambda_i | \lambda_{i+1})$ $\quad \langle 1\,2 \rangle = 2,\ \langle 2\,3 \rangle = 1,$ $\quad [1\,2] = -1,\ [2\,3] = -1,$

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- $C \longleftrightarrow$ weighted planar bipartite graph Γ in a disk (pick Kasteleyn signs);
- $\lambda \subset C$ extends uniquely to a white-holomorphic function $\lambda^{\circ} : \mathbf{V}^{\circ}(\Gamma) \to \mathbb{C}$;
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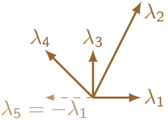

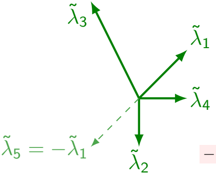

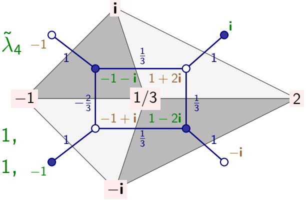

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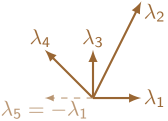

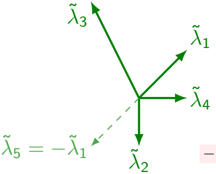

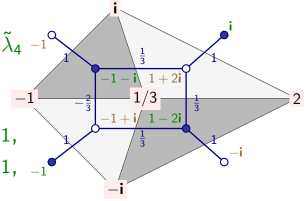

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Geometric condition	Momentum space	T-dual space
Embedded OCP: $\alpha_v^\circ = \alpha_v^\bullet = \pi$		

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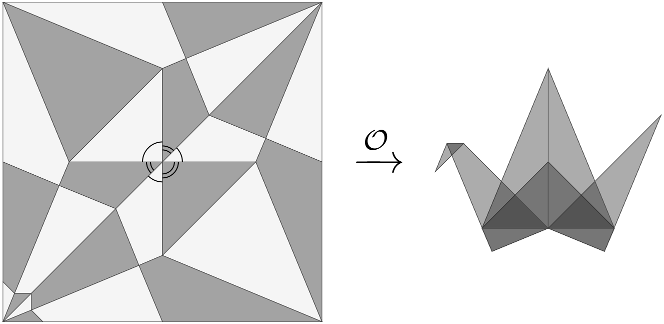

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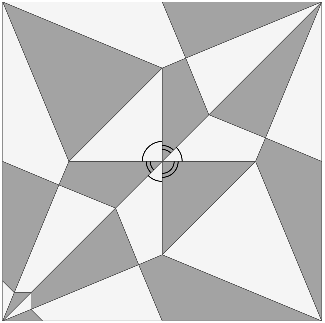
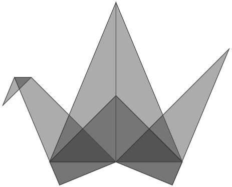


$\xrightarrow{\mathcal{O}}$



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- Define the magic projector $Q_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ with kernel λ,

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Geometric condition	Momentum space	T-dual space
Embedded OCP: $\alpha_v^\circ = \alpha_v^\bullet = \pi$	$\lambda \subset C \subset \tilde\lambda^\perp$: $(\lambda,\tilde\lambda) \in \mathcal{M}_{k,n}^{(\text{tree})}$, $C \in \mathrm{Gr}_{\geqslant 0}(k,n)$	$D = Q_\lambda(C)$, $\nu = Q_\lambda^{-1}(\tilde\lambda)$ $\nu \in \mathcal{A}_{k-2,n}^{(\text{tree})}$, $D \in \mathrm{Gr}_{\geqslant 0}(k-2,n)$
Black faces are embedded with correct orientation		$D \in \mathrm{Gr}_{\geqslant 0}(k-2,n)$

