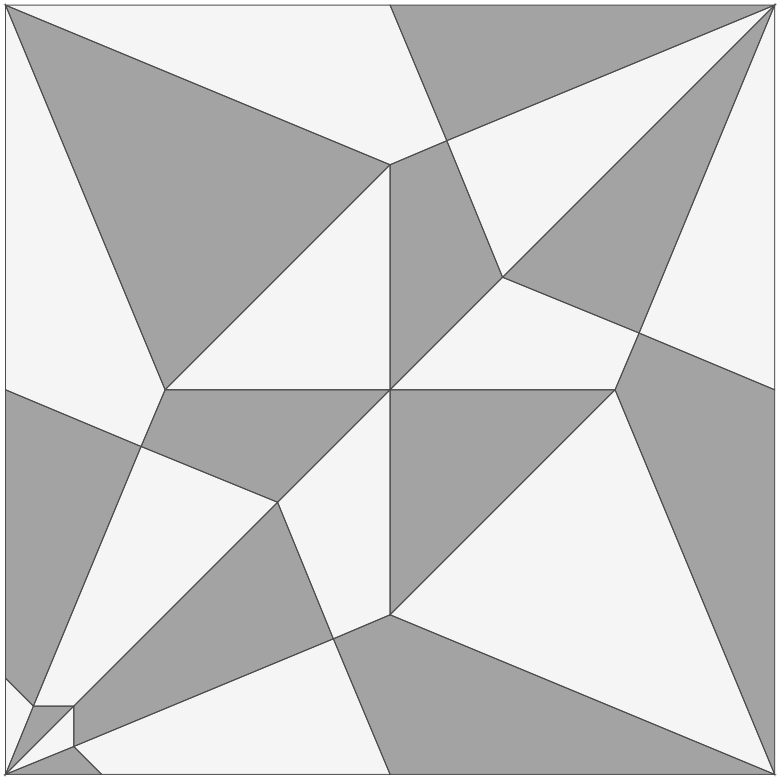
Amplituhedra and origami

Pavel Galashin (UCLA/Cornell)

Perfectly matched perspectives on statistical mechanics, combinatorics and geometry
A conference in honor of Richard Kenyon

June 16, 2025

Origami crease patterns (OCP)



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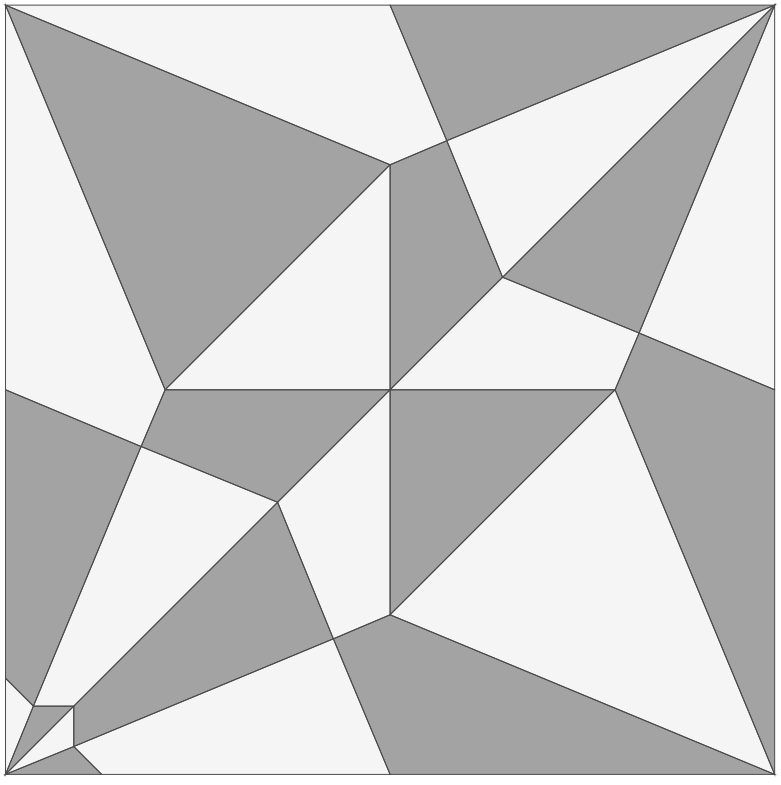

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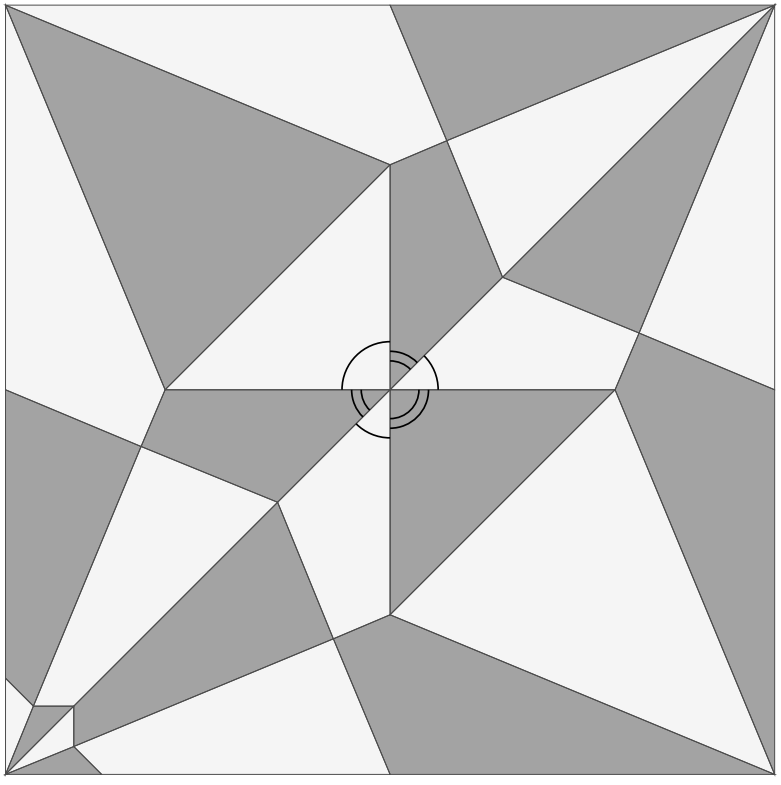

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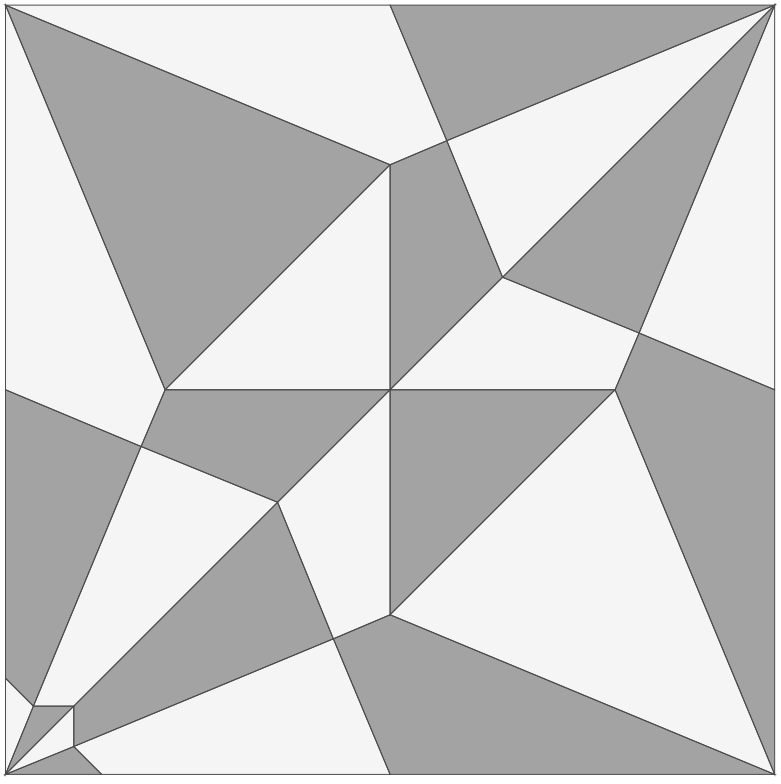

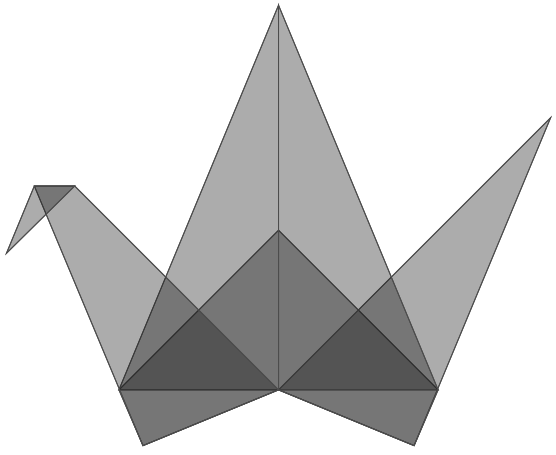

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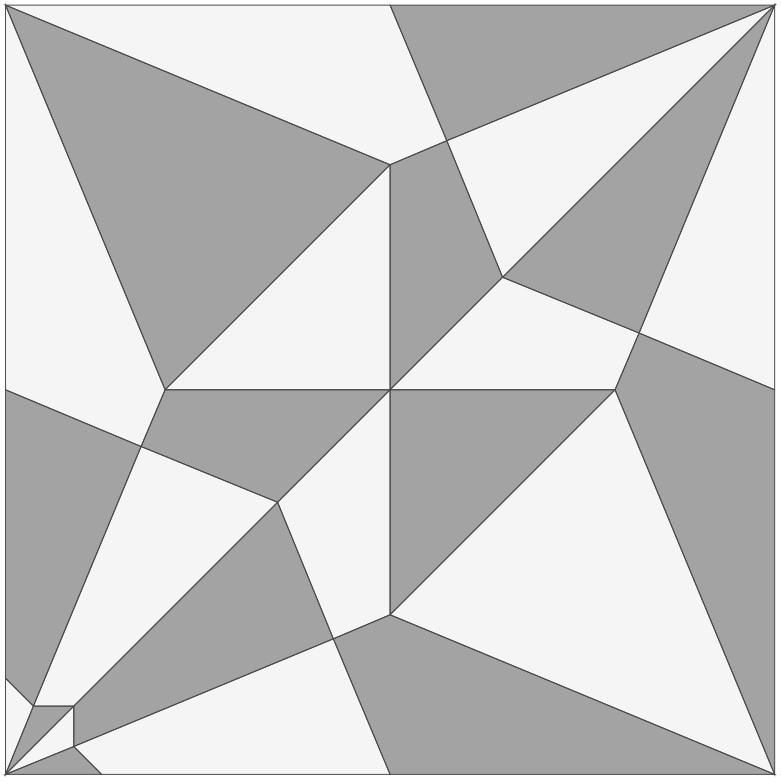

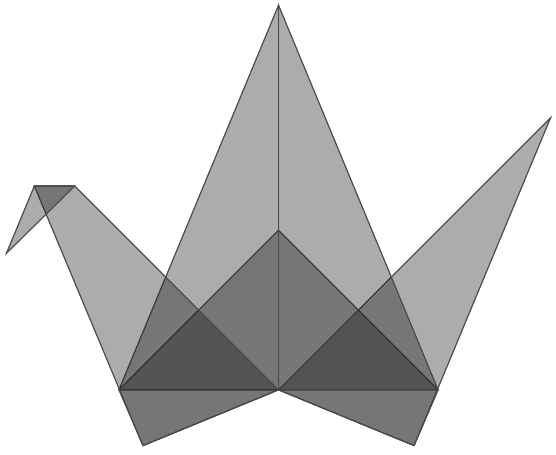

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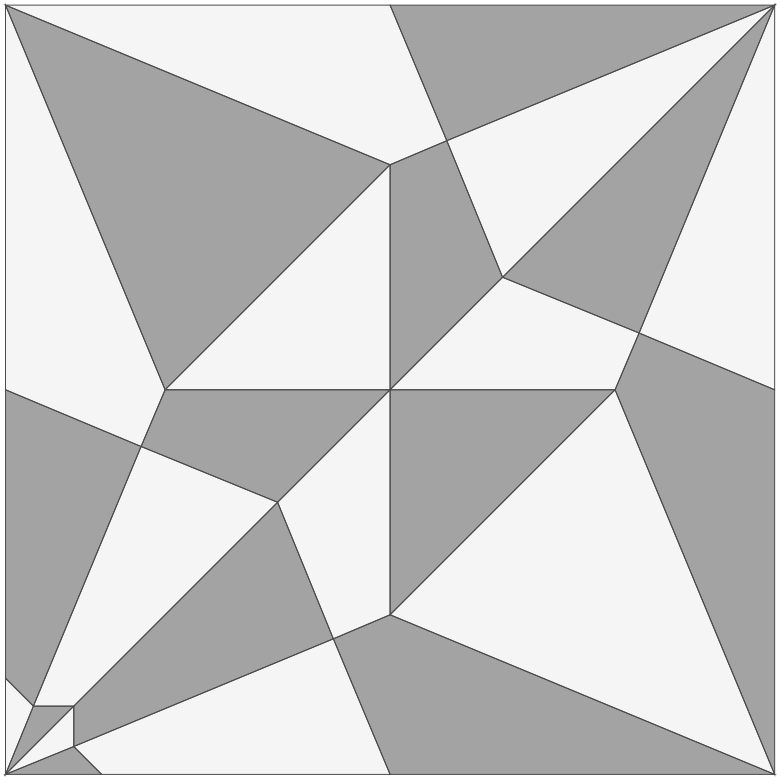

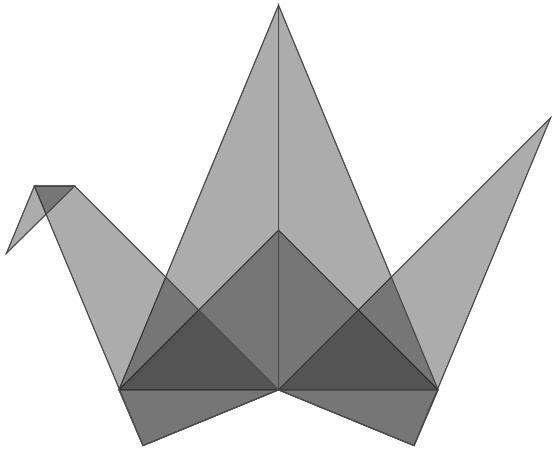

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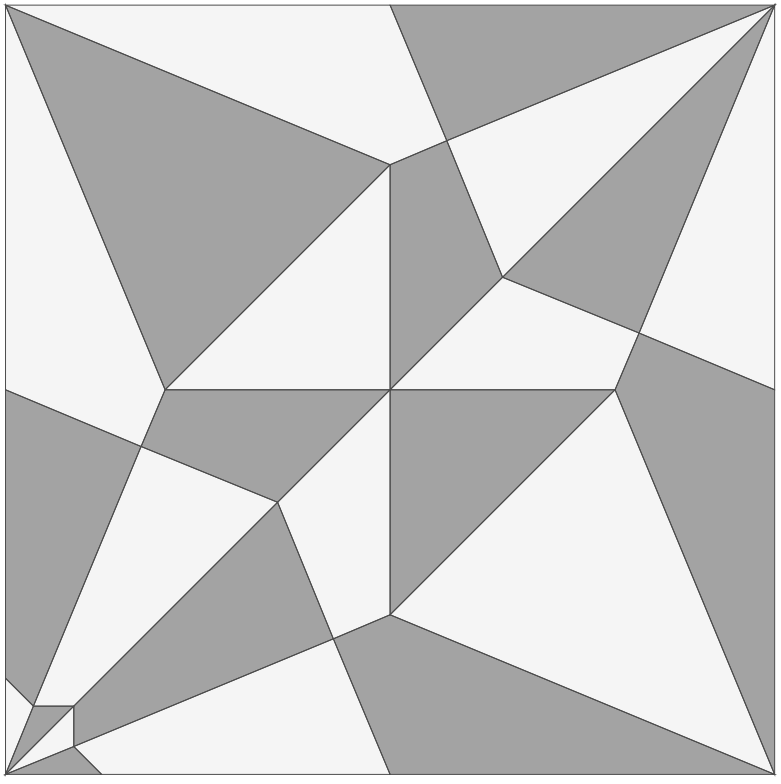

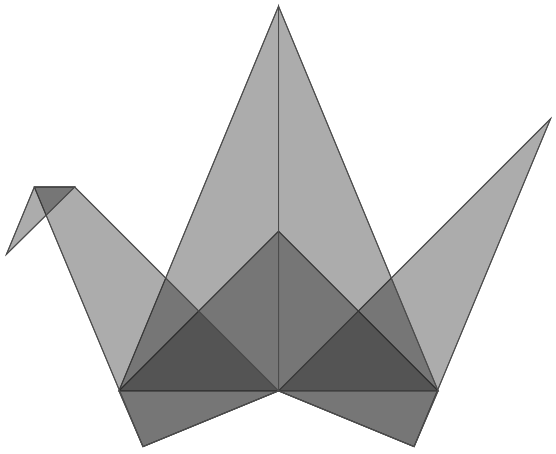

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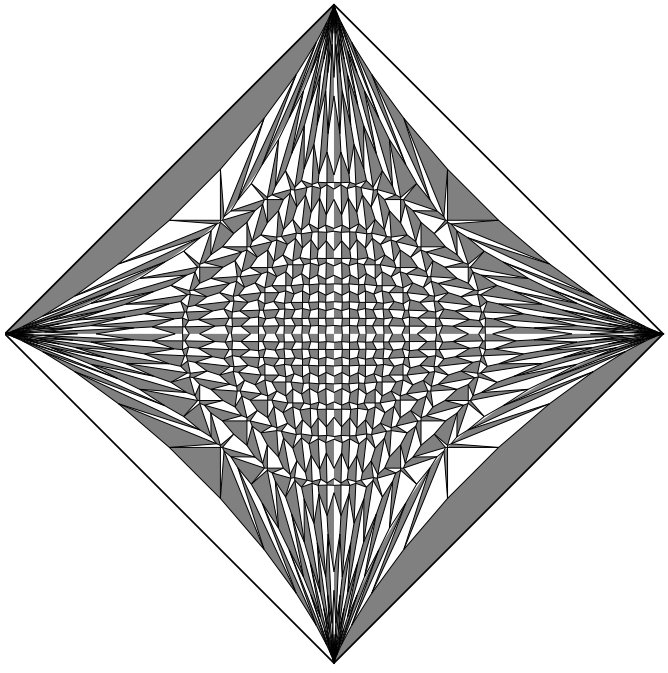
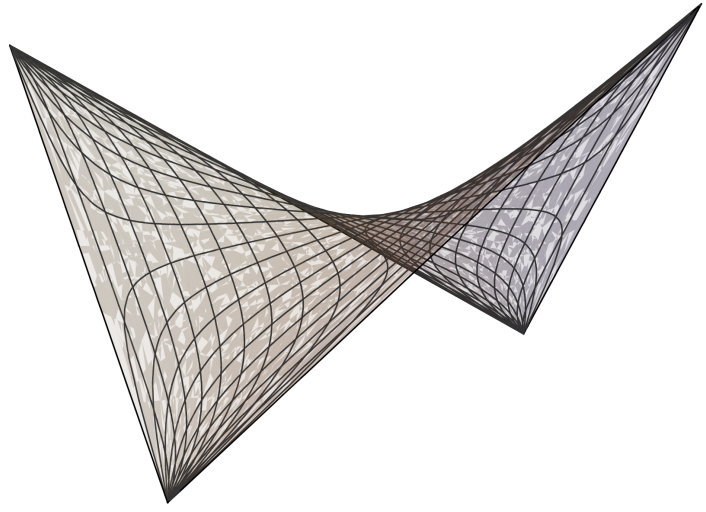
$$\sum_{i=1}^{n} \alpha_i^\circ = (n-k-1)\pi, \quad \sum_{i=1}^{n} \alpha_i^\bullet = (k-1)\pi.$$

Convergence results for dimer model observables on OCPs:

[Kenyon–Lam–Ramassamy–Russkikh '18], [Chelkak–Laslier–Russkikh '21]





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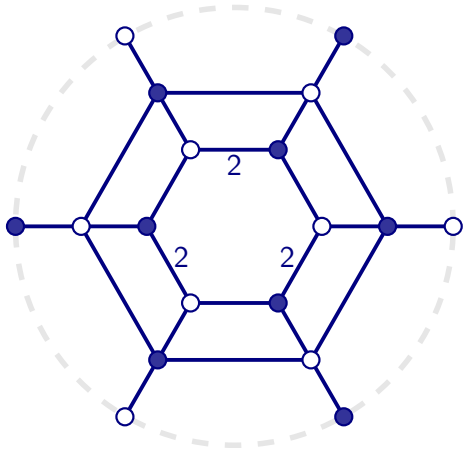

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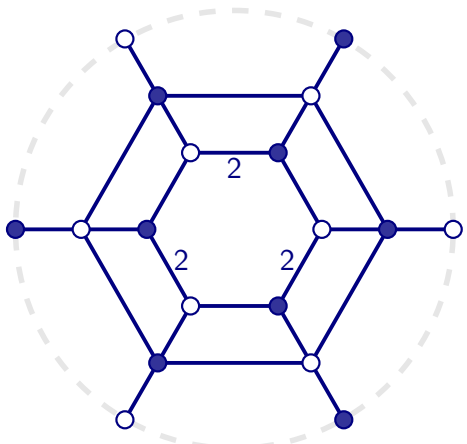

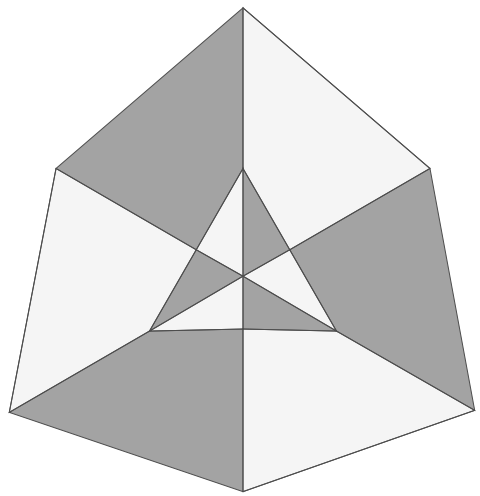

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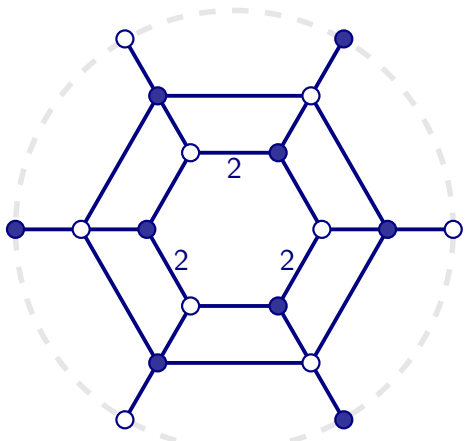

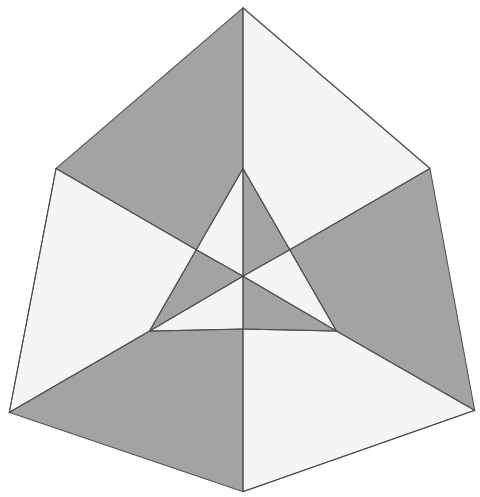

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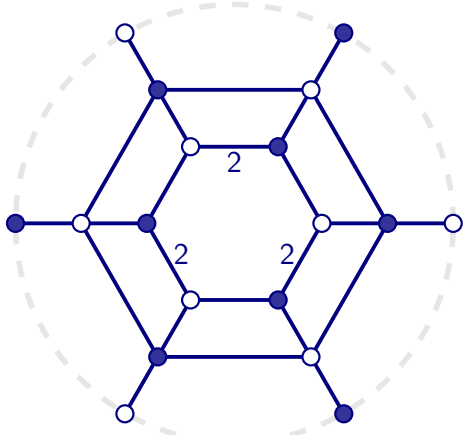
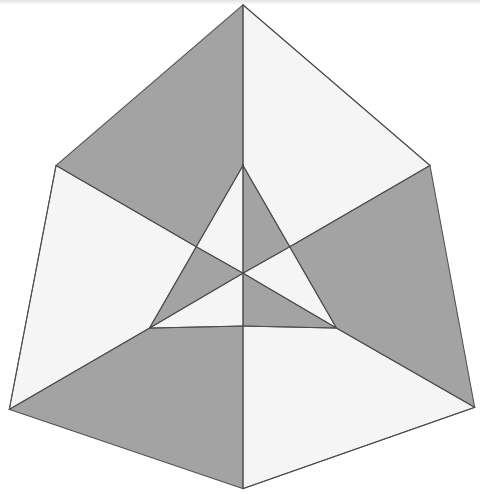
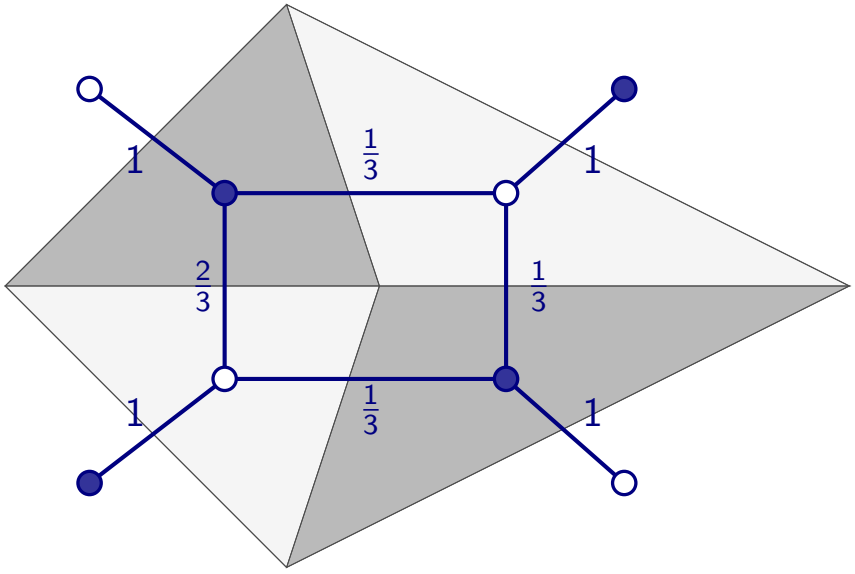
Proof.

Total positivity + physics of scattering amplitudes. □



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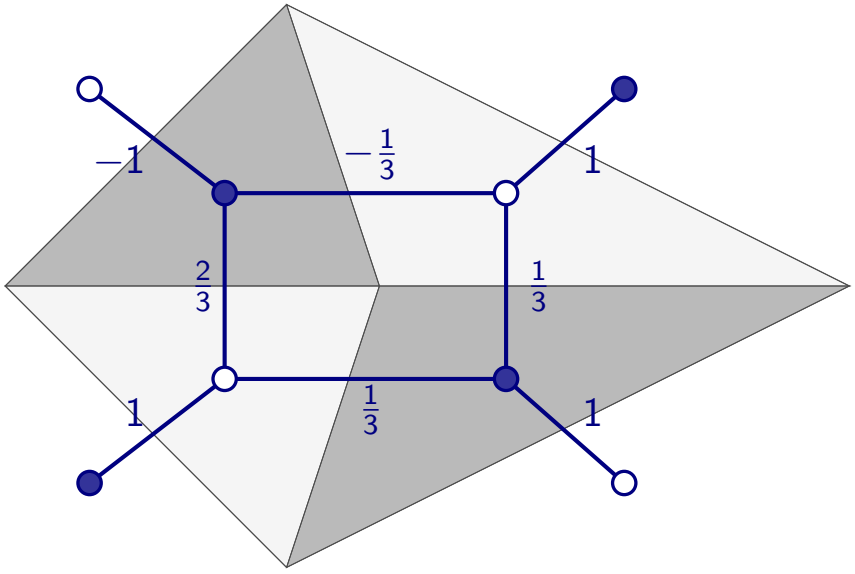

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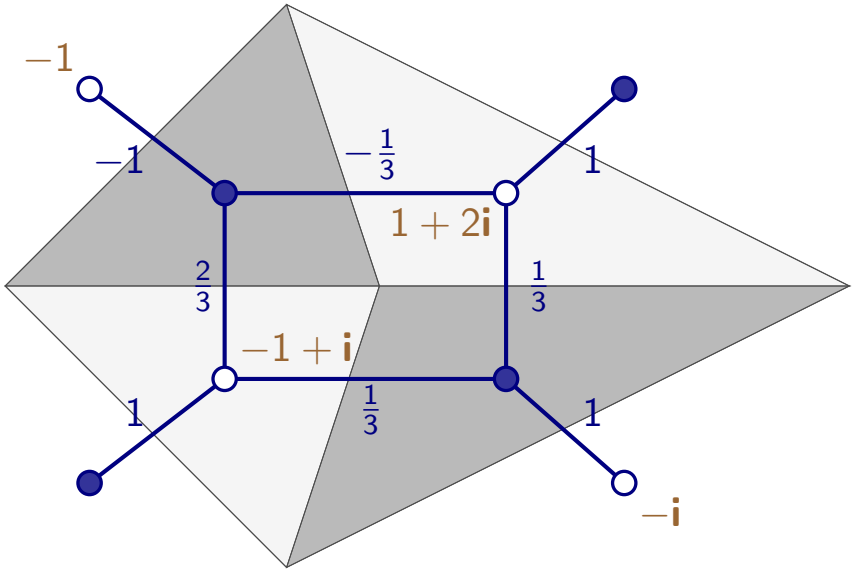

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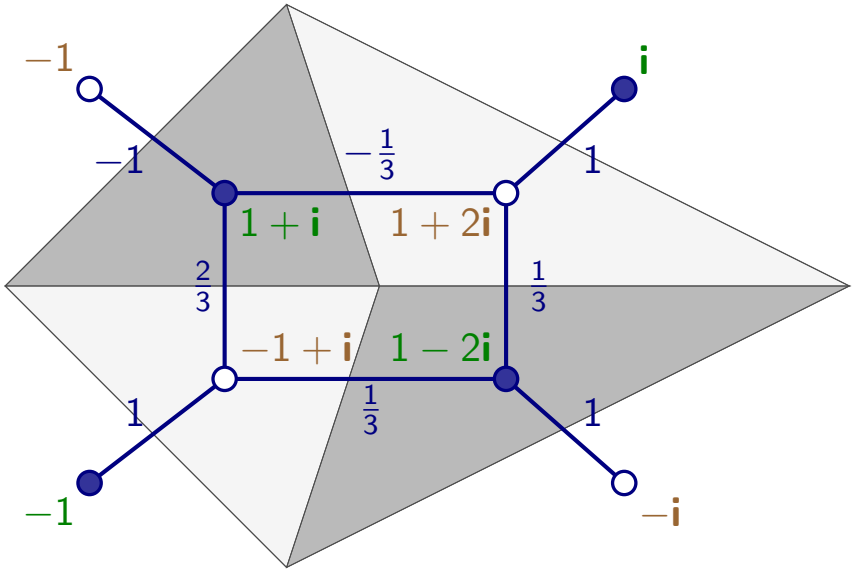

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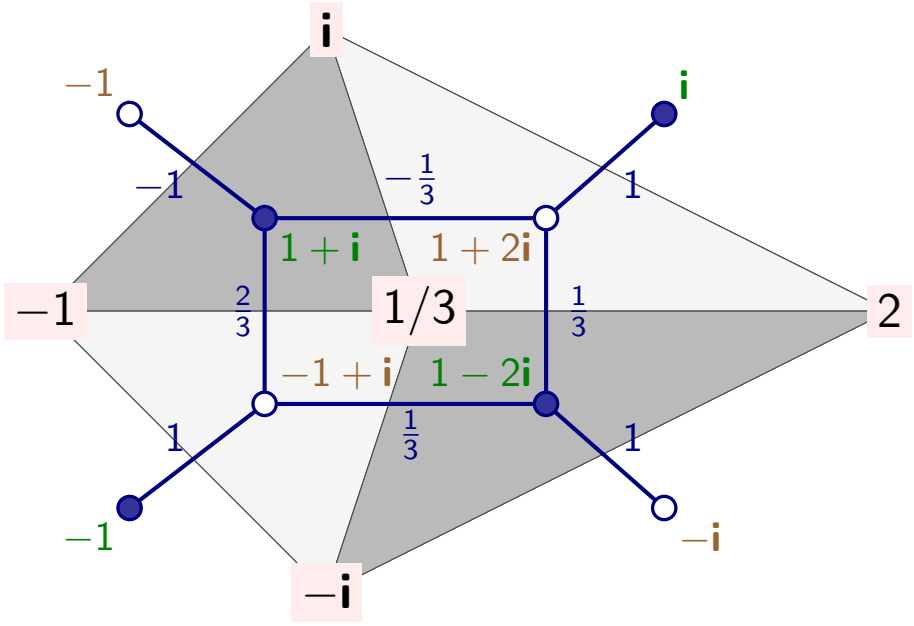

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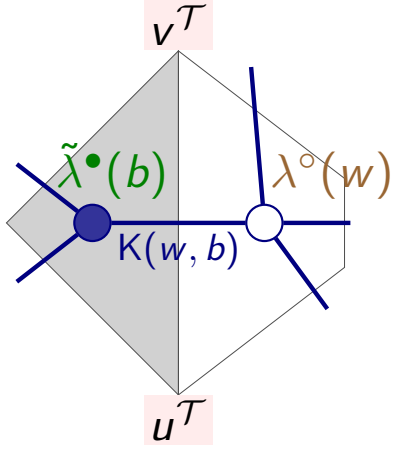

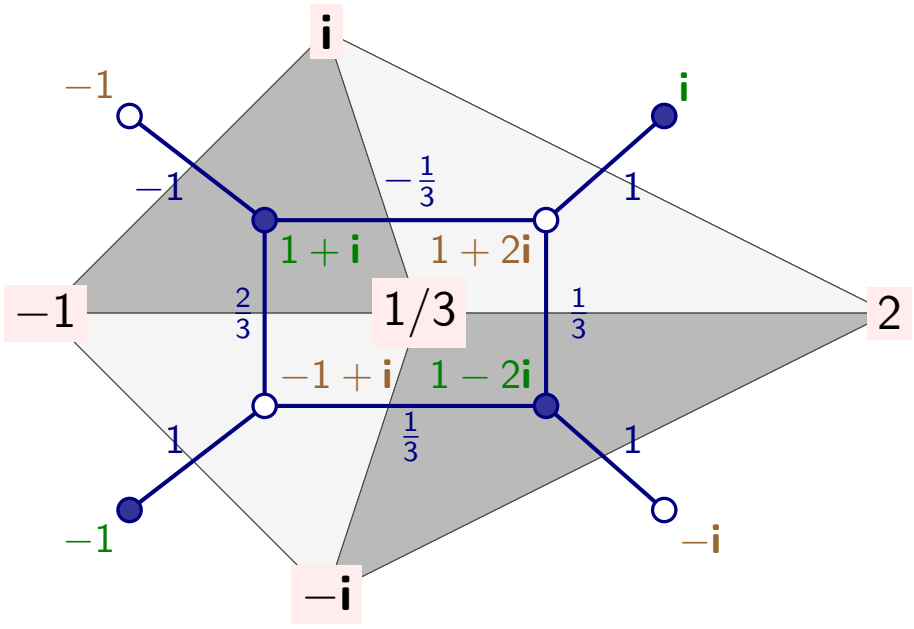

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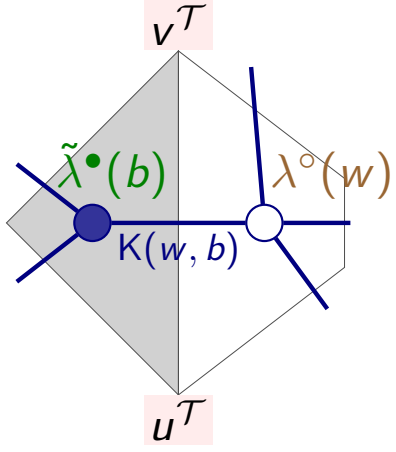

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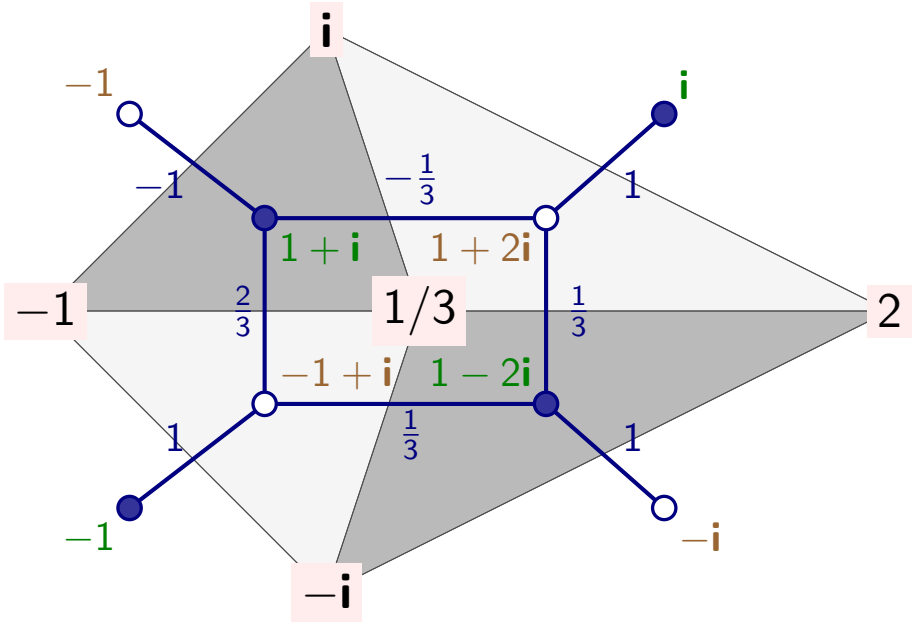

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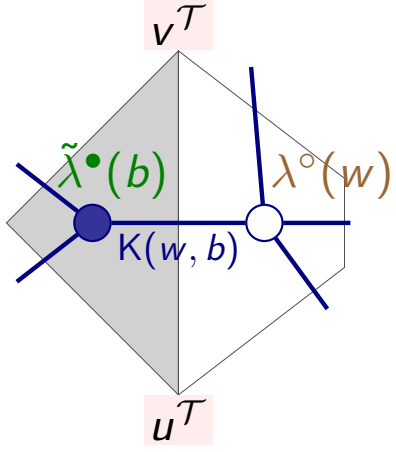

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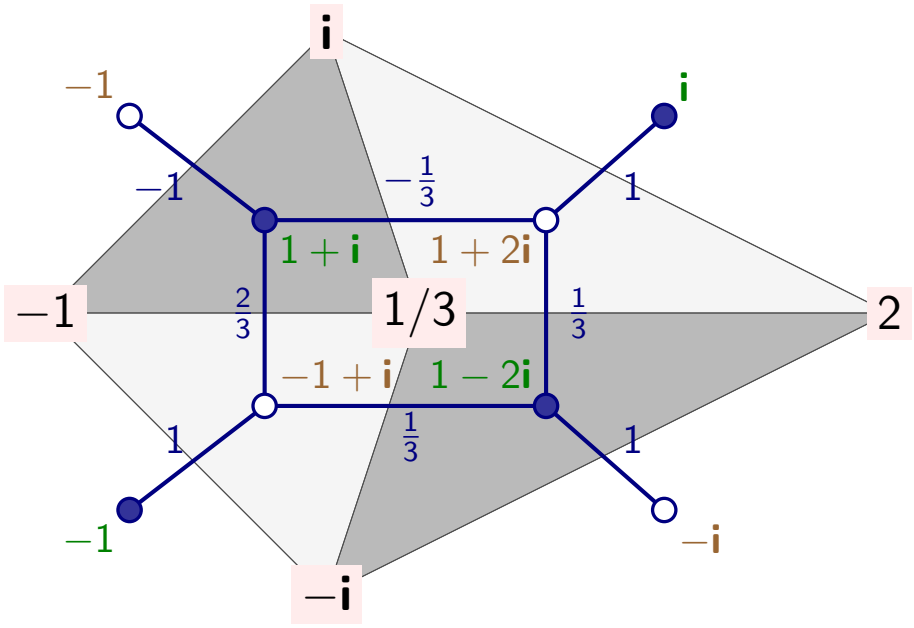

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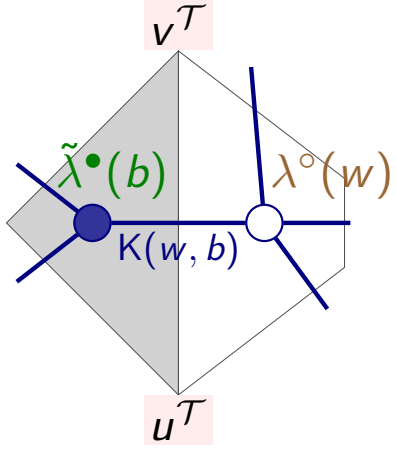

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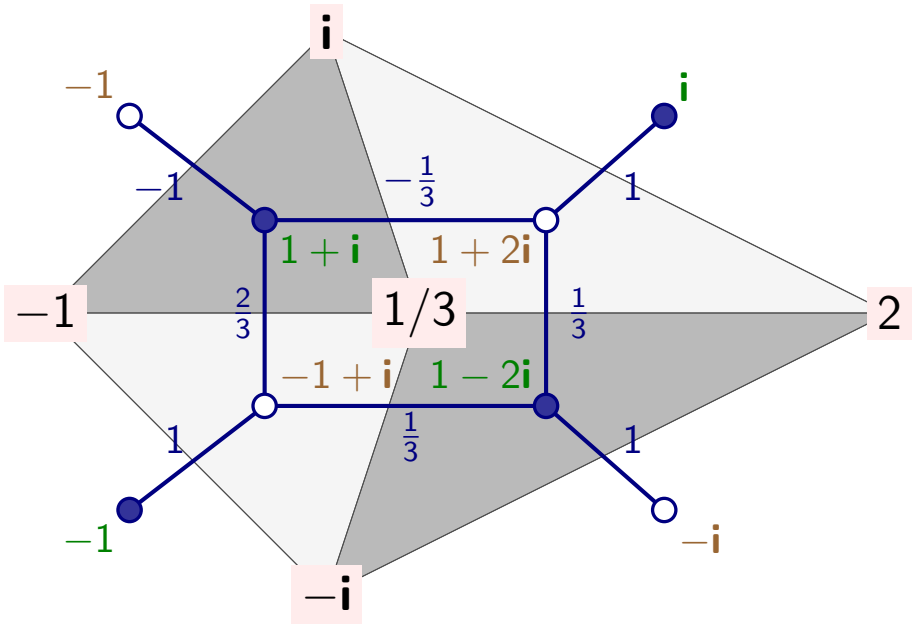
Easy: angle condition is satisfied mod 2π



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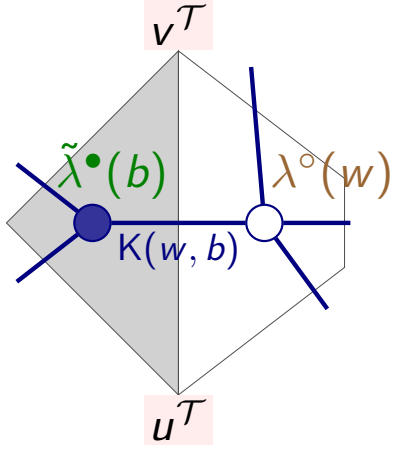

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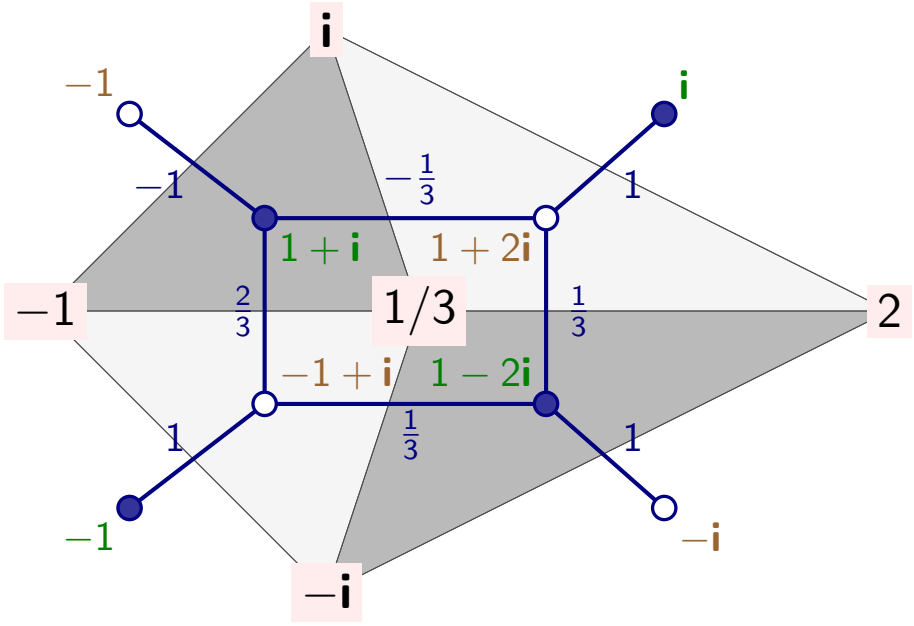
Hard: how to construct $(\lambda^\circ, \tilde{\lambda}^\bullet)$ such that $\int \lambda^\circ \tilde{\lambda}^\bullet \, dz$ is a valid, embedded OCP?



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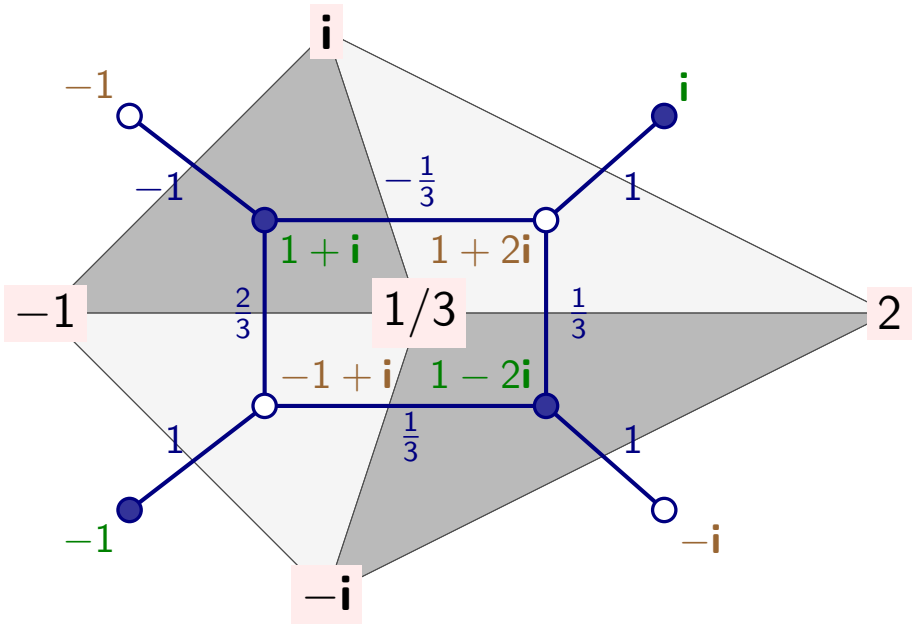

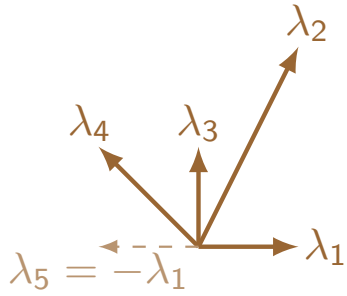

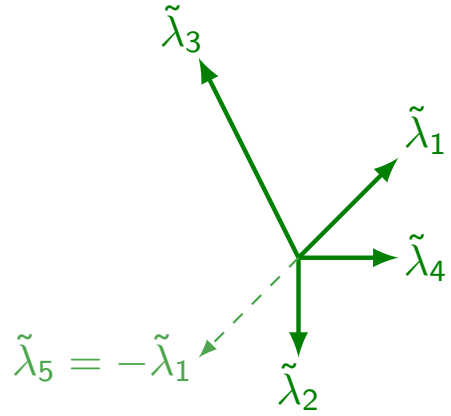

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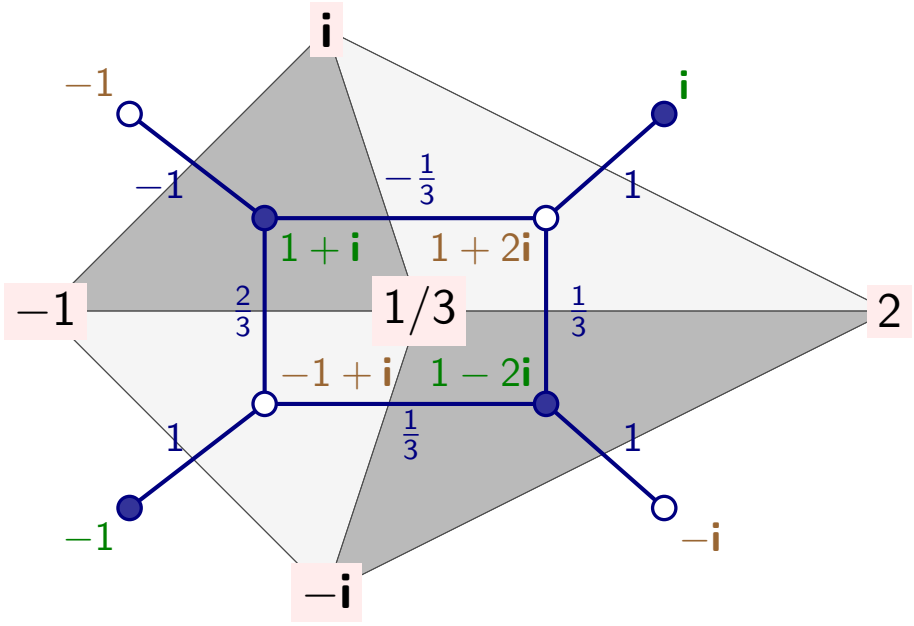