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- Repeat for white faces. □

Geometric condition	Momentum space	T-dual space
Embedded OCP: $\alpha_v^\circ = \alpha_v^\bullet = \pi$	$\lambda \subset C \subset \tilde{\lambda}^\perp$: $(\lambda, \tilde{\lambda}) \in \mathcal{M}_{k,n}^{(\text{tree})}$, $C \in \mathrm{Gr}_{\geqslant 0}(k, n)$	$D = Q_\lambda(C)$, $\nu = Q_\lambda^{-1}(\tilde{\lambda})$ $\nu \in \mathcal{A}_{k-2,n}^{(\text{tree})}$, $D \in \mathrm{Gr}_{\geqslant 0}(k-2, n)$
Black faces are embedded with correct orientation		$D \in \mathrm{Gr}_{\geqslant 0}(k-2, n)$

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Geometric condition	Momentum space	T-dual space
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Black faces are embedded with correct orientation		$D \in \mathrm{Gr}_{\geqslant 0}(k-2,n)$
Crystallization algorithm works		

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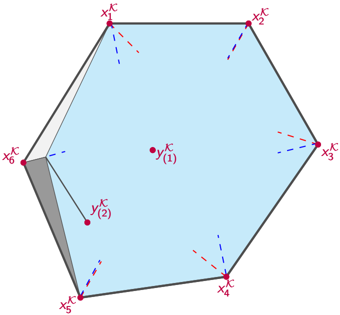

Theorem (G.'25+)

The crystallization algorithm works. For generic L-punctured polygons $(\mathcal{P}, \mathbf{y})$, it outputs a valid, embedded OCP.

Geometric condition	Momentum space	T-dual space
Embedded OCP: $\alpha_v^\circ = \alpha_v^\bullet = \pi$	$\lambda \subset C \subset \tilde{\lambda}^\perp$: $(\lambda, \tilde{\lambda}) \in \mathcal{M}_{k,n}^{(\text{tree})}$, $C \in \mathrm{Gr}_{\geqslant 0}(k,n)$	$D = Q_\lambda(C)$, $\nu = Q_\lambda^{-1}(\tilde{\lambda})$ $\nu \in \mathcal{A}_{k-2,n}^{(\text{tree})}$, $D \in \mathrm{Gr}_{\geqslant 0}(k-2,n)$
Black faces are embedded with correct orientation		$D \in \mathrm{Gr}_{\geqslant 0}(k-2,n)$
Crystallization algorithm works	BCFW cells triangulate $\mathcal{M}_{k,n}^{(L)}$	

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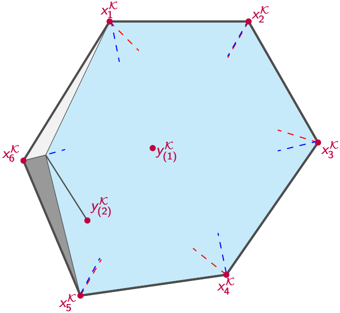

Conjecture (BCFW triangulation conjecture [AHT'14])

(1) BCFW cells triangulate tree T-dual amplituhedron.
(2) BCFW cells triangulate loop T-dual amplituhedron.

(3) BCFW cells triangulate tree momentum amplituhedron.
(4) BCFW cells triangulate loop momentum amplituhedron.

Theorem (G.'25+)

All 4 conjectures are true.

Geometric condition	Momentum space	T-dual space
Embedded OCP: $\alpha_v^\circ = \alpha_v^\bullet = \pi$	$\lambda \subset C \subset \tilde{\lambda}^\perp$: $(\lambda, \tilde{\lambda}) \in \mathcal{M}_{k,n}^{(\text{tree})}$, $C \in \mathrm{Gr}_{\geqslant 0}(k,n)$	$D = Q_\lambda(C)$, $\nu = Q_\lambda^{-1}(\tilde{\lambda})$ $\nu \in \mathcal{A}_{k-2,n}^{(\text{tree})}$, $D \in \mathrm{Gr}_{\geqslant 0}(k-2,n)$
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$$Q_\lambda(x_1,\dots,x_n) = (y_1,\dots,y_n), \quad y_i = \langle i\, i+1\rangle x_{i-1} + \langle i+1\, i-1\rangle x_i + \langle i-1\, i\rangle x_{i+1}.$$