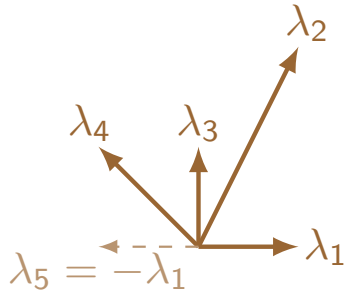

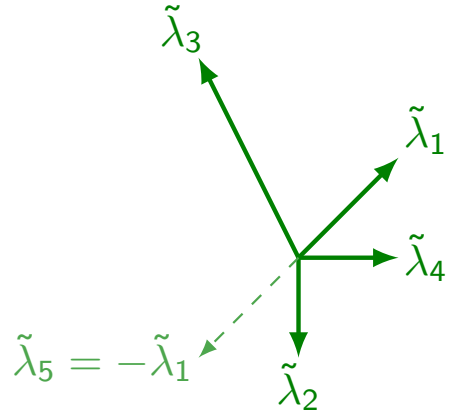

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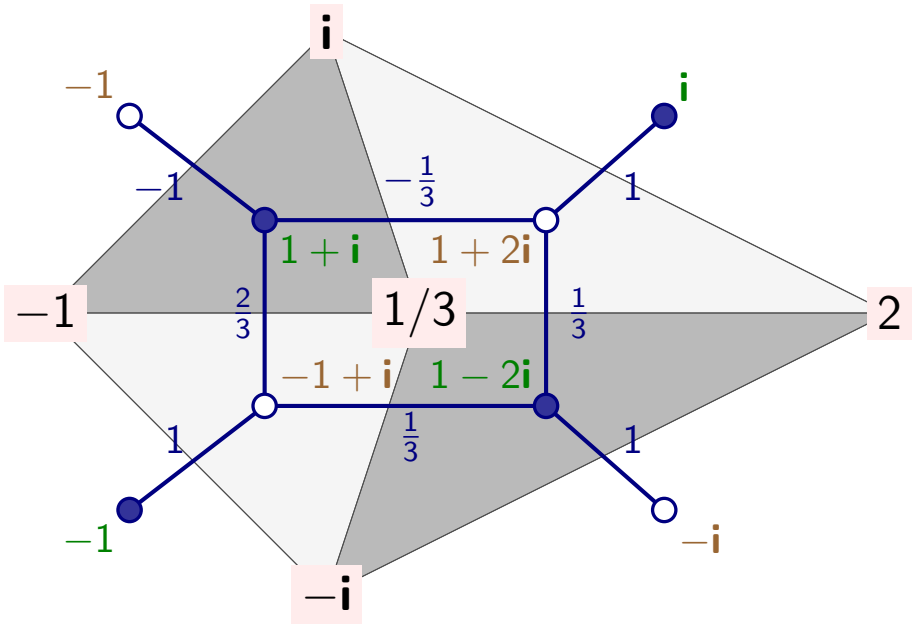

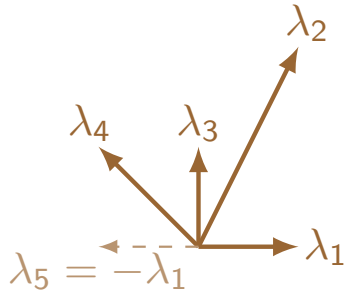

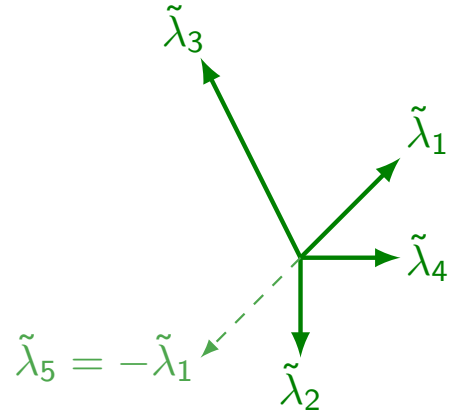

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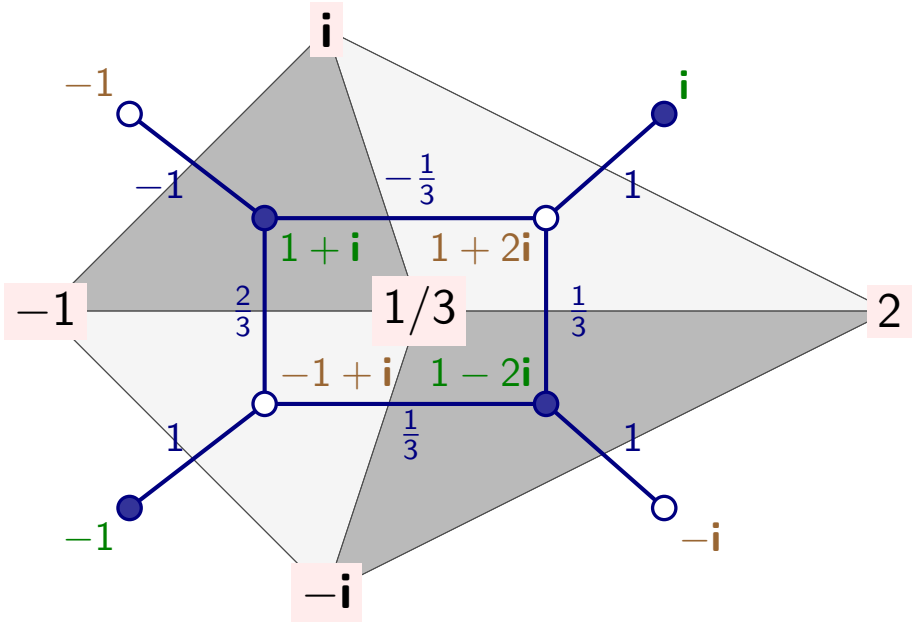

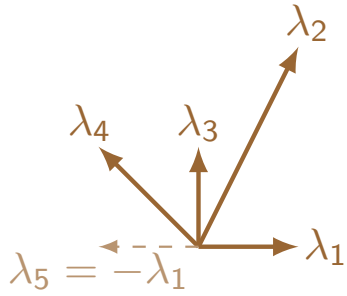

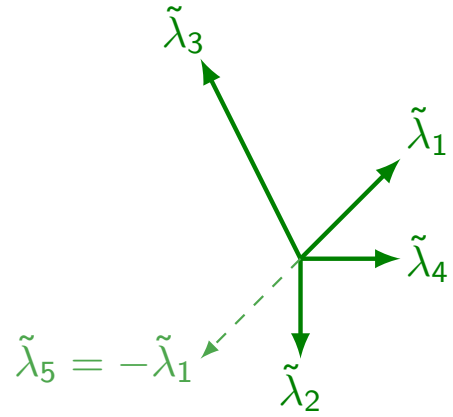

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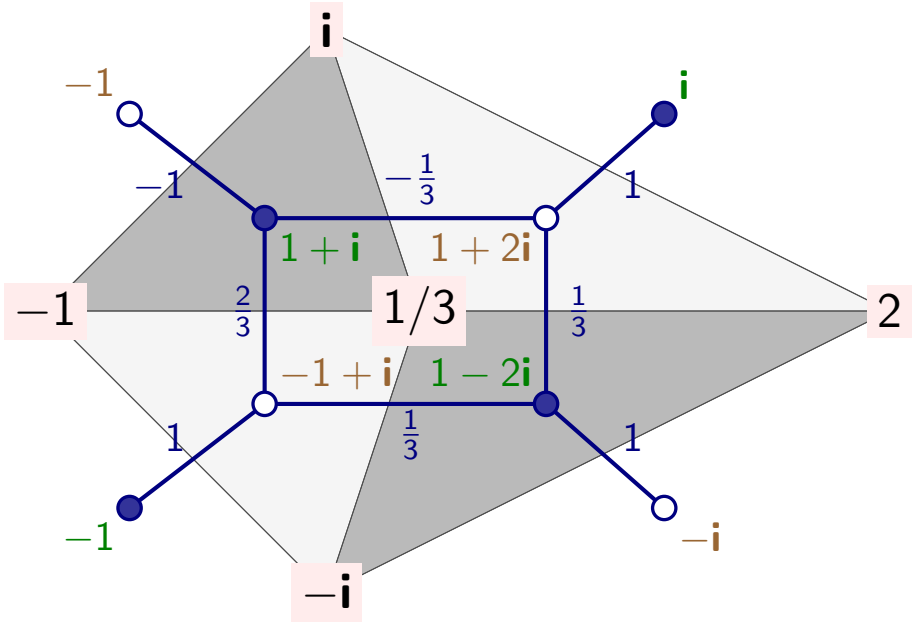

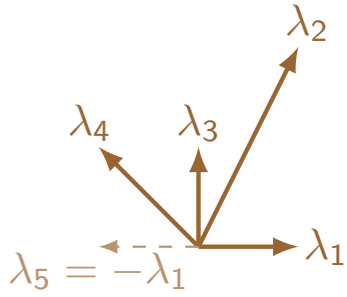

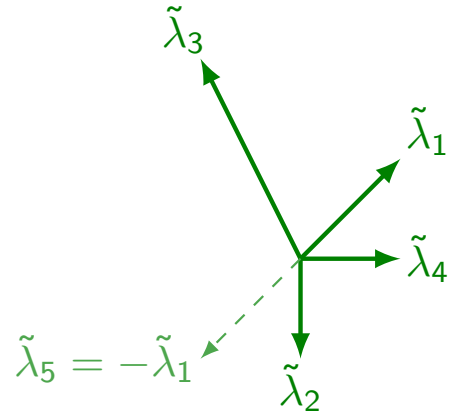

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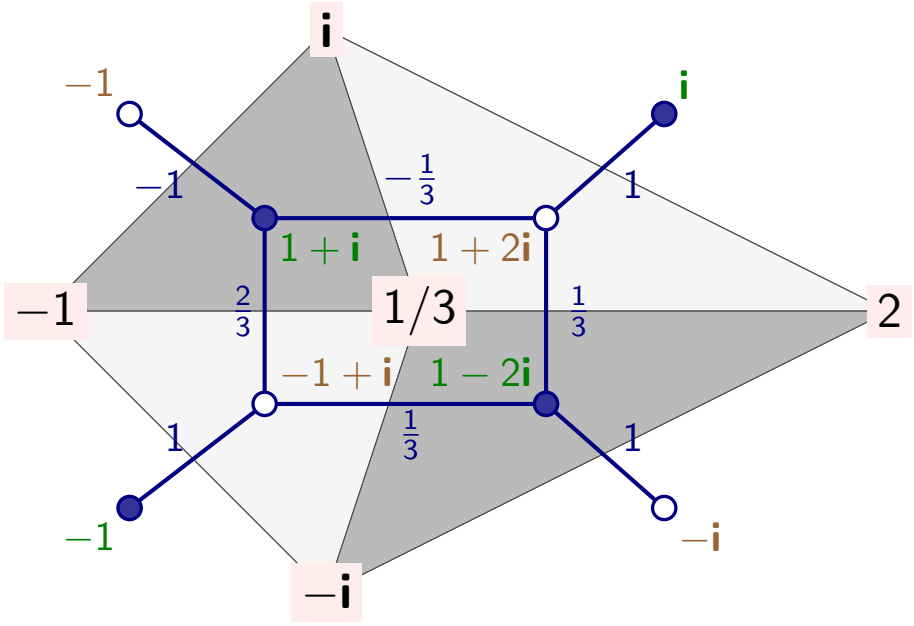

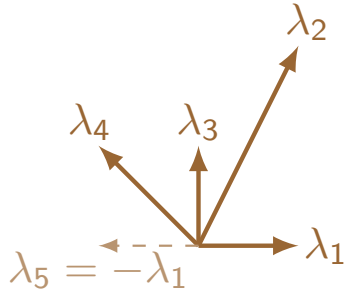

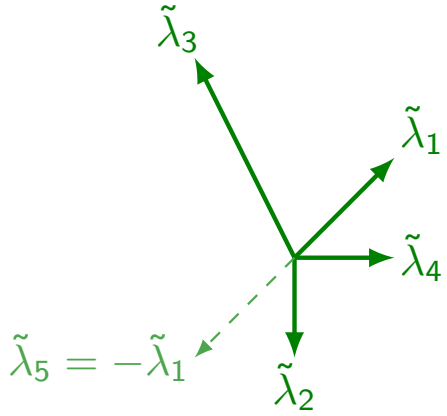

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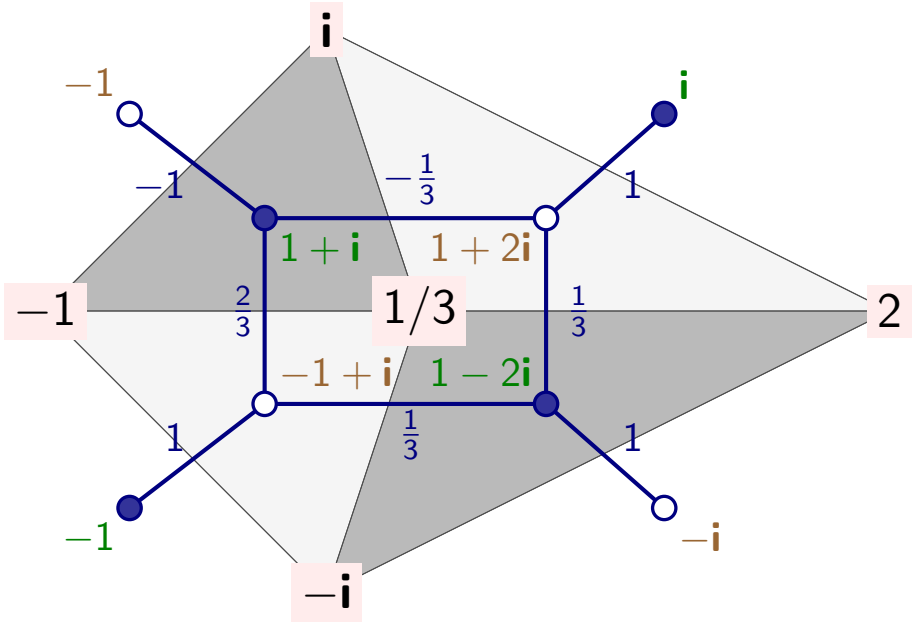

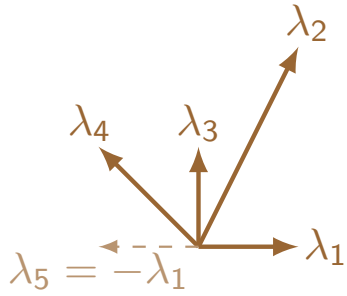

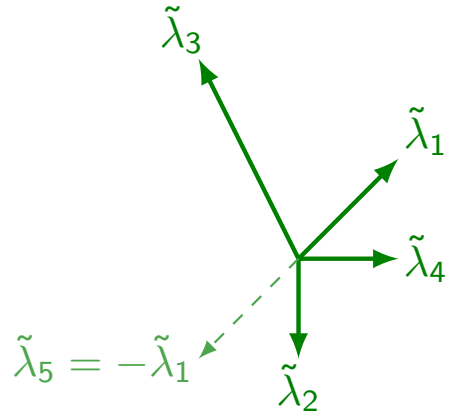

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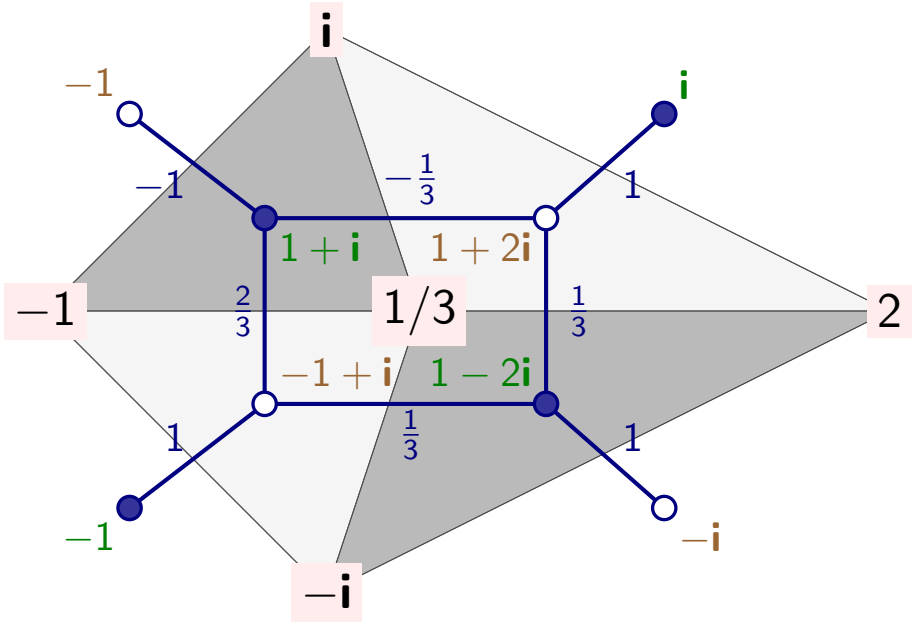

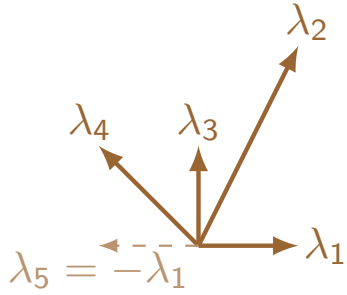

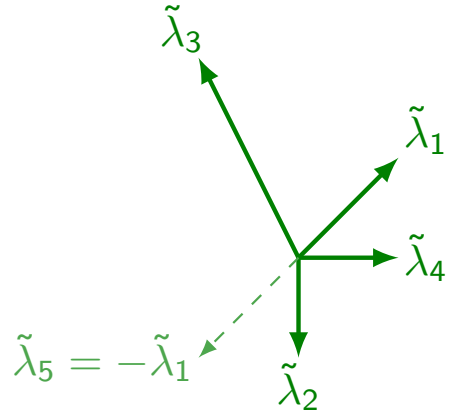

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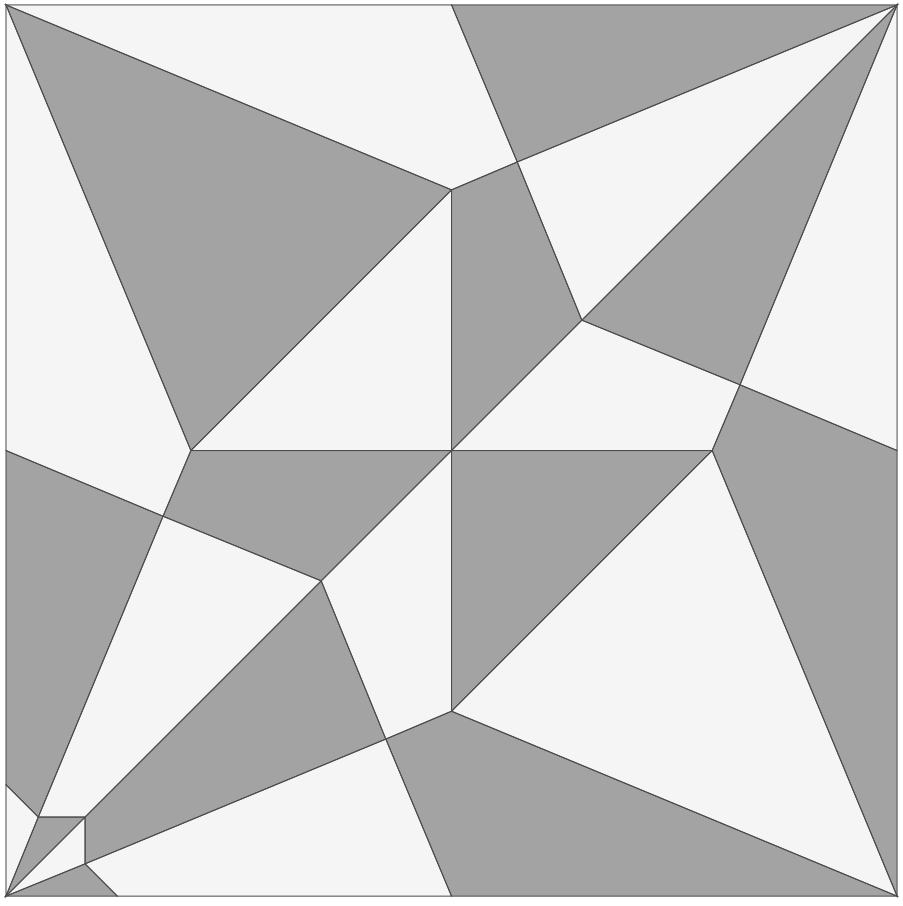
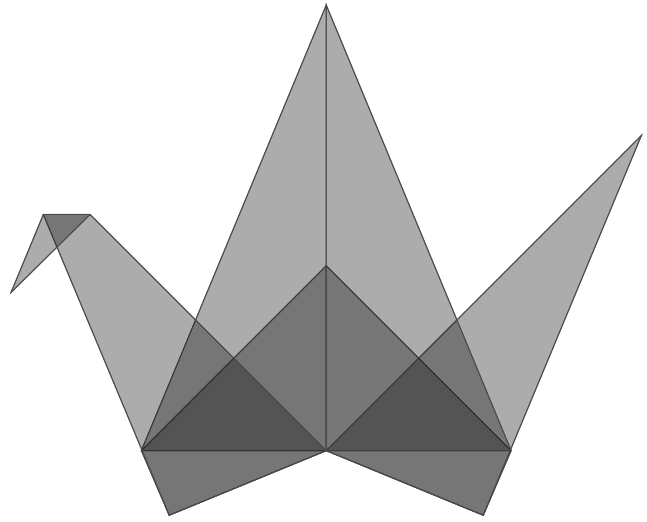
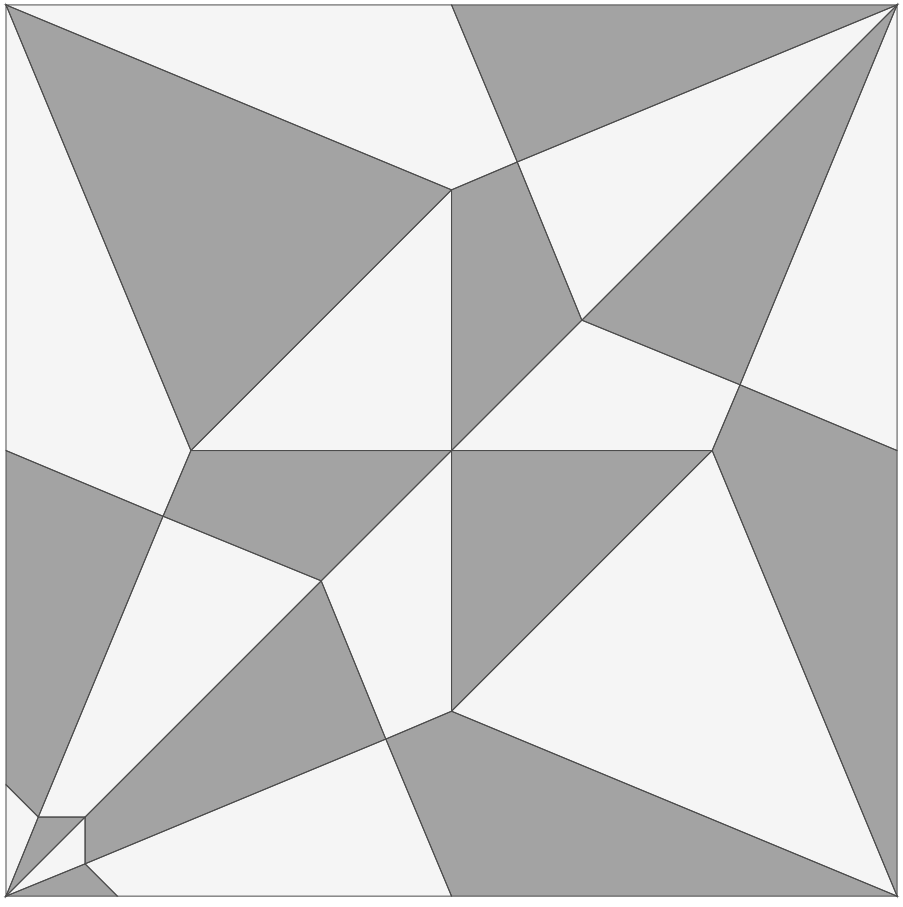
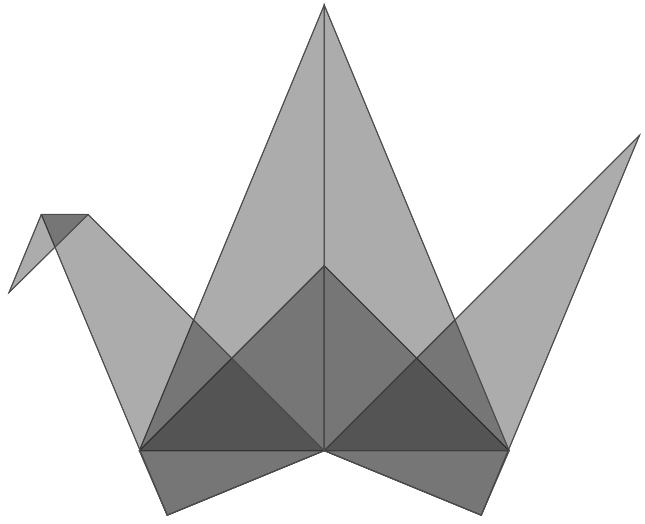




- OCP + its origami folding = 2-dimensional discrete PL surface in $\mathbb{R}^{2,2}$.





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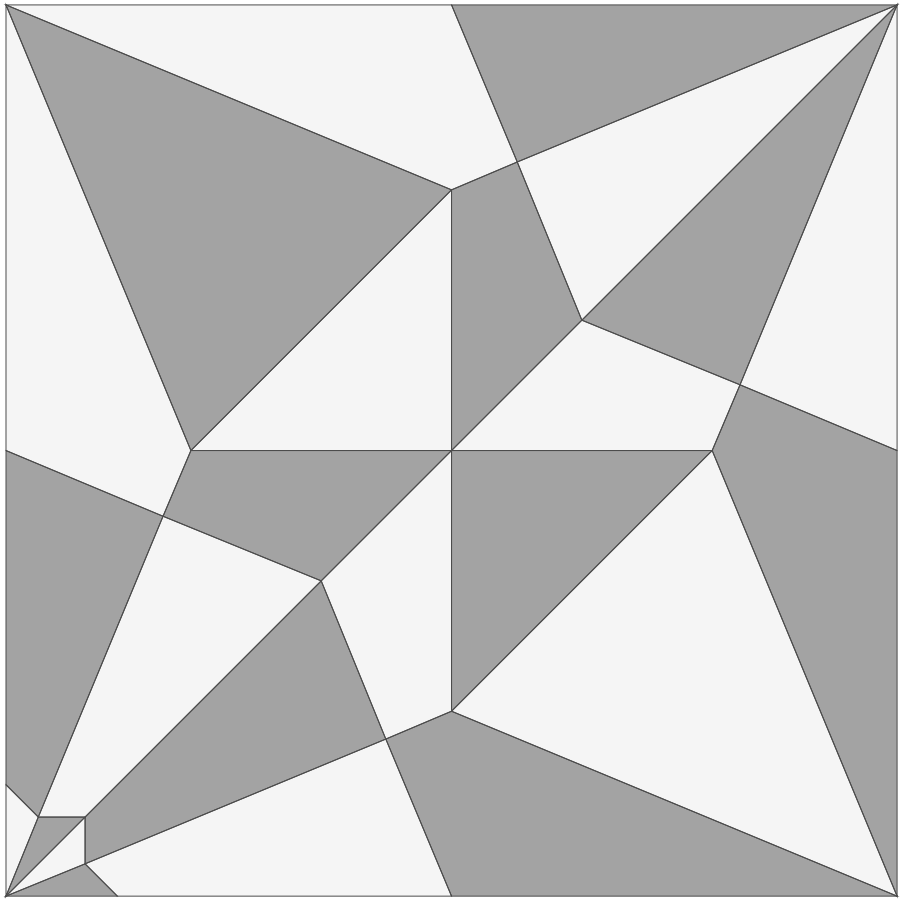

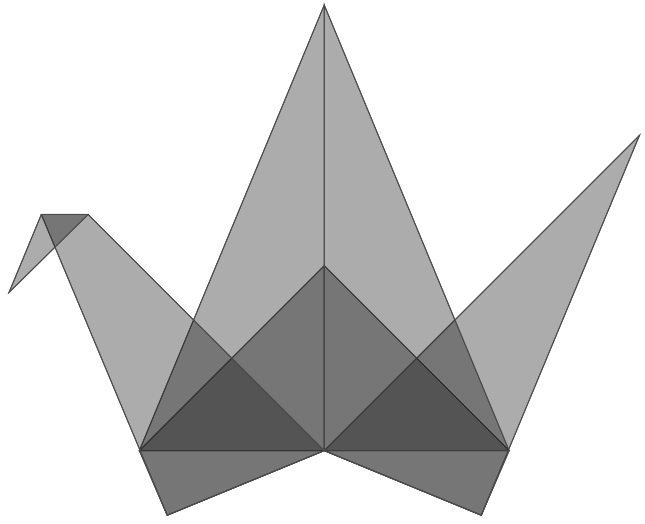

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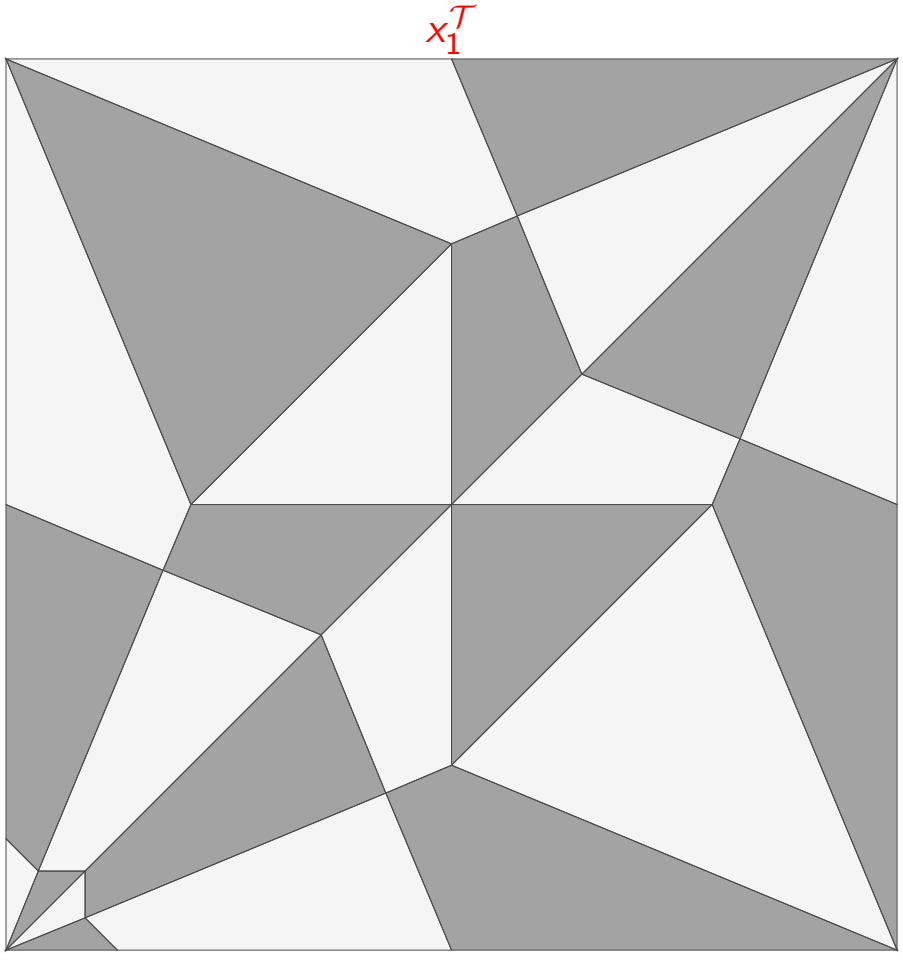

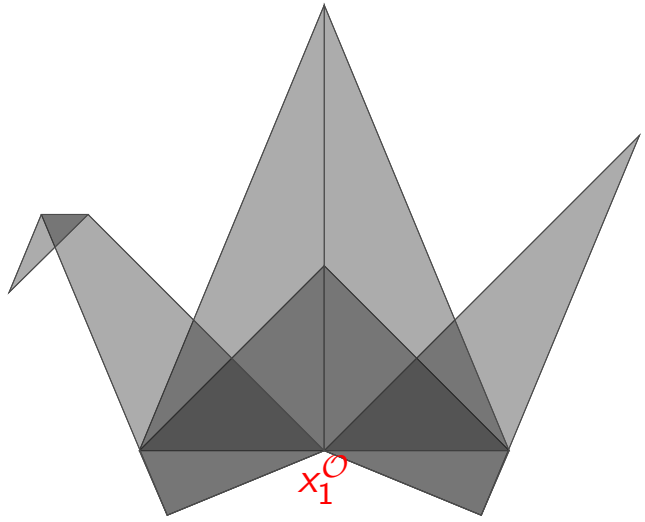

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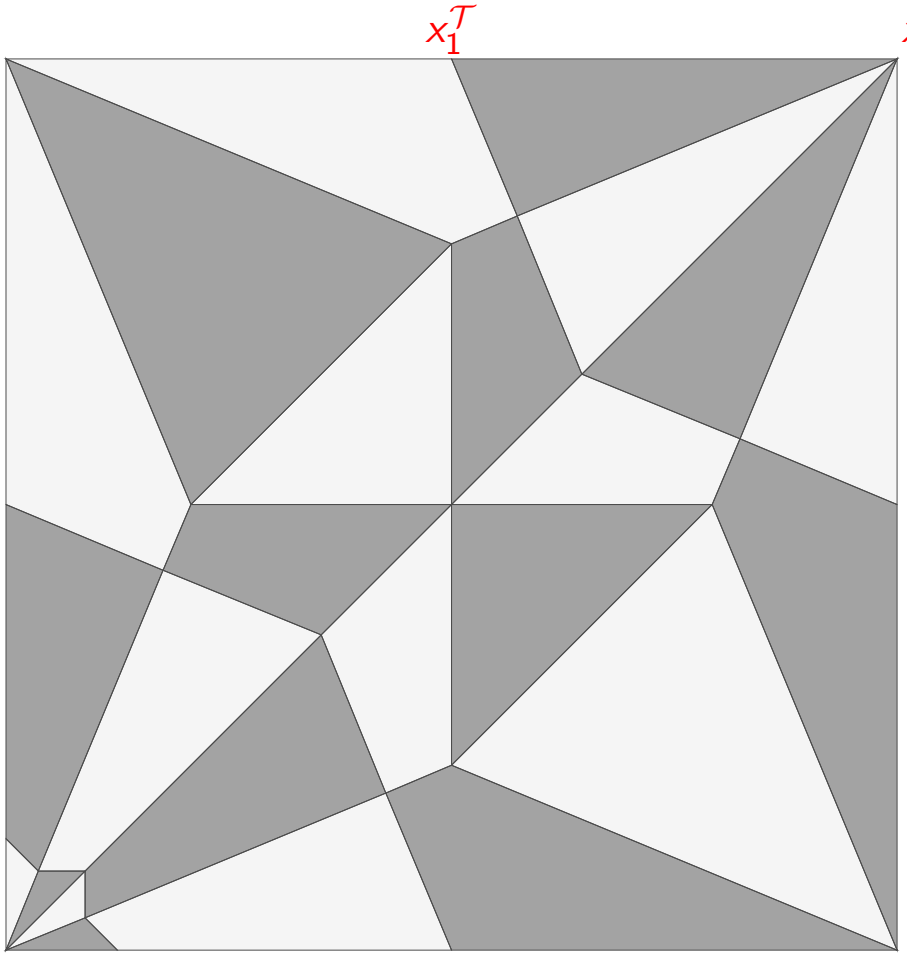

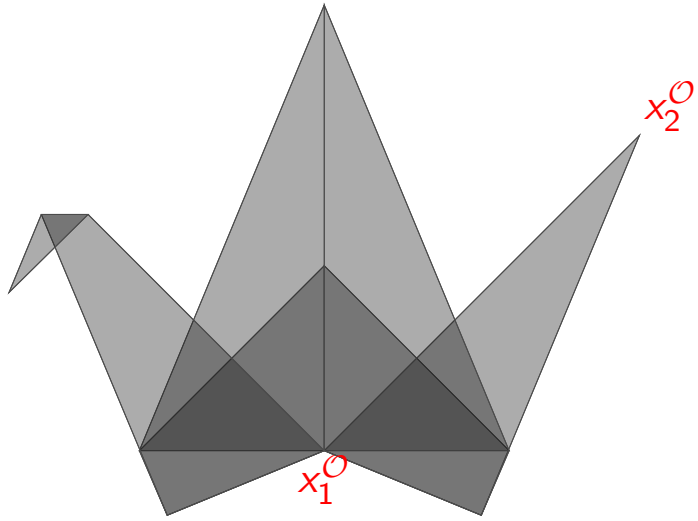

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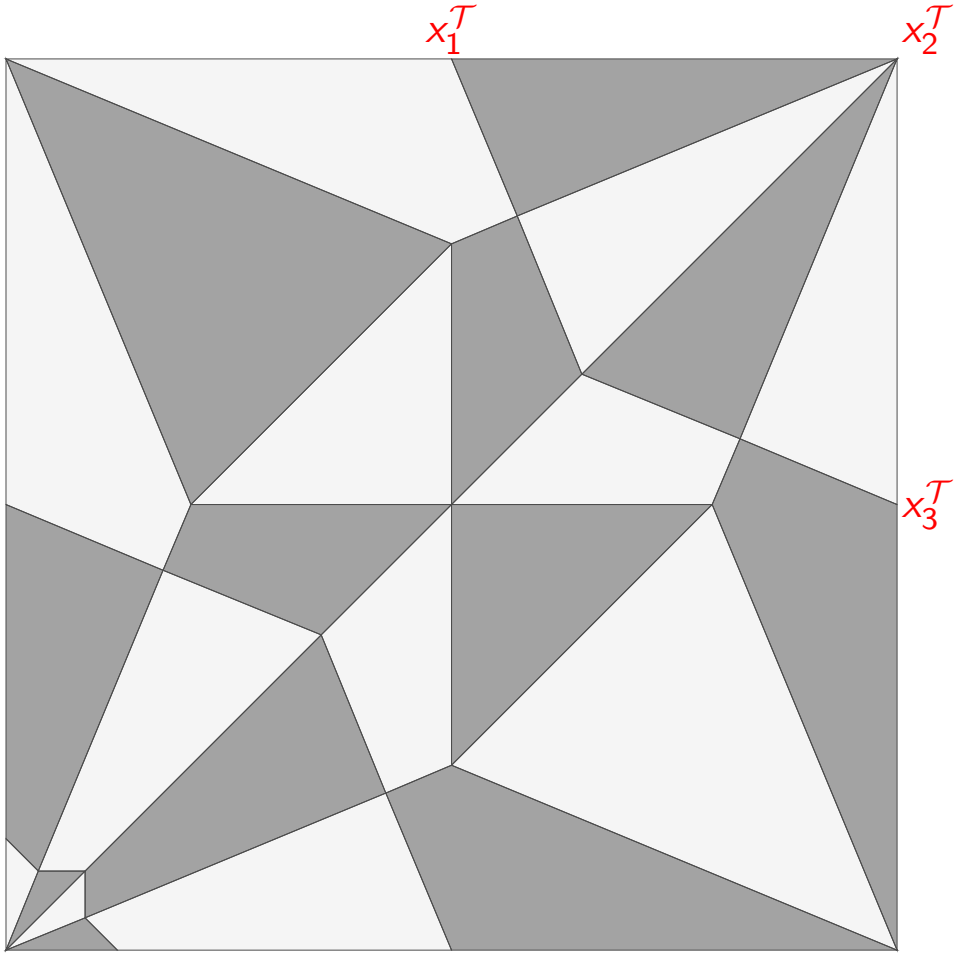

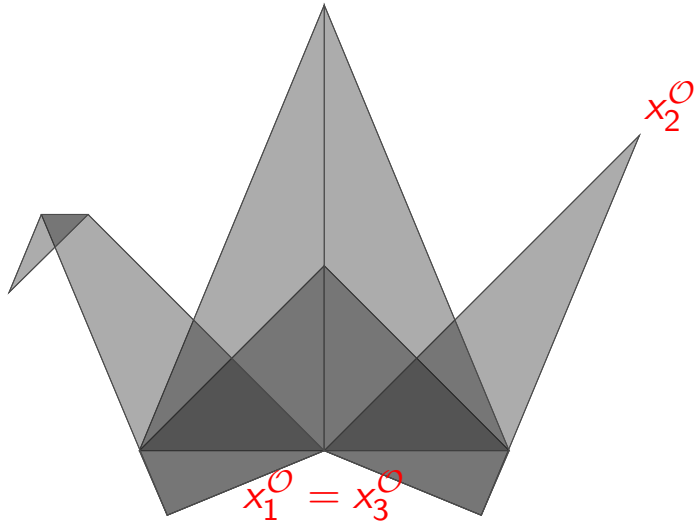

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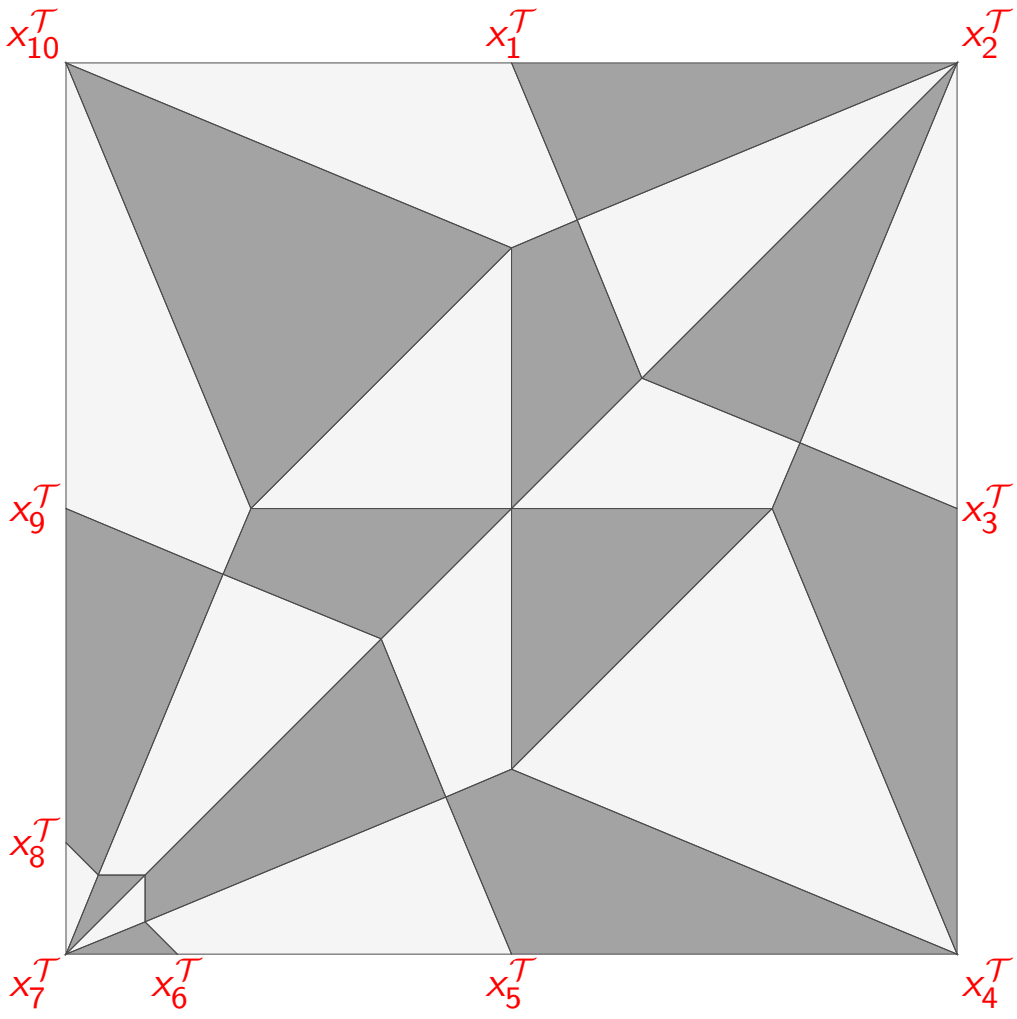

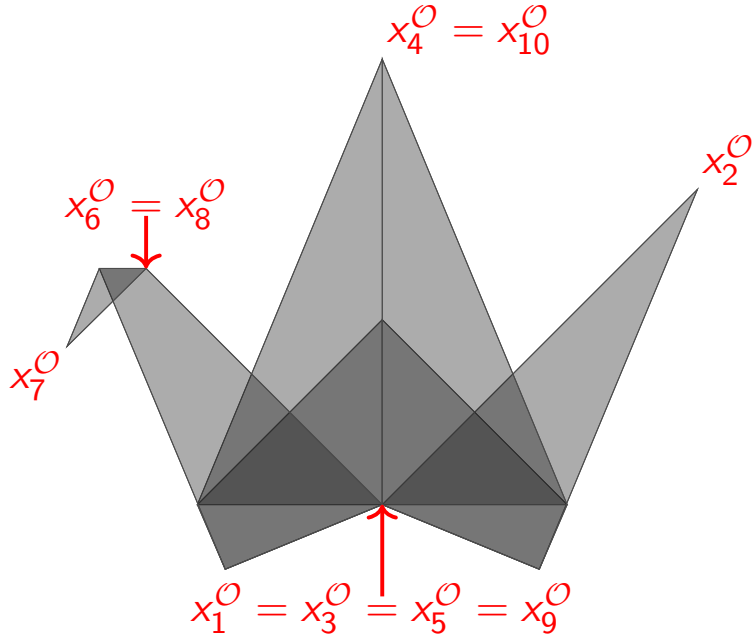

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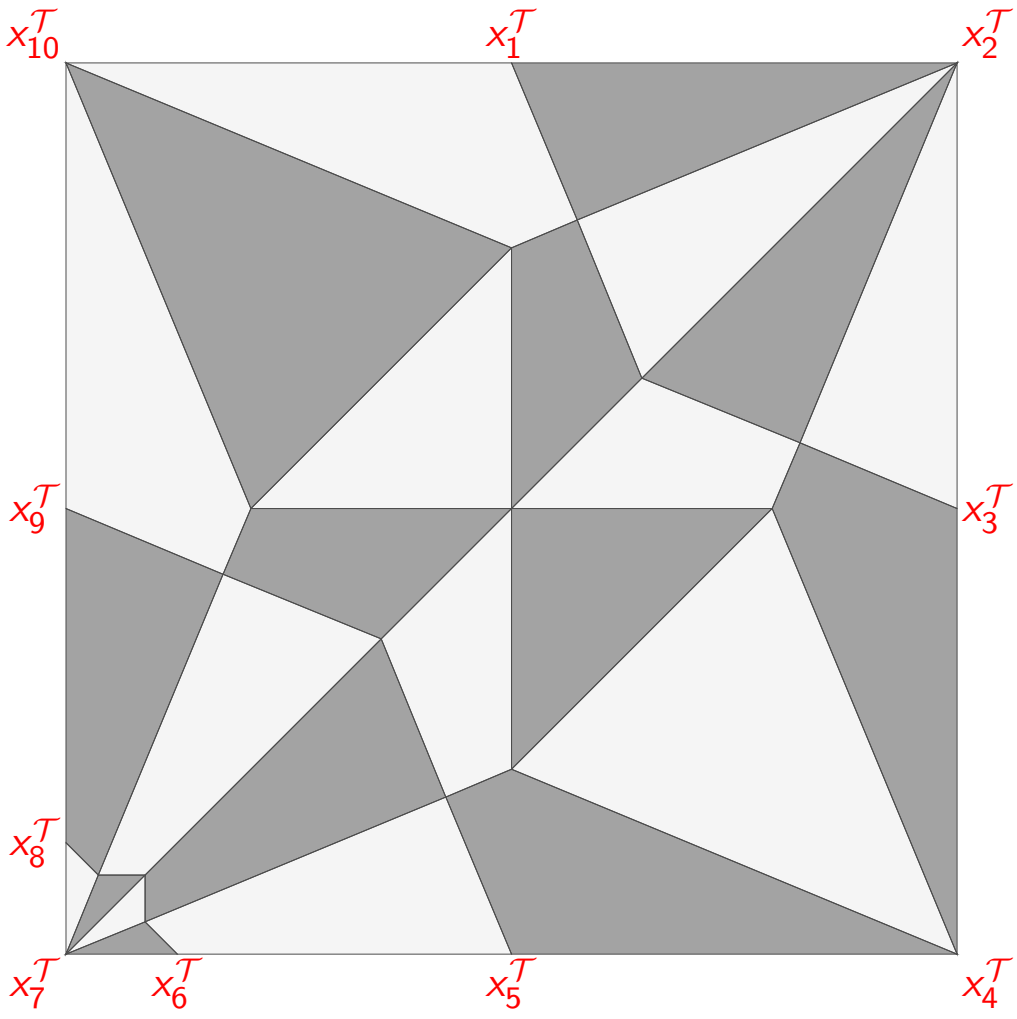