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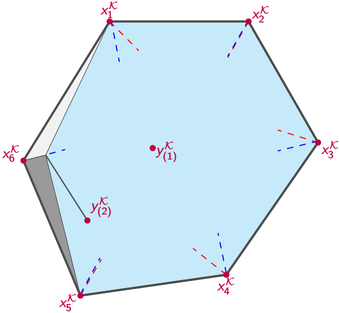

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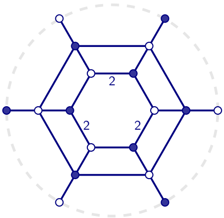

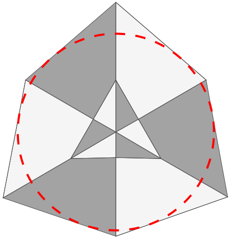

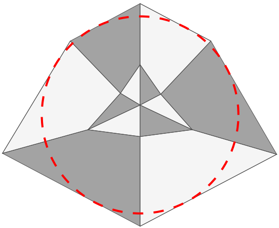

Geometric condition	Momentum space	T-dual space
Embedded OCP: $\alpha_v^\circ = \alpha_v^\bullet = \pi$	$\lambda \subset C \subset \tilde{\lambda}^\perp$: $(\lambda, \tilde{\lambda}) \in \mathcal{M}_{k,n}^{(\mathrm{tree})}$, $C \in \mathrm{Gr}_{\geqslant 0}(k,n)$	$D = Q_\lambda(C)$, $\nu = Q_\lambda^{-1}(\tilde{\lambda})$ $\nu \in \mathcal{A}_{k-2,n}^{(\mathrm{tree})}$, $D \in \mathrm{Gr}_{\geqslant 0}(k-2,n)$
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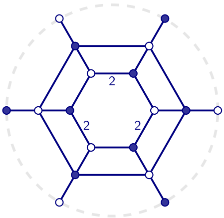

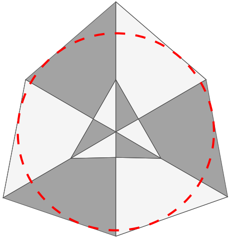

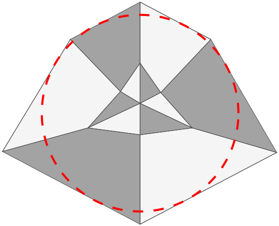

Geometric condition	Momentum space	T-dual space
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Perfect OCP: $n = 2k$, circle tangent to boundary, boundary angles: $\alpha_i^\circ = \alpha_i^\bullet$	$\tilde{\lambda} = Q_\lambda(\mathrm{alt}(\lambda))$	$\nu = \mathrm{alt}(\nu)$

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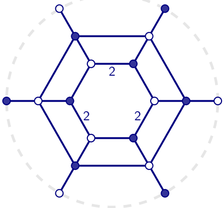

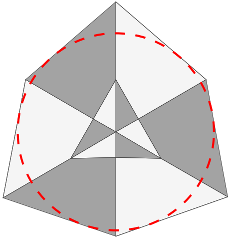

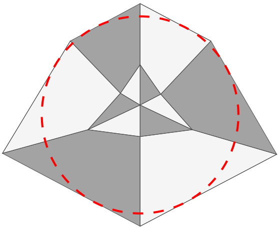

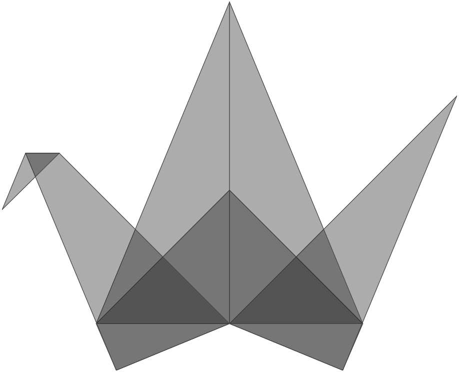

Thanks!