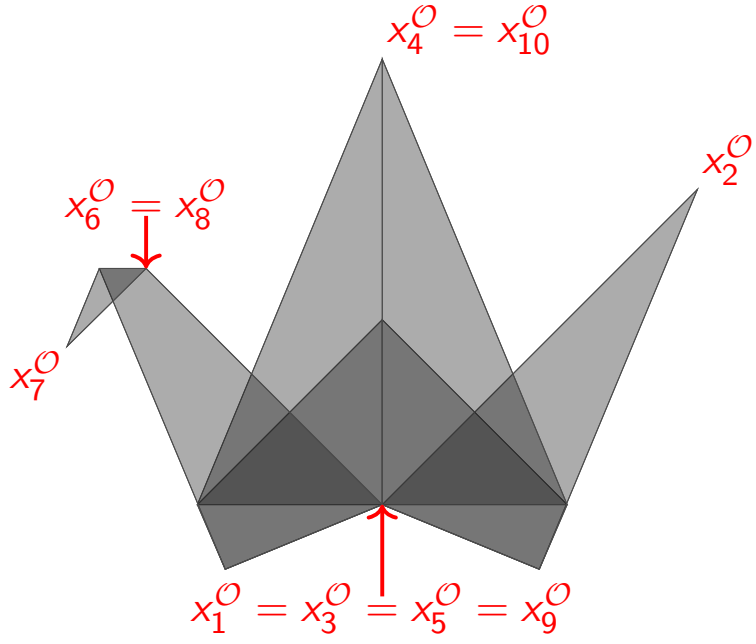

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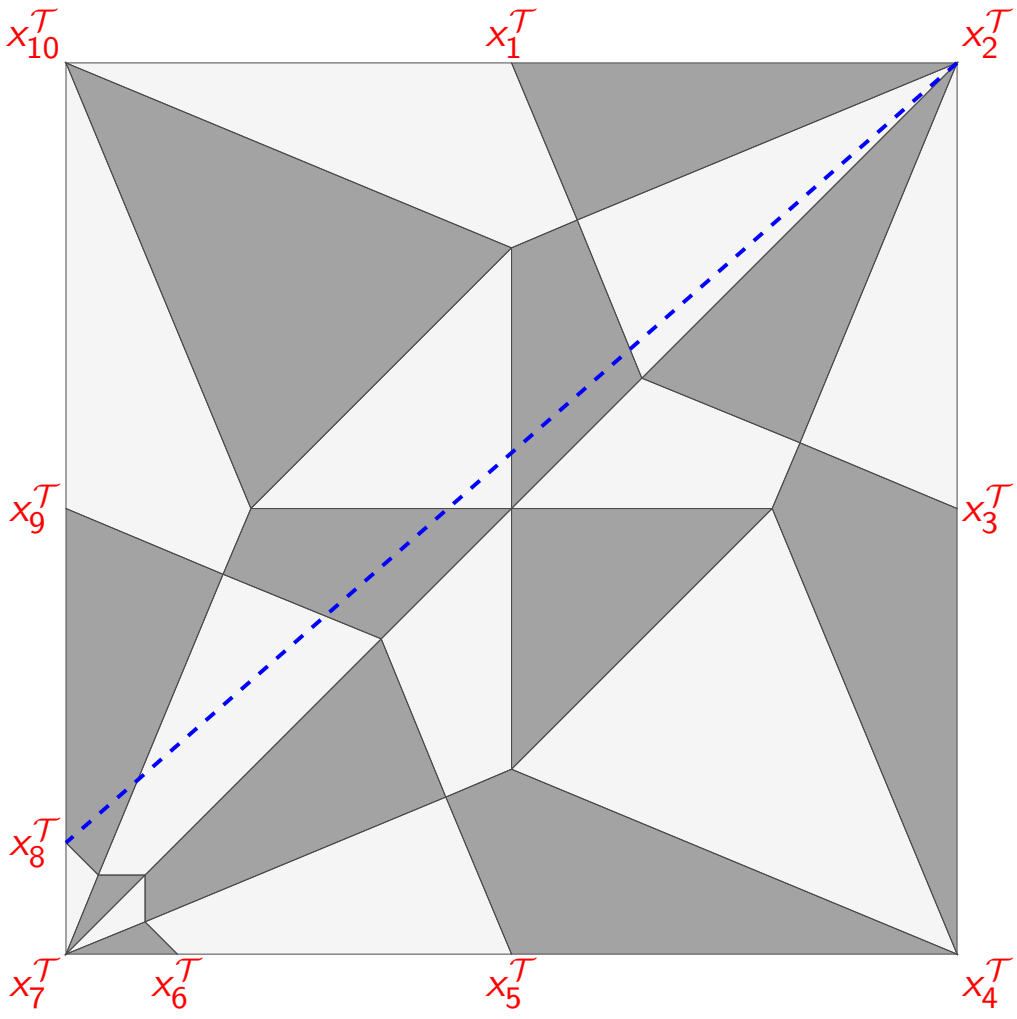

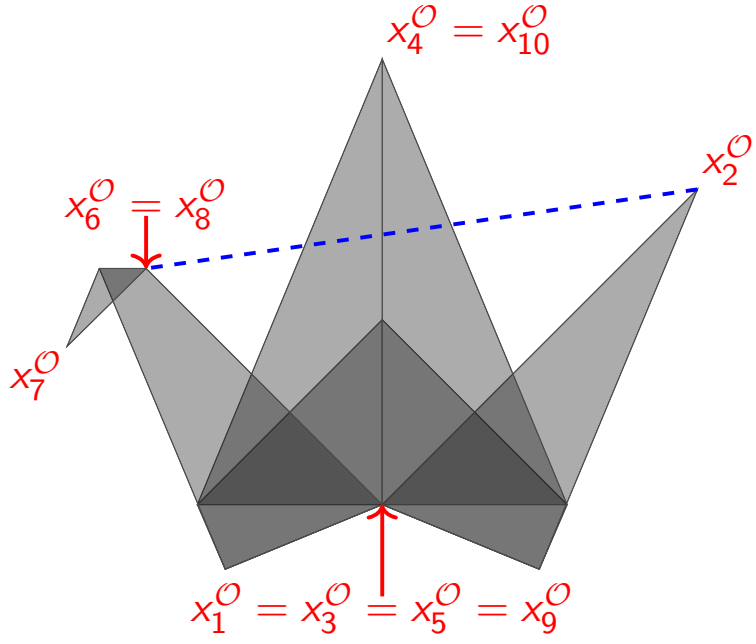

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- Minkowski distance vanishes on each white/black face.
- Origami map does not increase distances: $(x_i - x_j)^2 \geqslant 0$, i.e., $|x_i^{\mathcal{T}} - x_j^{\mathcal{T}}| \geqslant |x_i^{\mathcal{O}} - x_j^{\mathcal{O}}|$.

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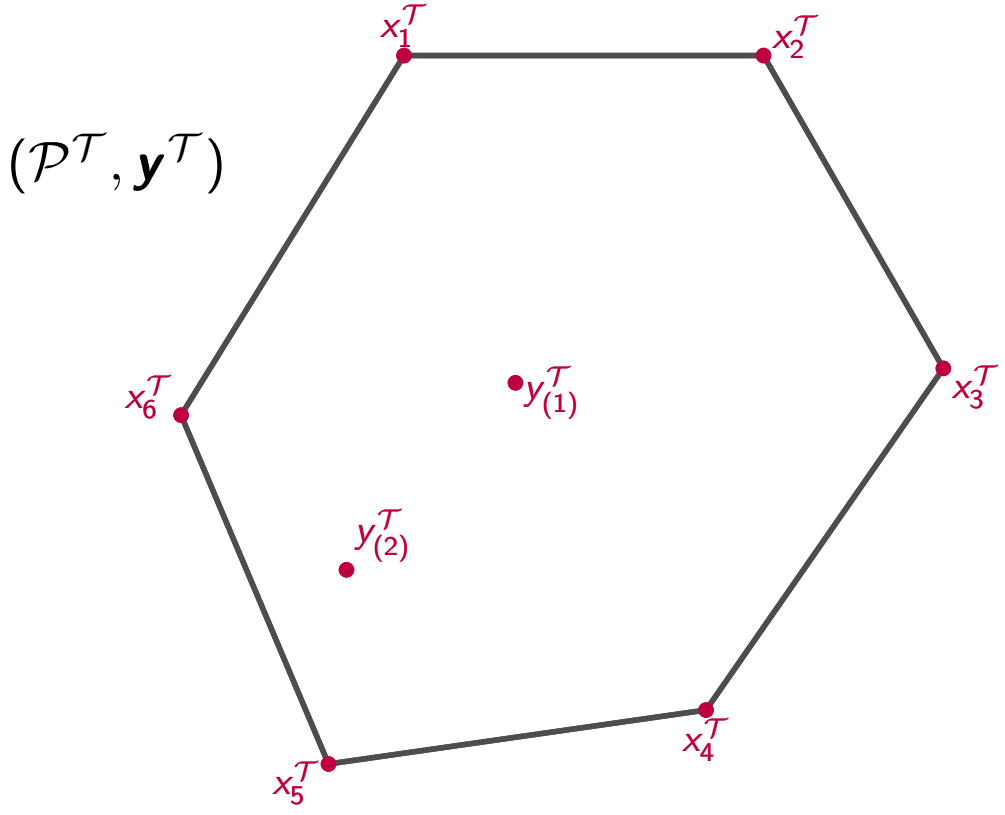
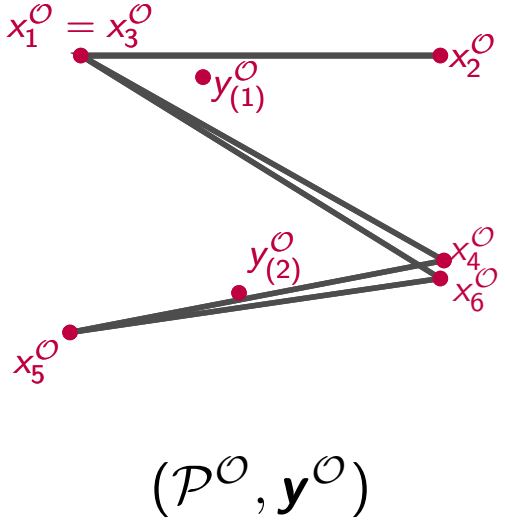
Definition

Let $L \geqslant 0$. An *L*-punctured polygon is a pair $(\mathcal{P}, \boldsymbol{y})$, where $\mathcal{P} = (x_1, x_2, \ldots, x_n) \in (\mathbb{R}^{2,2})^n$ and $\boldsymbol{y} = (y_{(1)}, y_{(2)}, \ldots, y_{(L)}) \in (\mathbb{R}^{2,2})^L$, such that





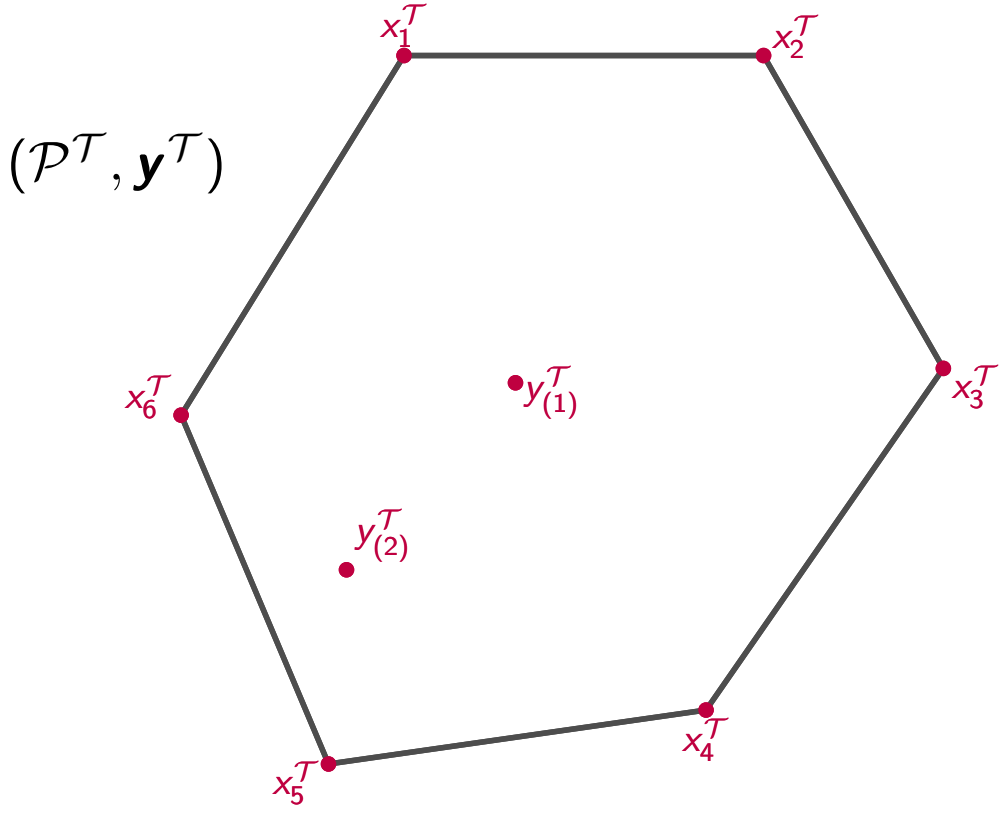
- Origami map does not increase distances: $(x_i - x_j)^2 \geqslant 0$, i.e., $|x_i^{\mathcal{T}} - x_j^{\mathcal{T}}| \geqslant |x_i^{\mathcal{O}} - x_j^{\mathcal{O}}|$.

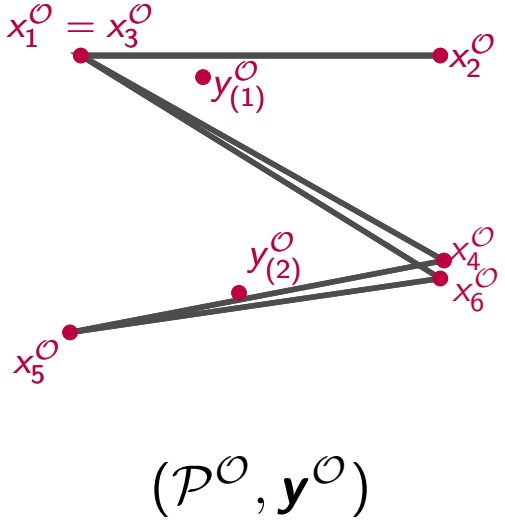
Definition

Let $L \geqslant 0$. An L-punctured polygon is a pair $(\mathcal{P}, \boldsymbol{y})$, where $\mathcal{P} = (x_1, x_2, \ldots, x_n) \in (\mathbb{R}^{2,2})^n$ and $\boldsymbol{y} = (y_{(1)}, y_{(2)}, \ldots, y_{(L)}) \in (\mathbb{R}^{2,2})^L$, such that

$$(x_i - x_{i-1})^2 = 0, \quad (x_i - x_j)^2 > 0, \quad (x_i - y_{(\rho)})^2 > 0, \quad (y_{(\rho)} - y_{(\gamma)})^2 > 0$$

for all non-adjacent $1 \leqslant i, j \leqslant n$ and all $1 \leqslant \rho \neq \gamma \leqslant L$,





- Origami map does not increase distances: $(x_i - x_j)^2 \geqslant 0$, i.e., $|x_i^{\mathcal{T}} - x_j^{\mathcal{T}}| \geqslant |x_i^{\mathcal{O}} - x_j^{\mathcal{O}}|$.

Definition

Let $L \geqslant 0$. An L-punctured polygon is a pair $(\mathcal{P}, \boldsymbol{y})$, where $\mathcal{P} = (x_1, x_2, \ldots, x_n) \in (\mathbb{R}^{2,2})^n$ and $\boldsymbol{y} = (y_{(1)}, y_{(2)}, \ldots, y_{(L)}) \in (\mathbb{R}^{2,2})^L$, such that

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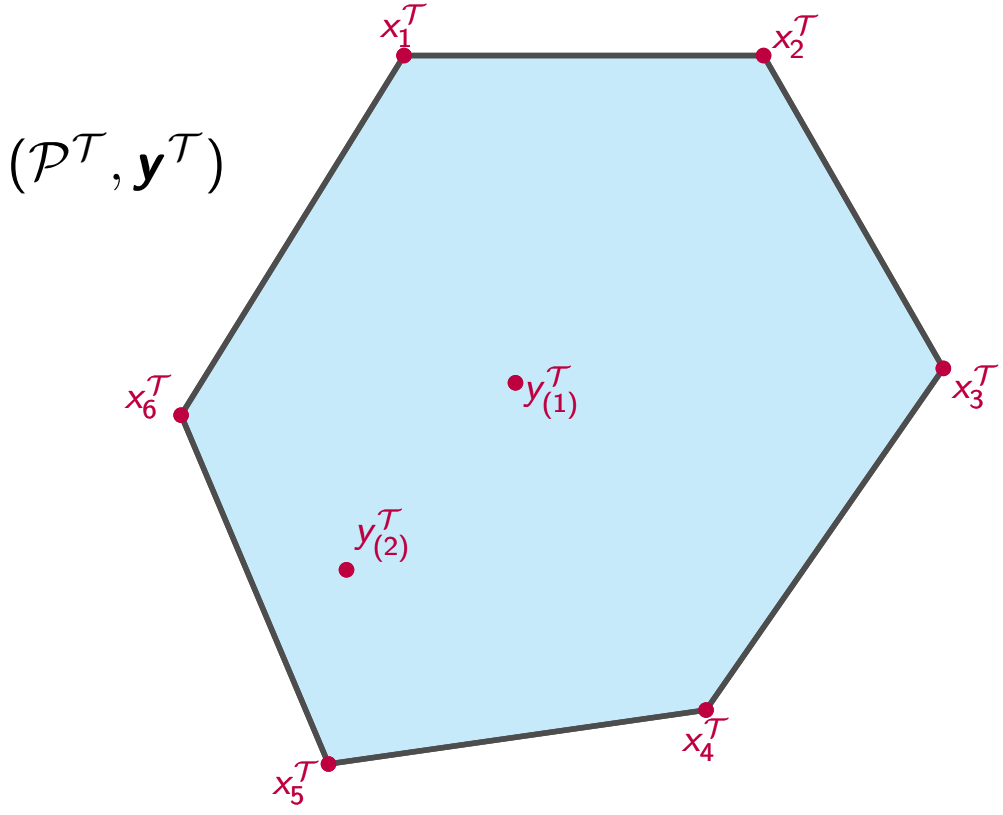
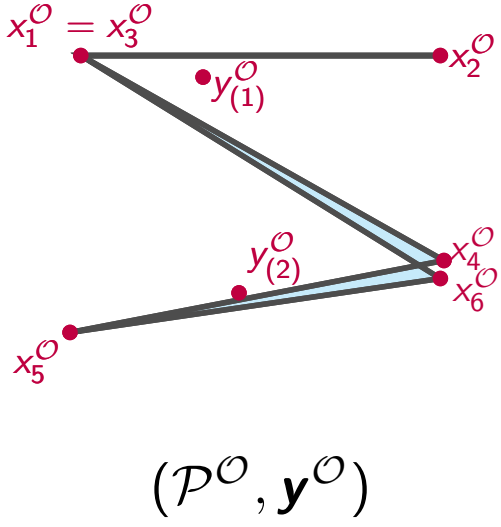
for all non-adjacent $1 \leqslant i, j \leqslant n$ and all $1 \leqslant \rho \neq \gamma \leqslant L$, and such that

each $y_{(\rho)}^{\mathcal{T}}$ is located inside the polygon $\mathcal{P}^{\mathcal{T}} = (x_1^{\mathcal{T}}, x_2^{\mathcal{T}}, \ldots, x_n^{\mathcal{T}})$.





$(\mathcal{P}^{\mathcal{O}}, \boldsymbol{y}^{\mathcal{O}})$

- Origami map does not increase distances: $(x_i - x_j)^2 \geqslant 0$, i.e., $|x_i^{\mathcal{T}} - x_j^{\mathcal{T}}| \geqslant |x_i^{\mathcal{O}} - x_j^{\mathcal{O}}|$.

Definition

Let $L \geqslant 0$. An L-punctured polygon is a pair $(\mathcal{P}, \boldsymbol{y})$, where $\mathcal{P} = (x_1, x_2, \ldots, x_n) \in (\mathbb{R}^{2,2})^n$ and $\boldsymbol{y} = (y_{(1)}, y_{(2)}, \ldots, y_{(L)}) \in (\mathbb{R}^{2,2})^L$, such that

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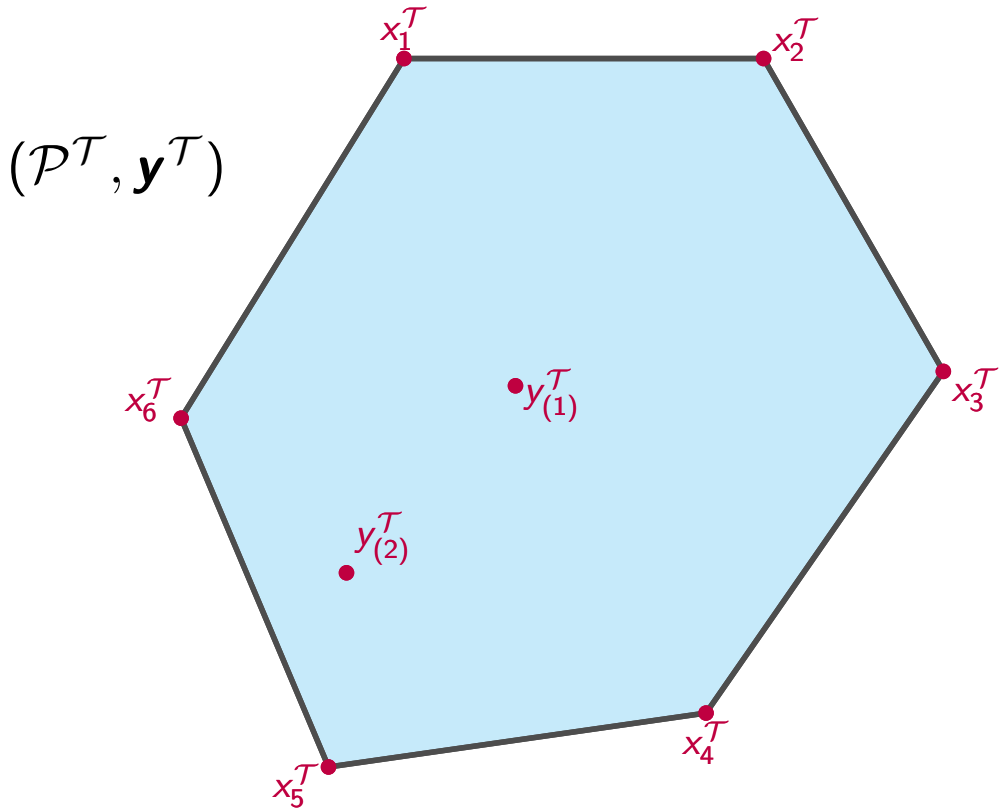
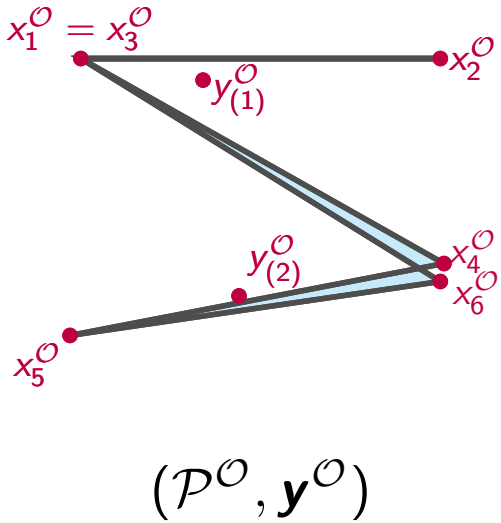
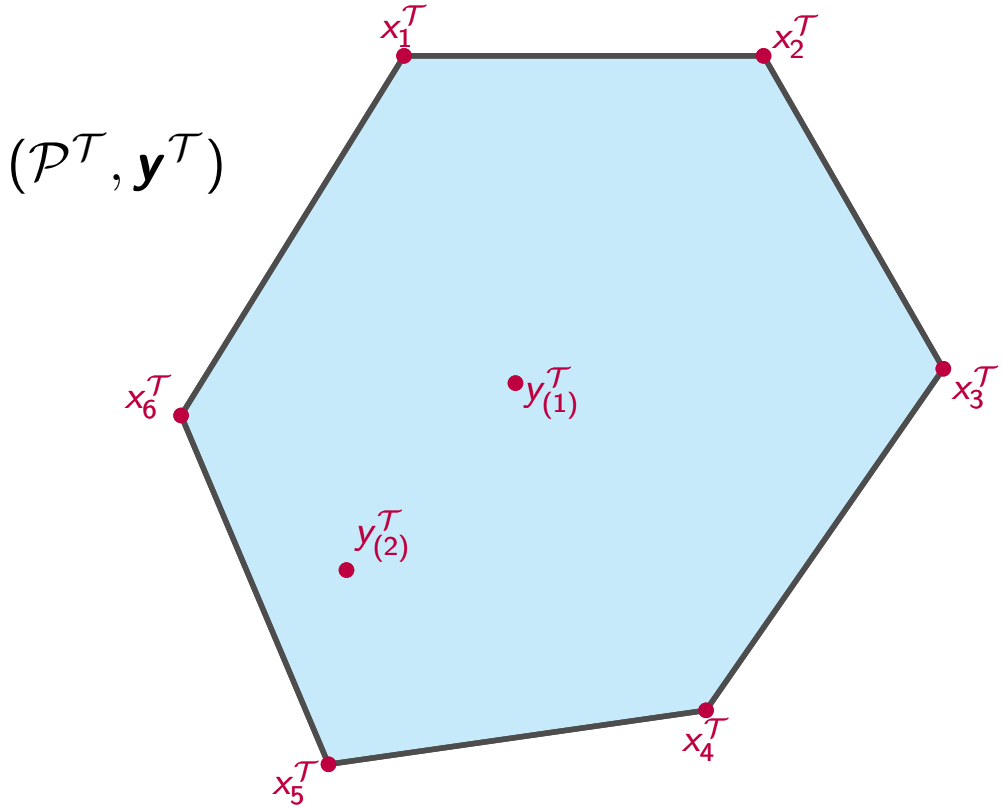
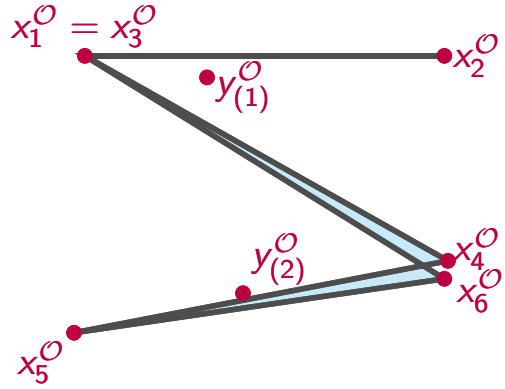




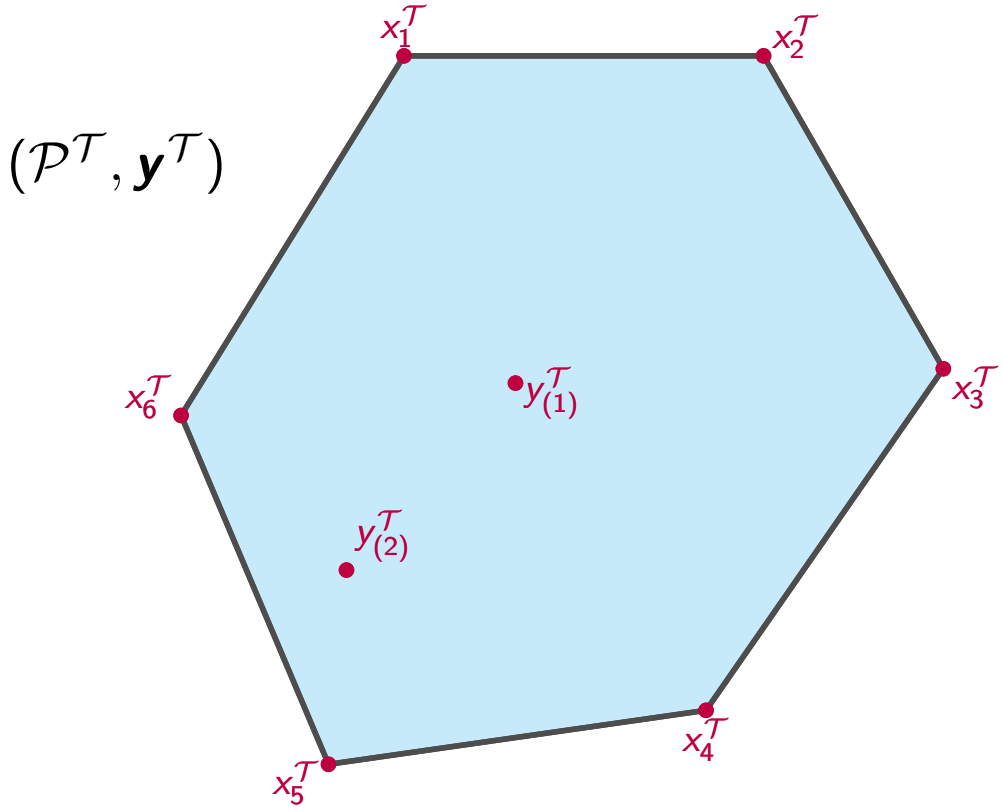
Problem: Find an OCP with boundary $\mathcal{P}$ passing through all points in $\boldsymbol{y}$.

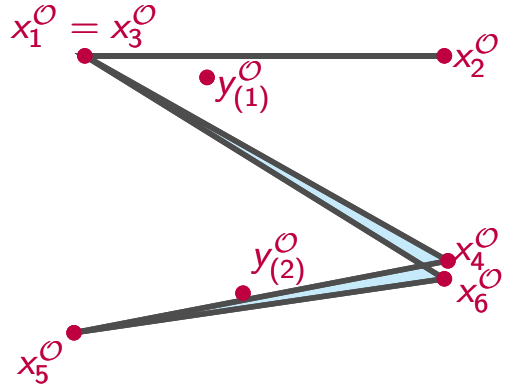




$(\mathcal{P}^{\mathcal{O}}, \boldsymbol{y}^{\mathcal{O}})$

Problem: Find an OCP with boundary $\mathcal{P}$ passing through all points in $\boldsymbol{y}$.



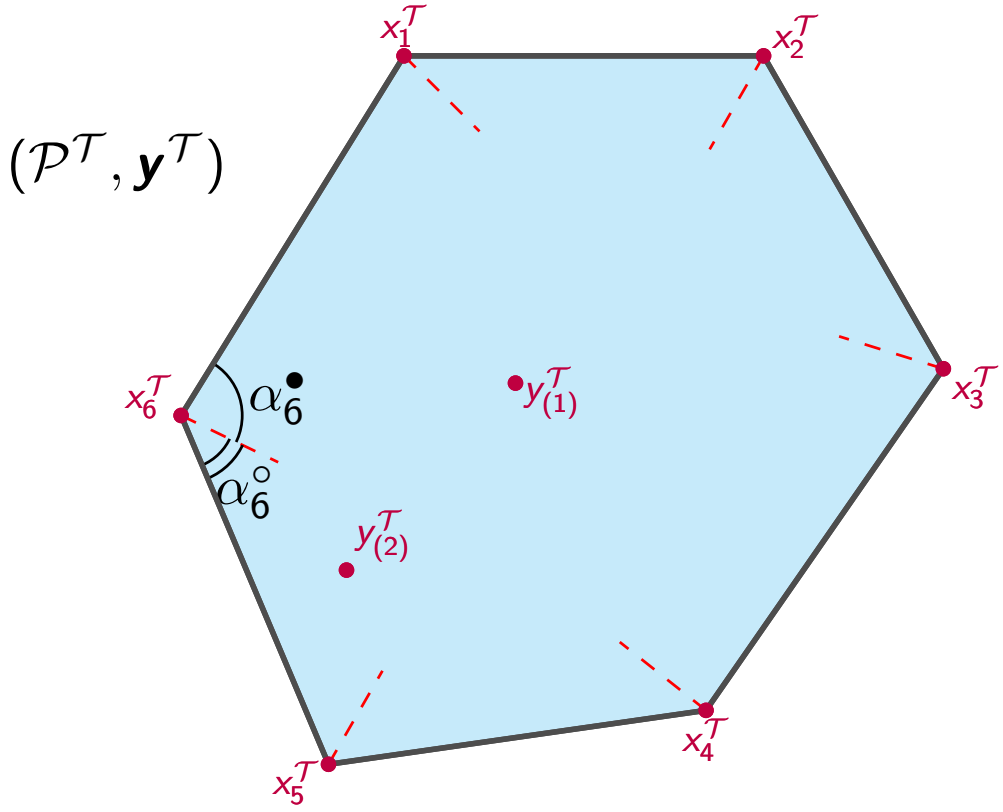


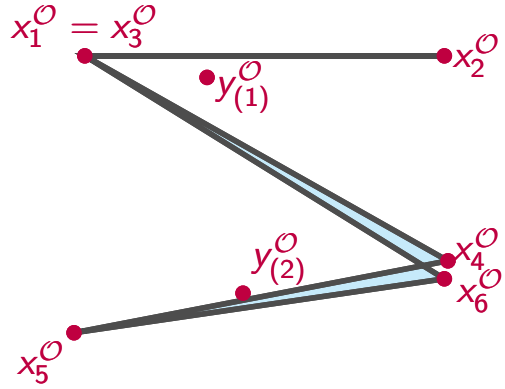
$(\mathcal{P}^{\mathcal{O}}, \boldsymbol{y}^{\mathcal{O}})$

Problem: Find an OCP with boundary $\mathcal{P}$ passing through all points in $\boldsymbol{y}$.

- Recover white/black angle sums $(\alpha_i^{\circ}, \alpha_i^{\bullet})$ from the geometry of $\mathcal{P}$:

$$\alpha_i^{\circ} + \alpha_i^{\bullet} = \alpha_i^{\mathcal{T}} \quad \text{and} \quad \alpha_i^{\circ} - \alpha_i^{\bullet} \equiv \alpha_i^{\mathcal{O}} \bmod 2\pi.$$





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Problem: Find an OCP with boundary $\mathcal{P}$ passing through all points in $\boldsymbol{y}$.

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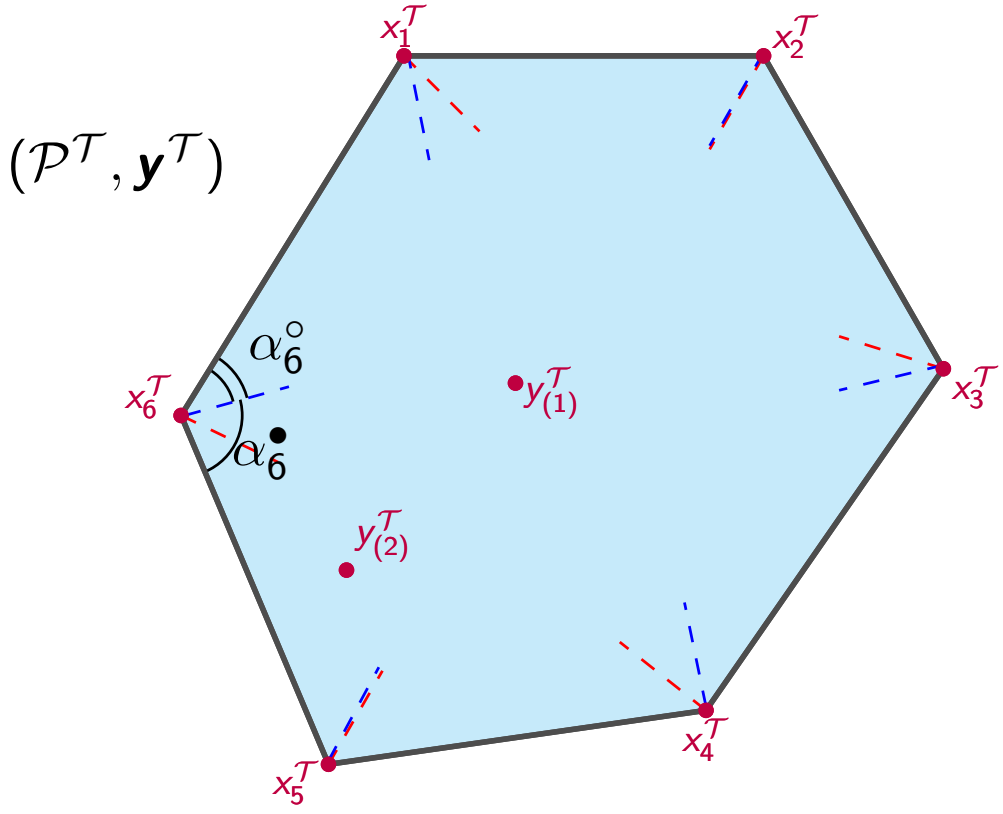
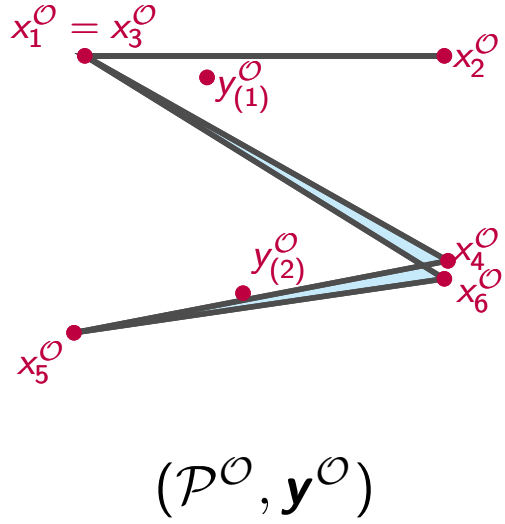
- Red folding ray splits the angle $\alpha_i^{\mathcal{T}}$ into angles $(\alpha_i^{\bullet}, \alpha_i^{\circ})$.



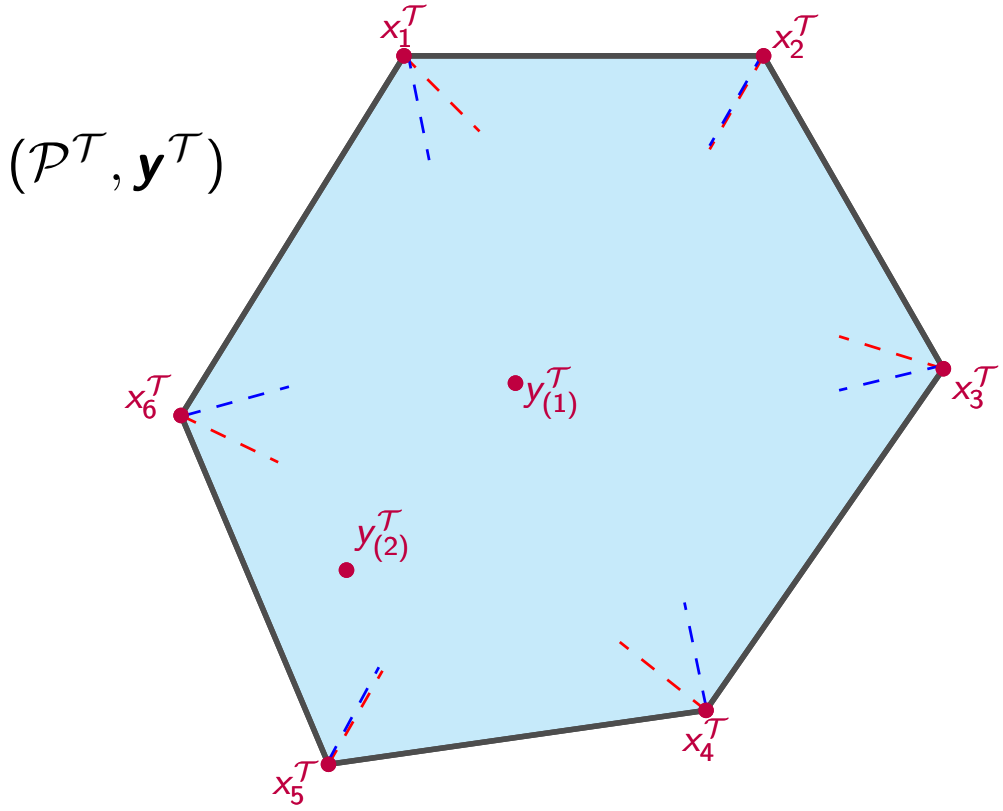


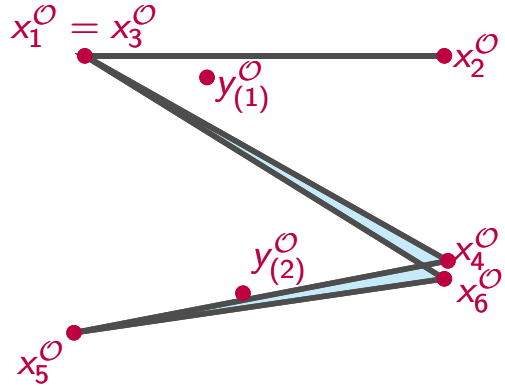
Problem: Find an OCP with boundary $\mathcal{P}$ passing through all points in $\boldsymbol{y}$.

- Recover white/black angle sums $(\alpha_i^{\circ}, \alpha_i^{\bullet})$ from the geometry of $\mathcal{P}$:

$$\alpha_i^{\circ} + \alpha_i^{\bullet} = \alpha_i^{\mathcal{T}} \quad \text{and} \quad \alpha_i^{\circ} - \alpha_i^{\bullet} \equiv \alpha_i^{\mathcal{O}} \bmod 2\pi.$$

- Red folding ray splits the angle $\alpha_i^{\mathcal{T}}$ into angles $(\alpha_i^{\bullet}, \alpha_i^{\circ})$.
- Blue folding ray splits the angle $\alpha_i^{\mathcal{T}}$ into angles $(\alpha_i^{\circ}, \alpha_i^{\bullet})$.

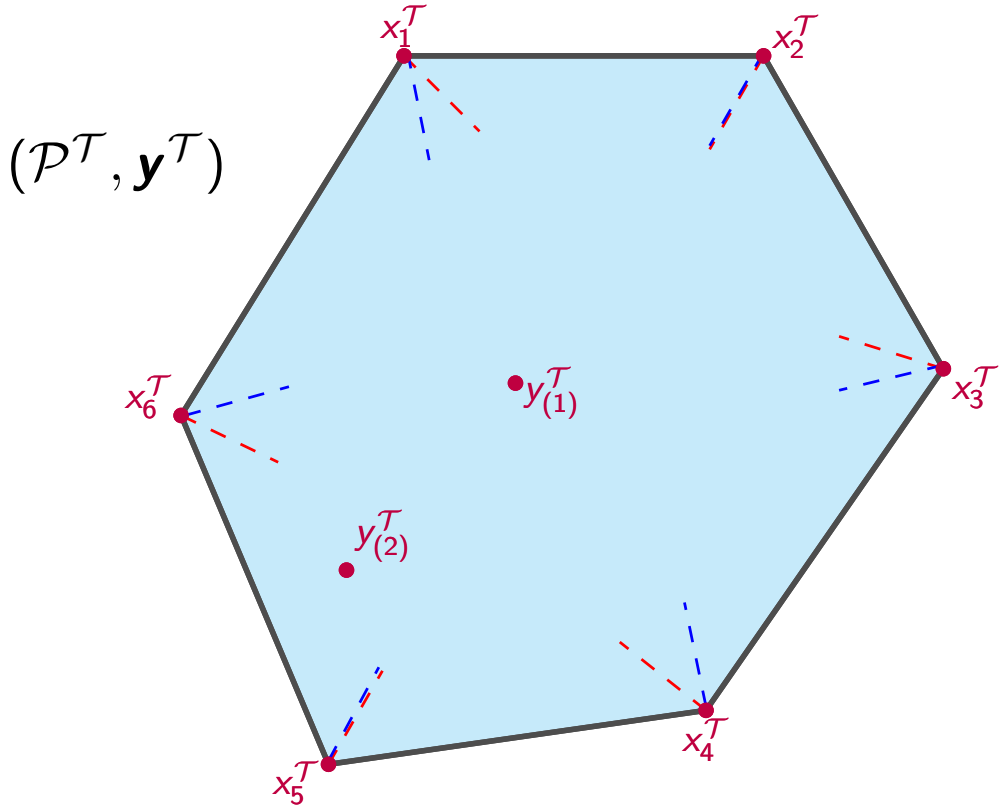


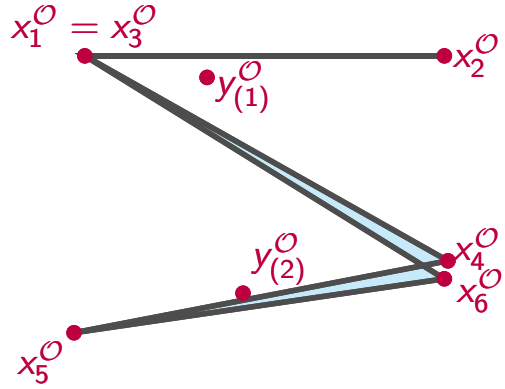


$(\mathcal{P}^{\mathcal{O}}, \boldsymbol{y}^{\mathcal{O}})$

Problem: Find an OCP with boundary $\mathcal{P}$ passing through all points in $\boldsymbol{y}$.

Algorithm (Crystallization algorithm)



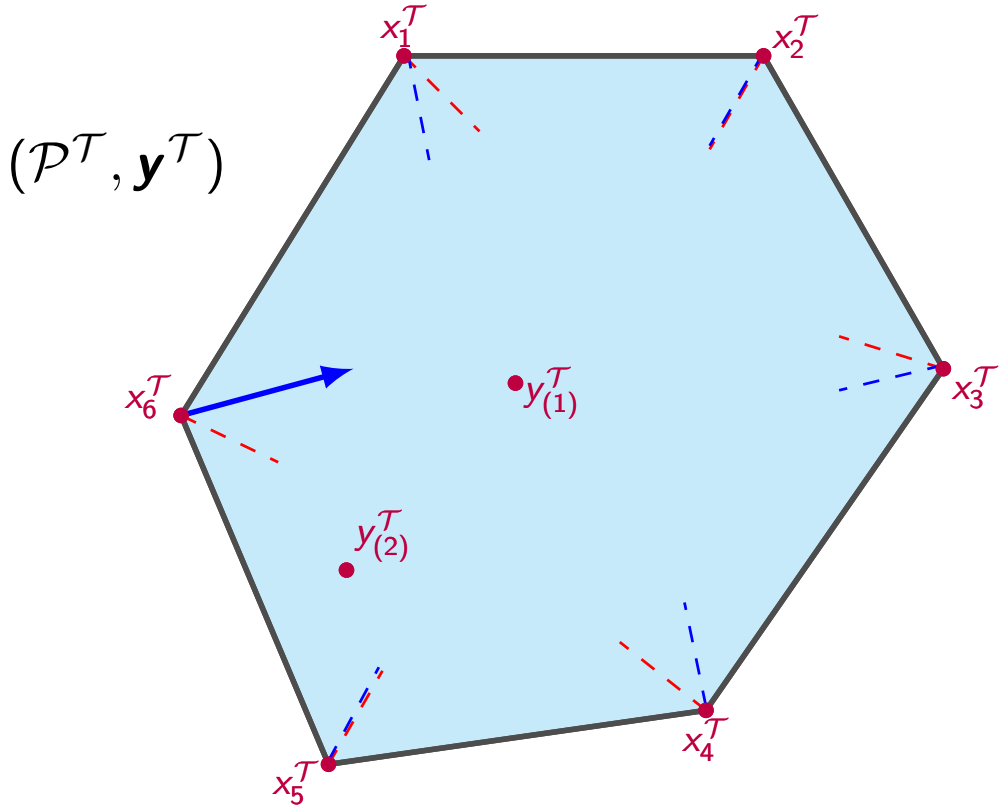


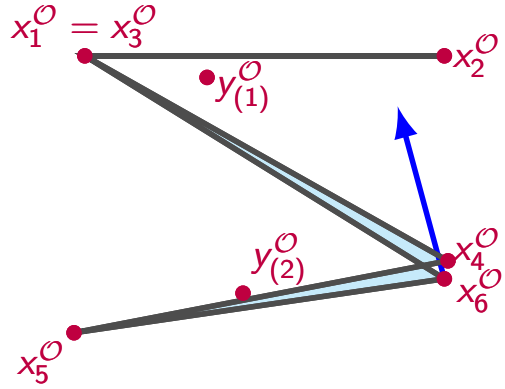
$(\mathcal{P}^{\mathcal{O}}, \boldsymbol{y}^{\mathcal{O}})$

Problem: Find an OCP with boundary $\mathcal{P}$ passing through all points in $\boldsymbol{y}$.

Algorithm (Crystallization algorithm)

1. *Choose* $1 \leqslant i \leqslant n$ *and a red or a blue folding ray* $R_i \in \mathbb{R}^{2,2}$.





$(\mathcal{P}^{\mathcal{O}}, \boldsymbol{y}^{\mathcal{O}})$

Problem: Find an OCP with boundary $\mathcal{P}$ passing through all points in $\boldsymbol{y}$.

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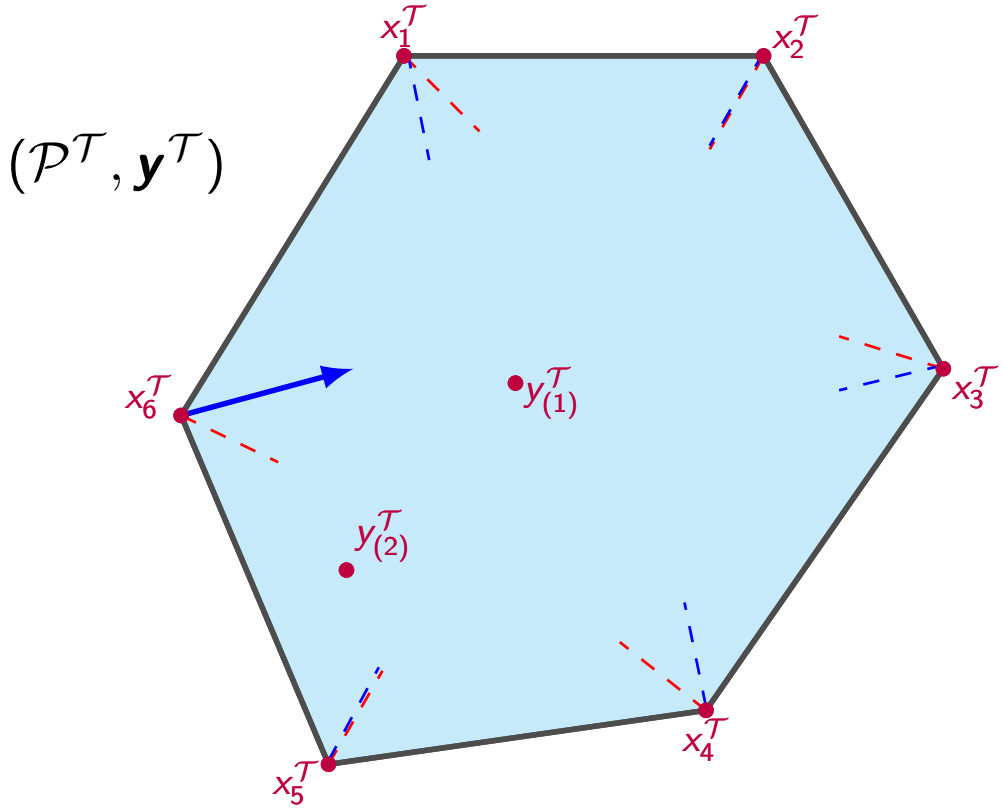
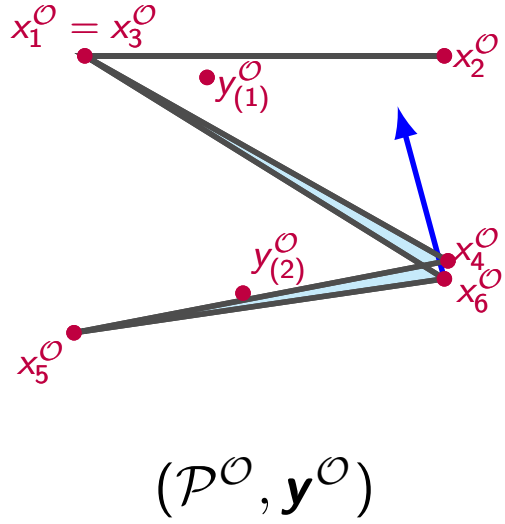
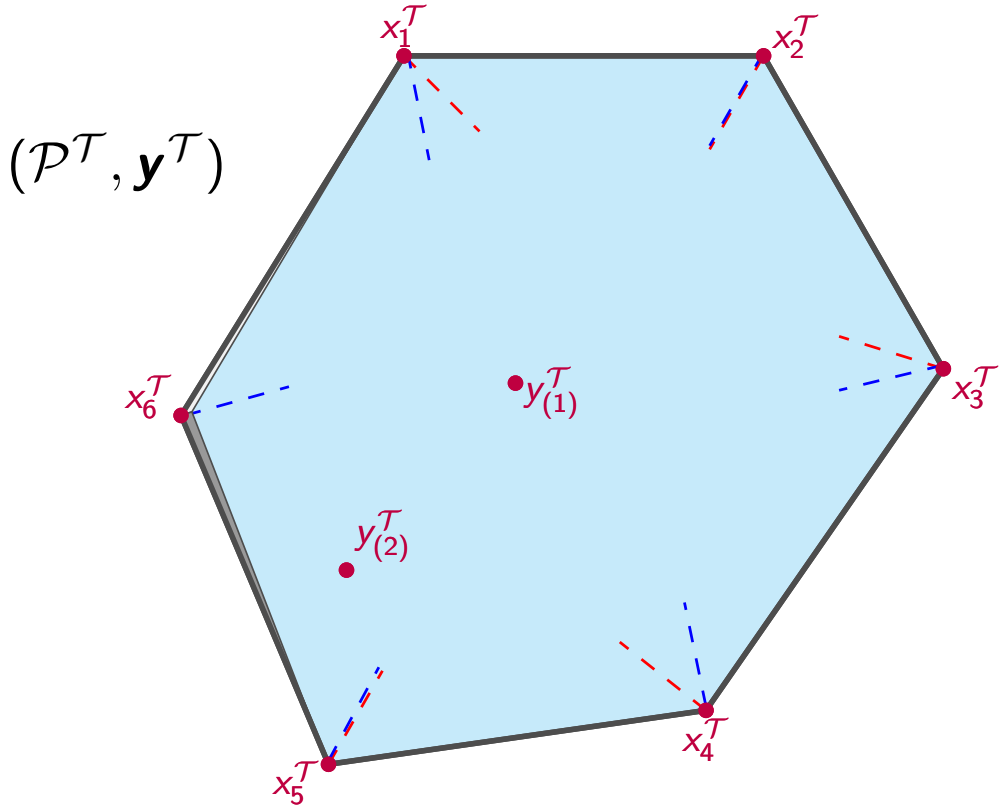


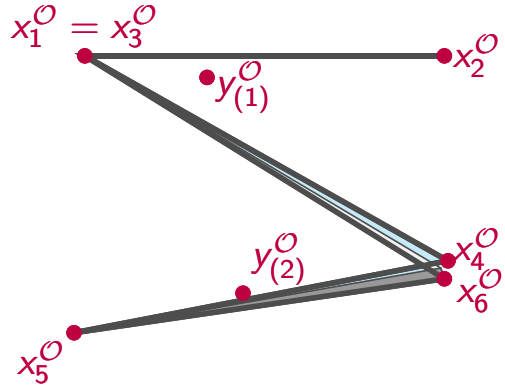


Problem: Find an OCP with boundary $\mathcal{P}$ passing through all points in $\boldsymbol{y}$.

Algorithm (Crystallization algorithm)

1. *Choose* $1 \leqslant i \leqslant n$ *and a red or a blue folding ray* $R_i \in \mathbb{R}^{2,2}$.
2. *Let* $x_i(t) = x_i + tR_i$, *and let* $t > 0$ *be minimal such that either* $(x_i(t) - x_j)^2 = 0$ *or* $\left(x_i(t) - y_{(\rho)}\right)^2 = 0$ *for some* $1 \leqslant j \leqslant n$ *or* $1 \leqslant \rho \leqslant L$.





$(\mathcal{P}^{\mathcal{O}}, \boldsymbol{y}^{\mathcal{O}})$

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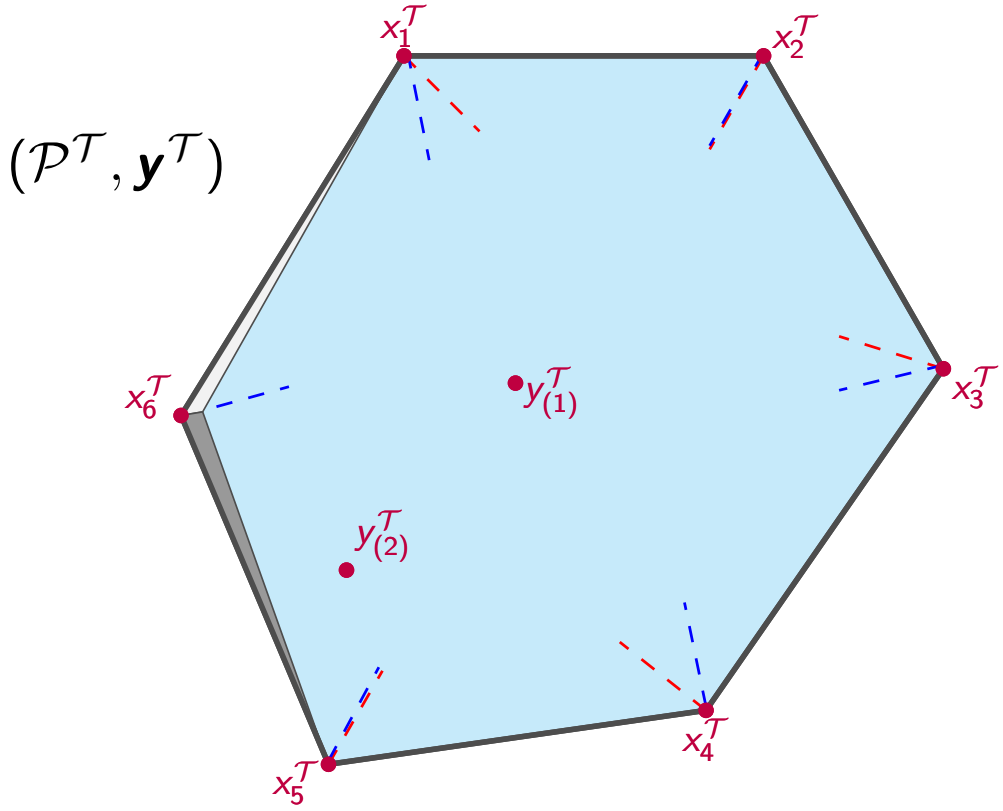

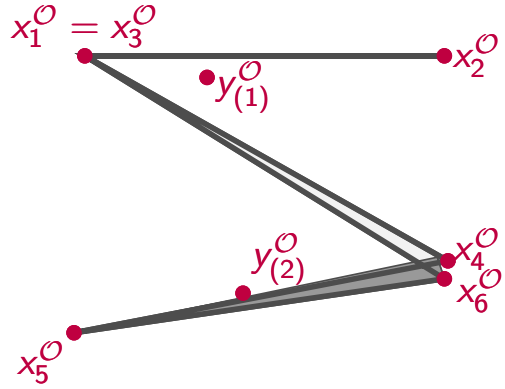

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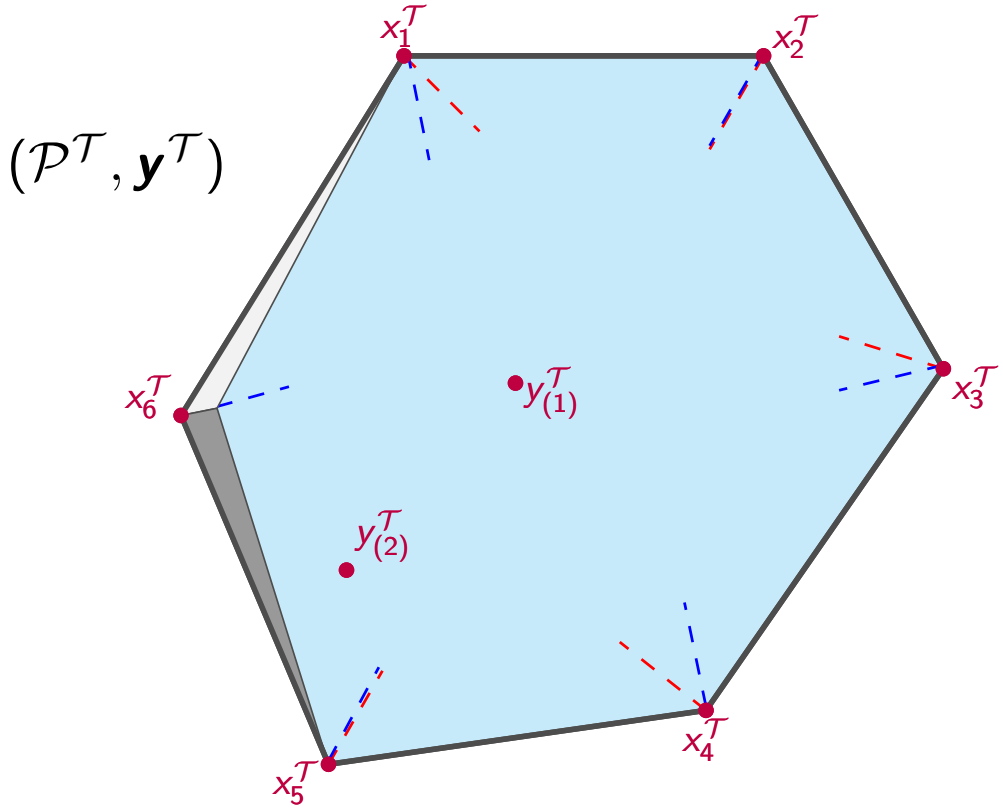

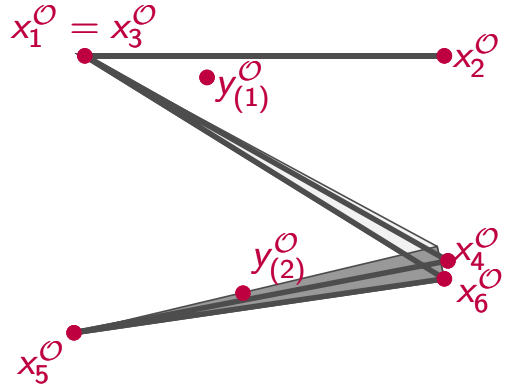

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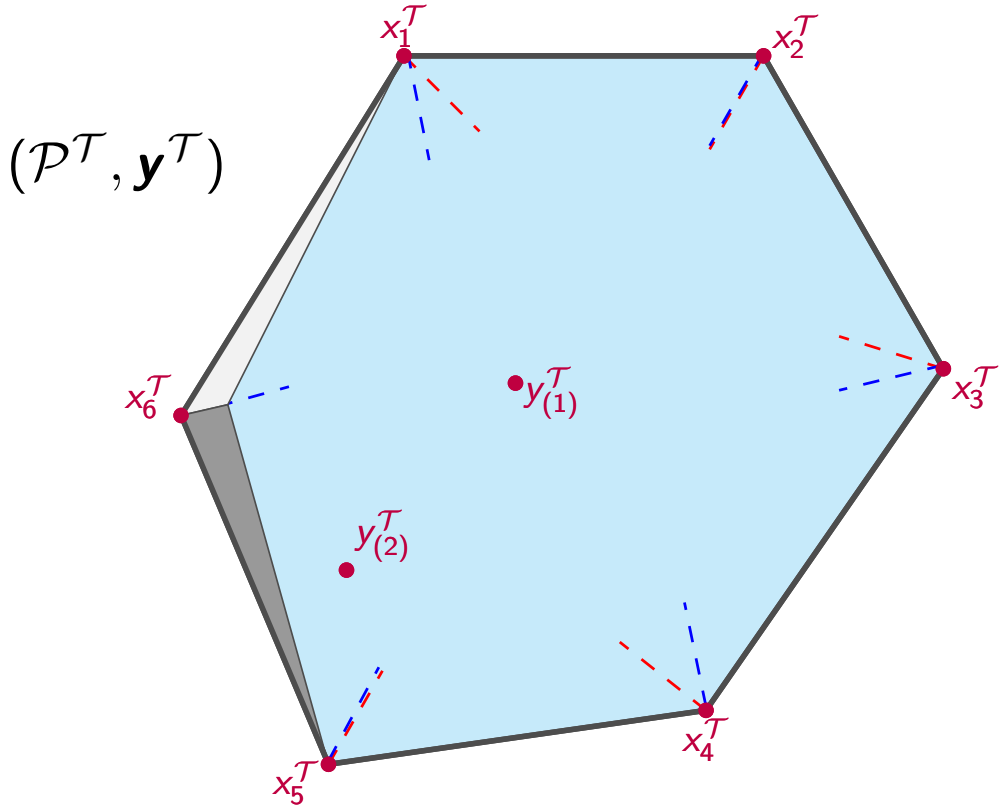

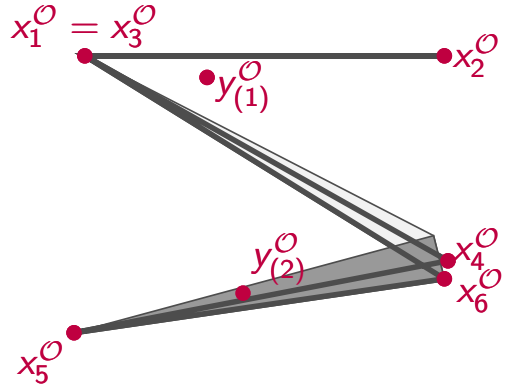

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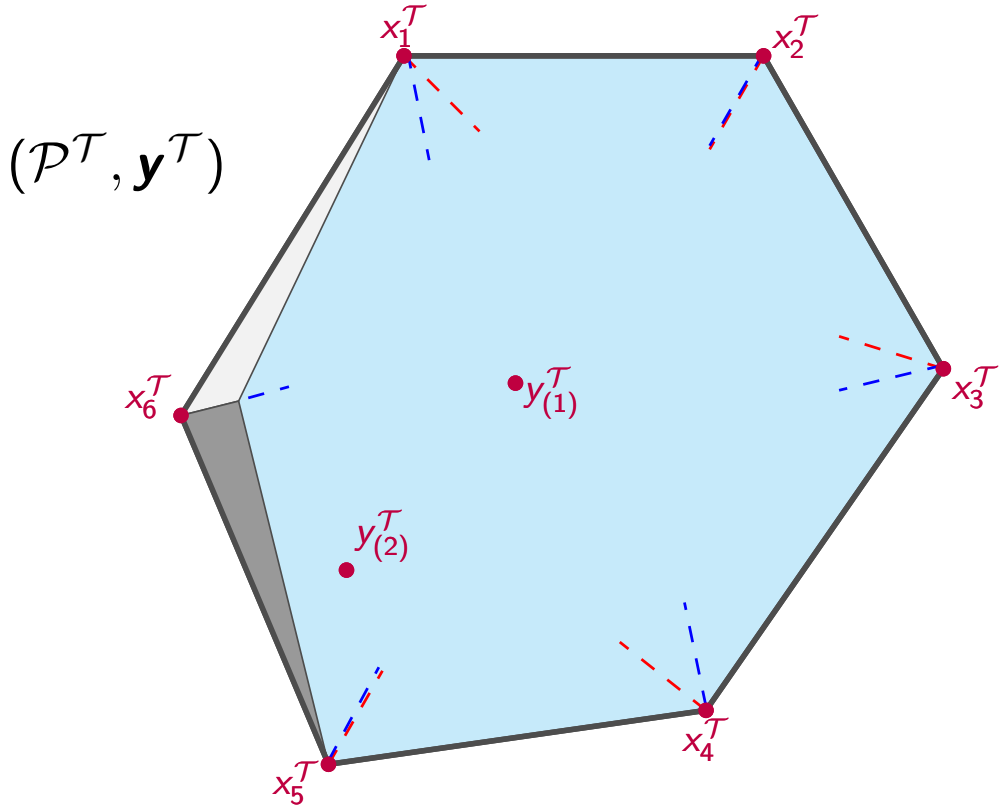

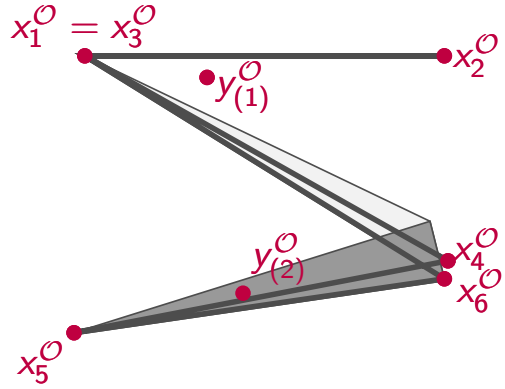

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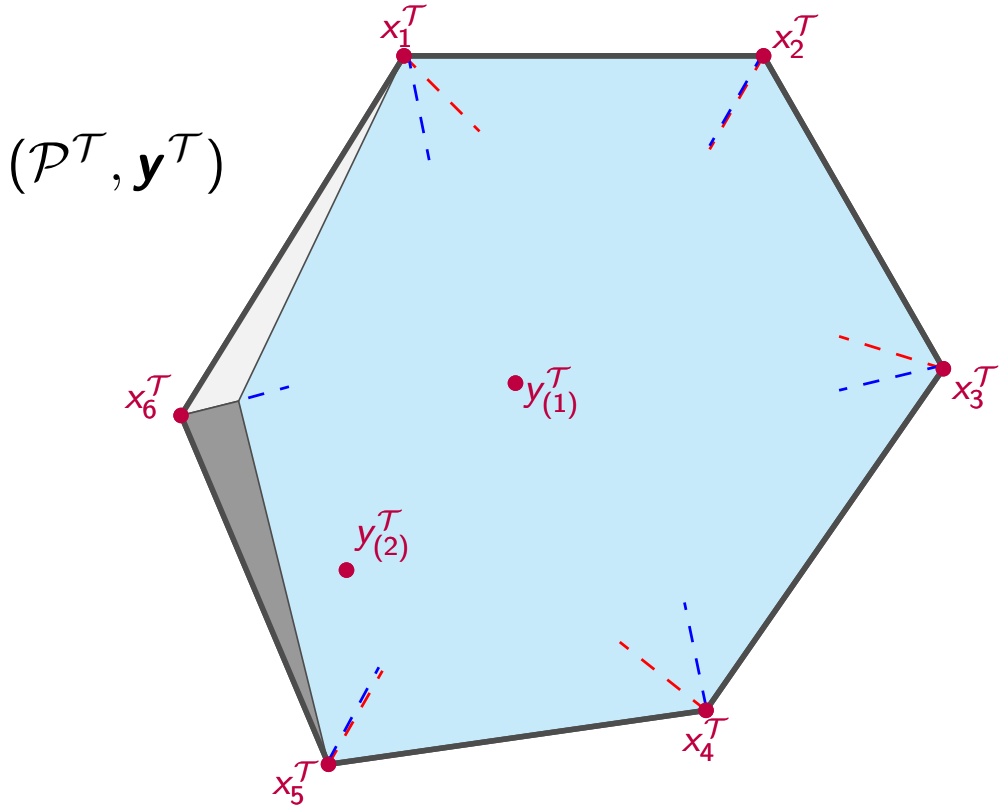

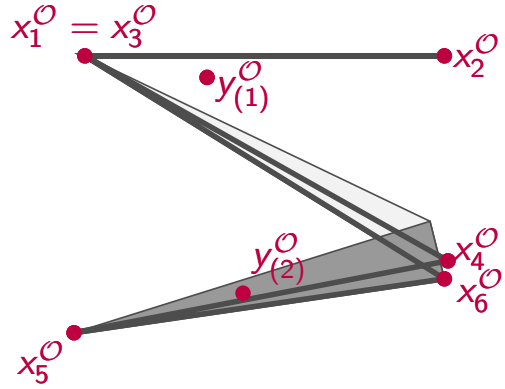

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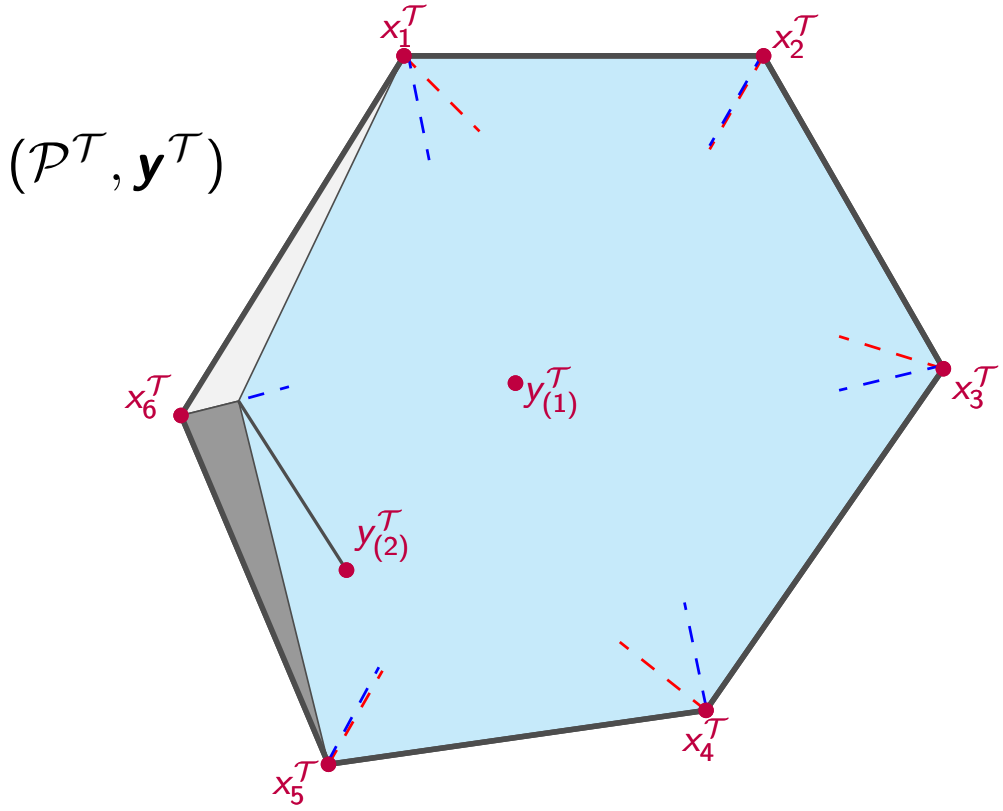

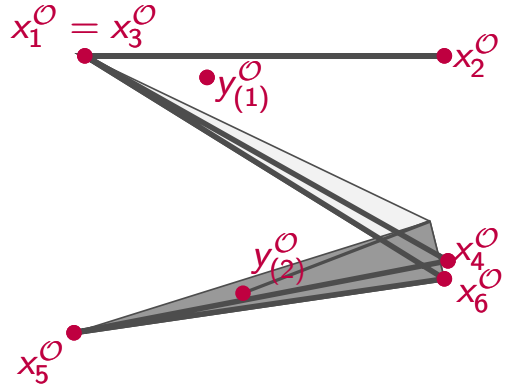

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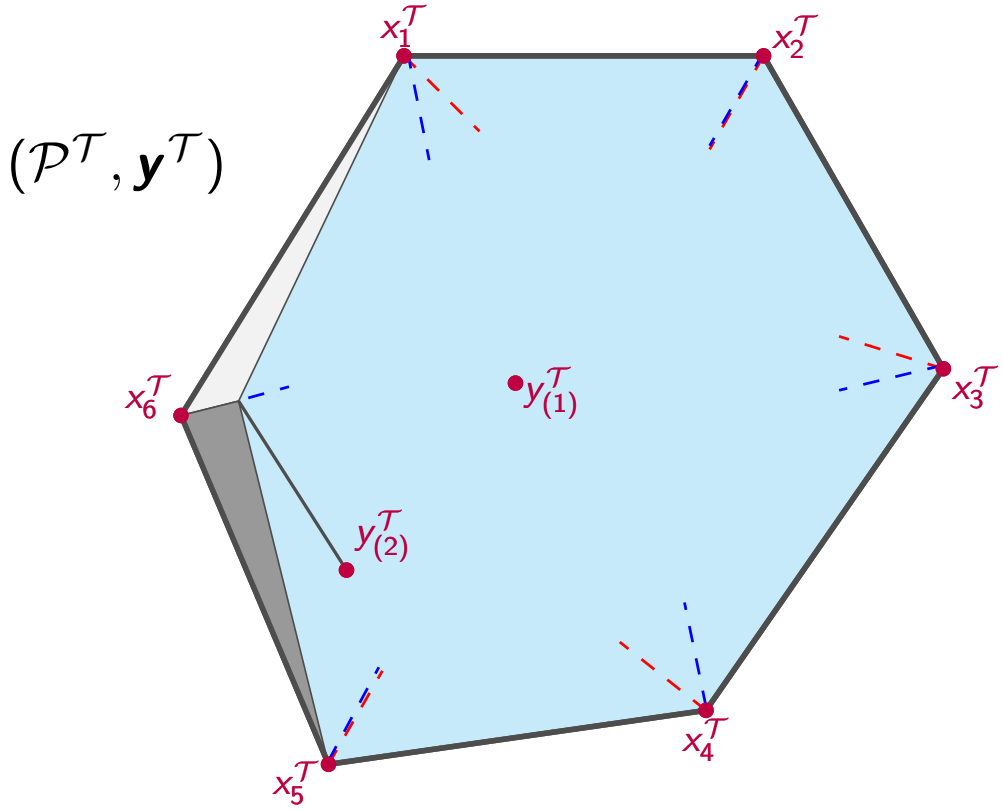

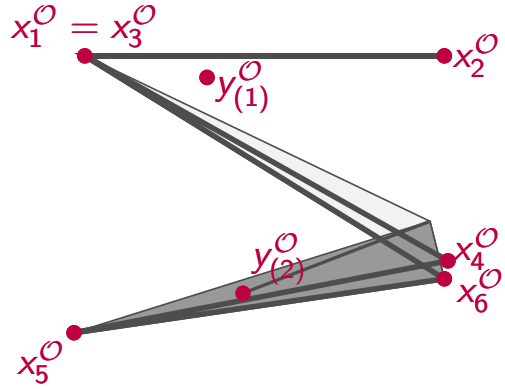

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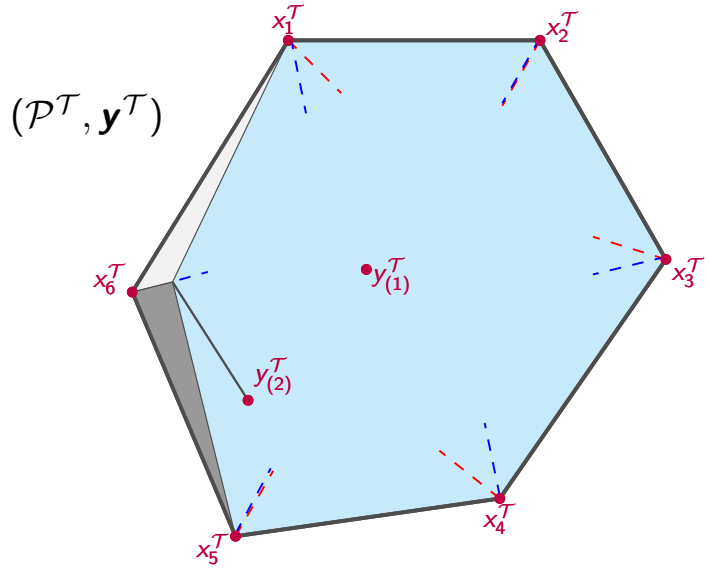

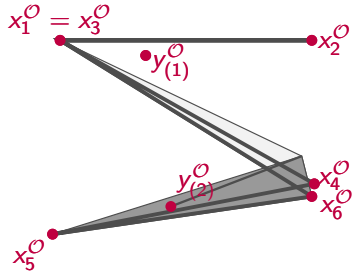

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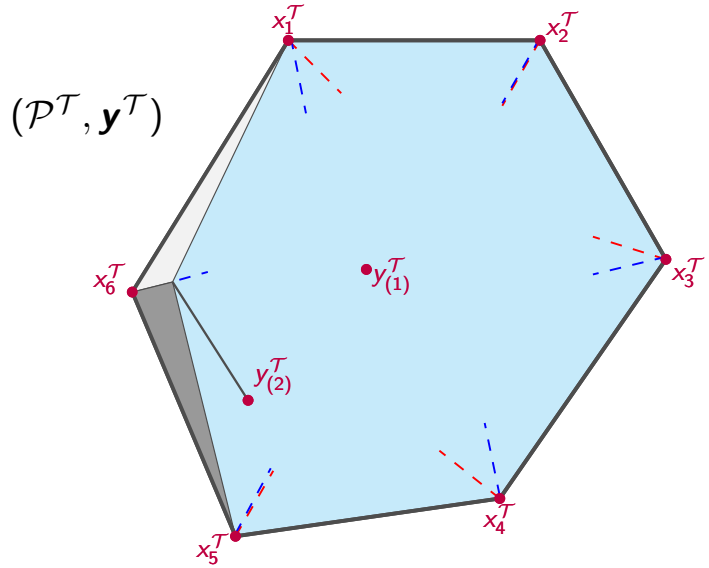

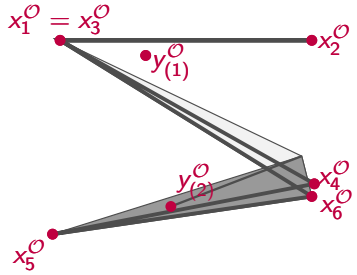

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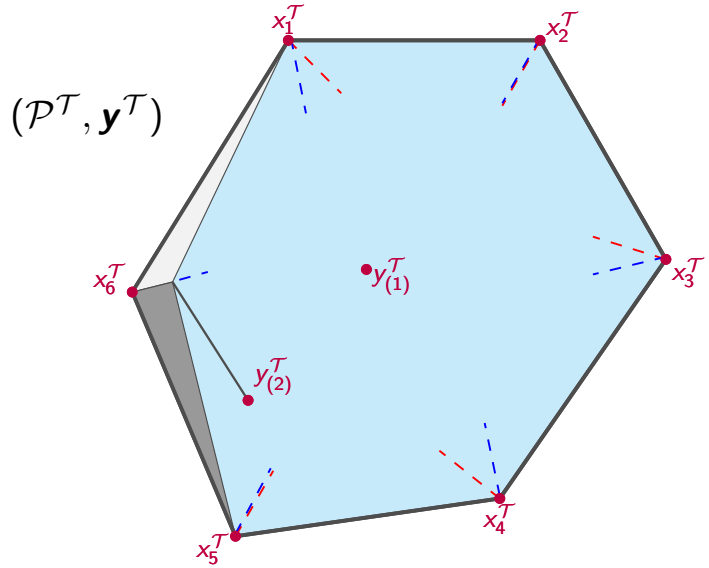

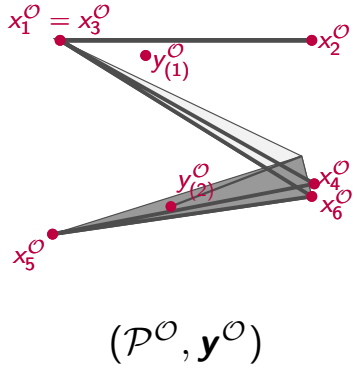

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For generic L-punctured polygons $(\mathcal{P}, \boldsymbol{y})$, *it outputs a valid, embedded OCP.*

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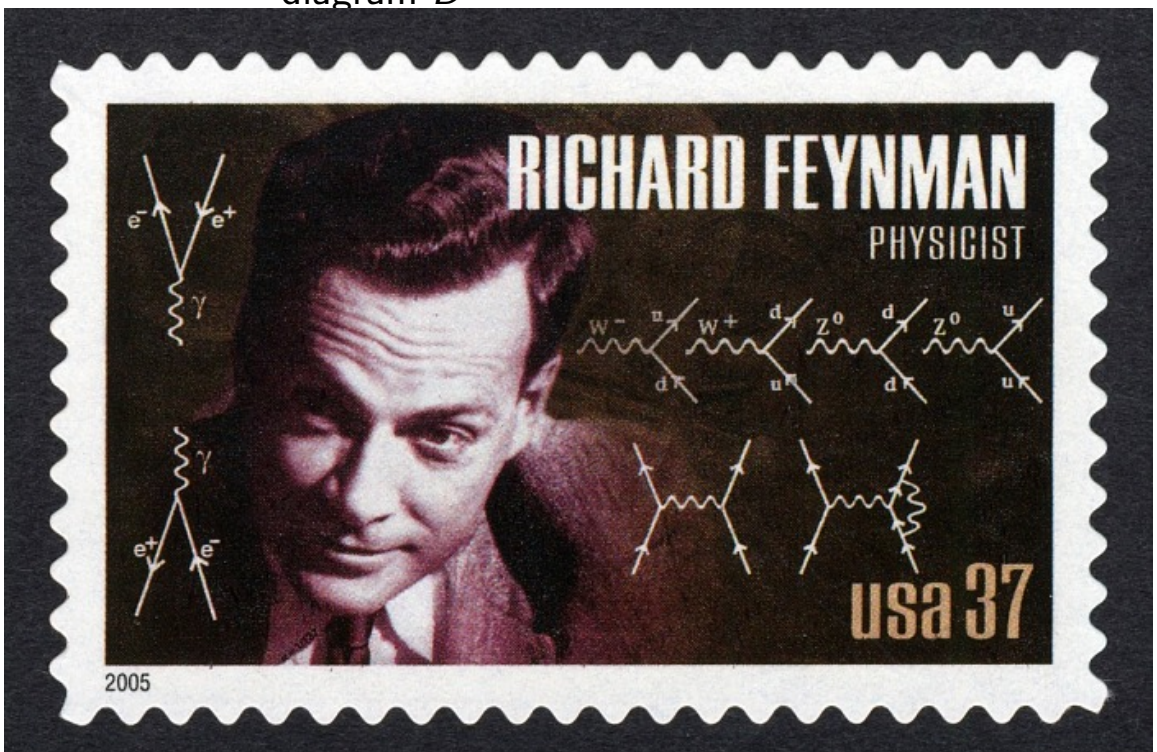

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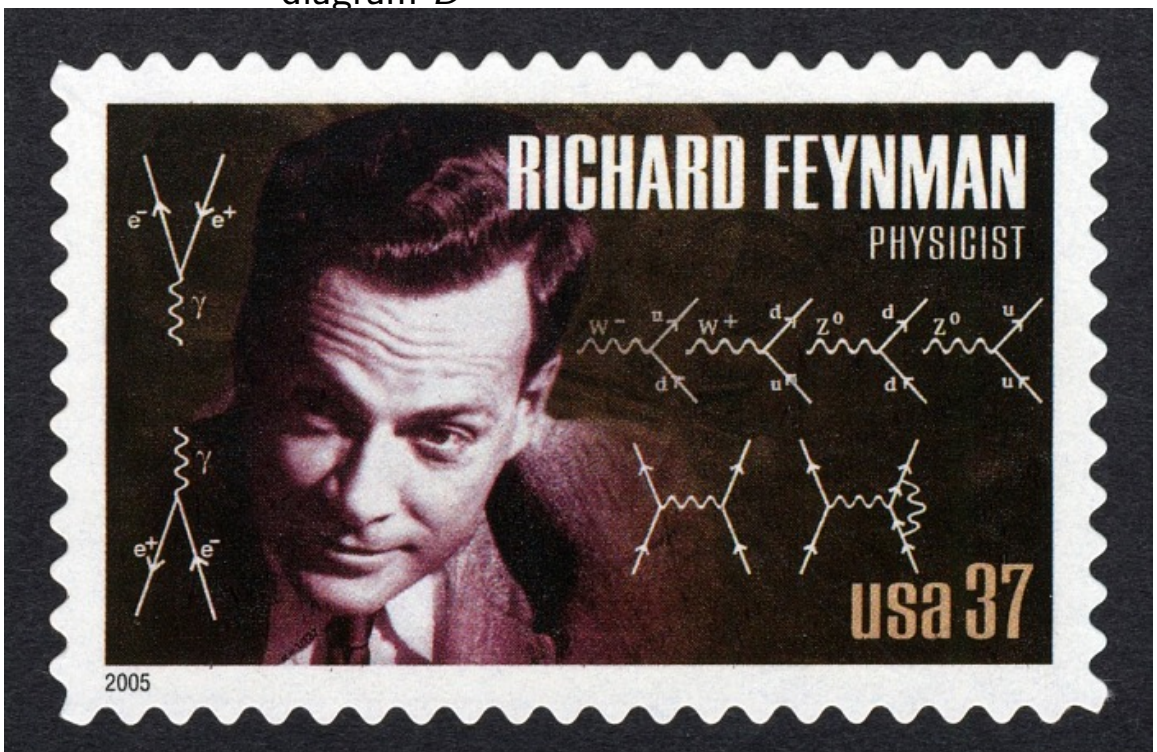

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$$A(P_1, P_2, \ldots, P_n) = \sum_{\substack{\text{Feynman} \\ \text{diagram } D}} A_D = A^{(\text{tree})} + \underset{L = |E| - |V| + 1}{A^{(L=1)}} + A^{(L=2)} + \cdots .$$

Define $\mathcal{P} = (x_1, \ldots, x_n)$ by $x_i - x_{i-1} = P_i$. Write $A(x_1, \ldots, x_n) = A(P_1, \ldots, P_n)$.

Algorithm (Britto–Cachazo–Feng–Witten (2005))

1. *Choose* $1 \leqslant i \leqslant n$ *and a red or a blue folding ray* $R_i \in \mathbb{R}^{2,2}$.
2. *Let* $x_i(z) = x_i + zR_i$ *for* $z \in \mathbb{C}$ *and set* $A^{(tree)}(z) := A^{(tree)}(x_1, \ldots, x_i(z), \ldots, x_n)$.

$$A^{(tree)}(0) = \sum_{\text{poles } z_j \text{ of } A^{(tree)}(z)} \text{Res}_{z=z_j} \frac{A^{(tree)}(z)}{z}.$$

3. *Compute each smaller scattering amplitude* $\text{Res}_{z=z_j} \frac{A^{(tree)}(z)}{z}$ *recursively.*

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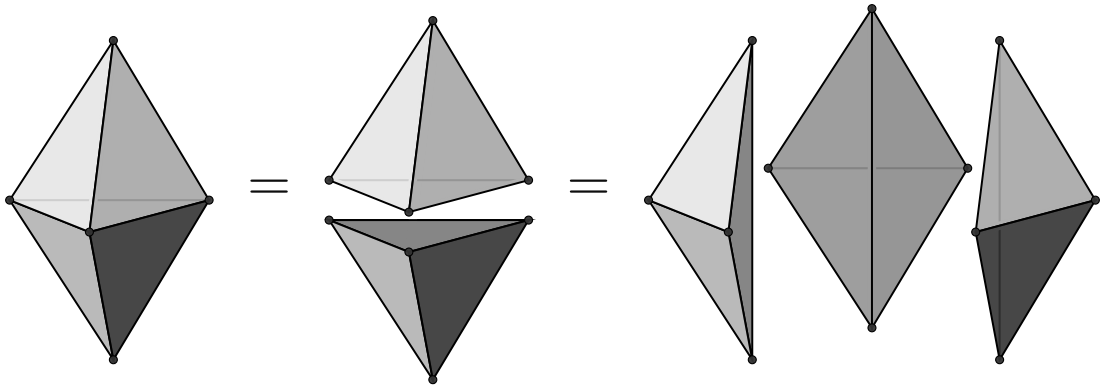

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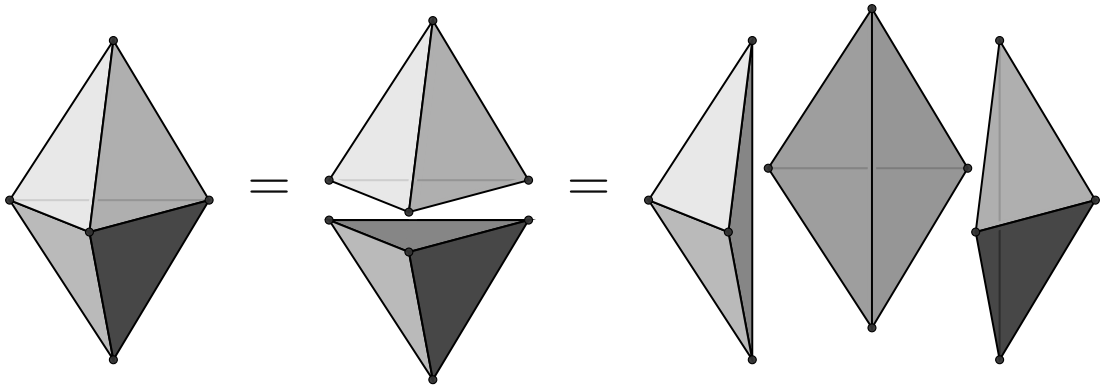

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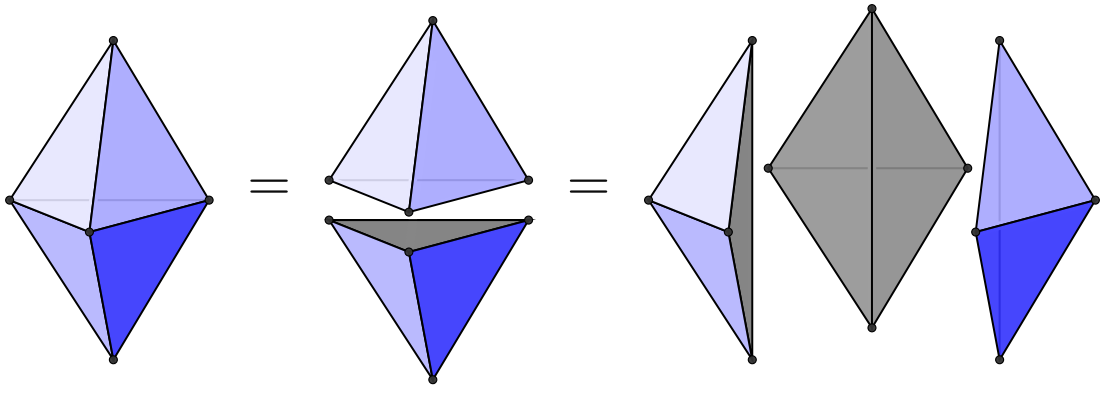

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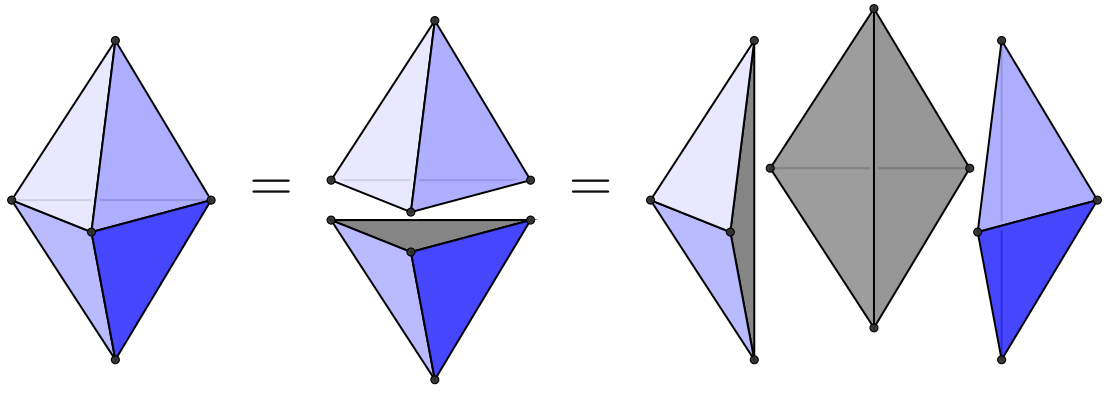

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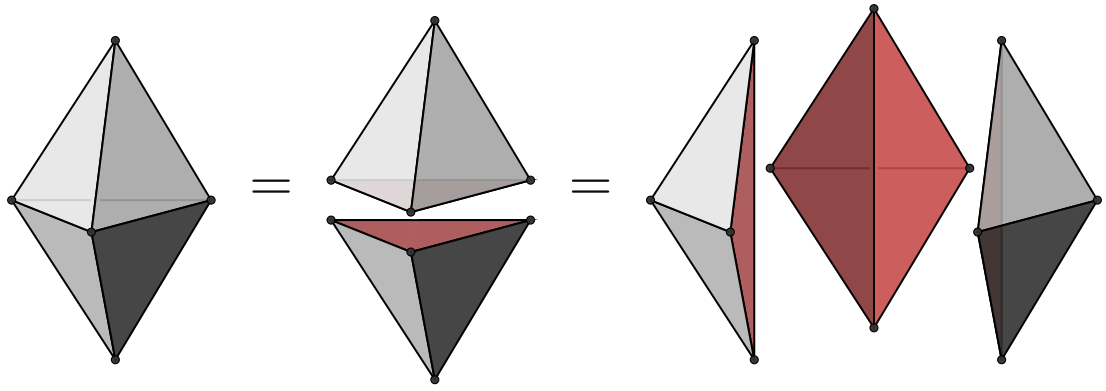

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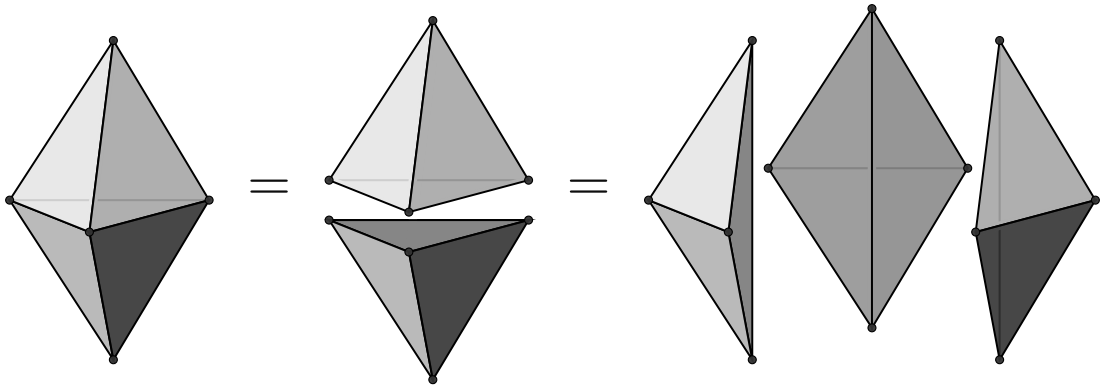

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(1) Standard BCFW cells triangulate tree momentum-twistor amplituhedron.
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Theorem (G.'25+)

All 8 conjectures are true.

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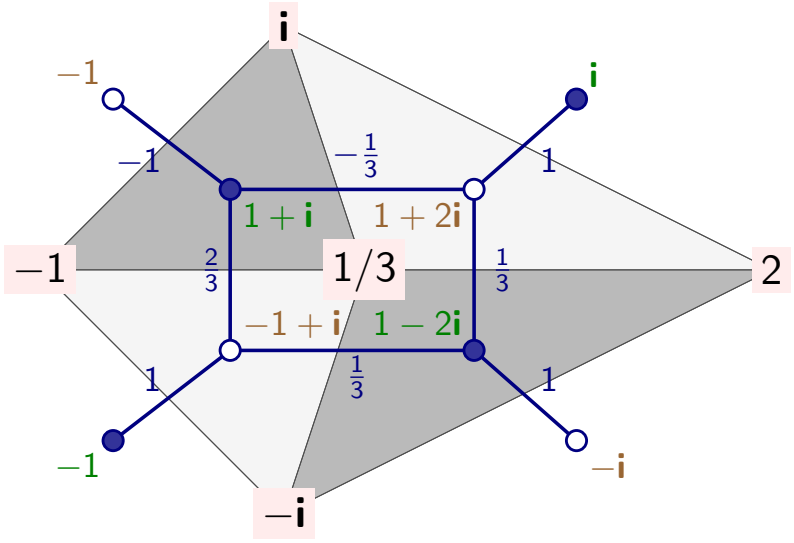
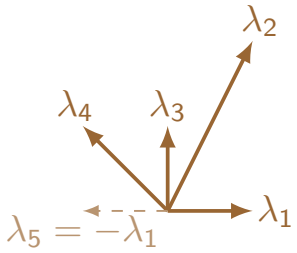
- With this formulation, (8) becomes equivalent to the previous result that the crystallization algorithm always outputs a valid, embedded OCP.
- Our approach also implies that (4) is equivalent to (8). □





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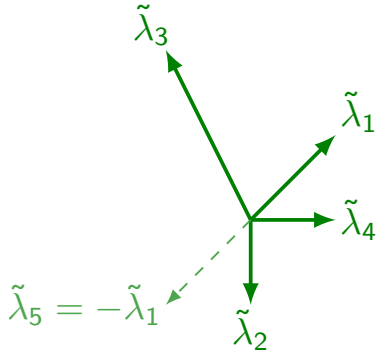

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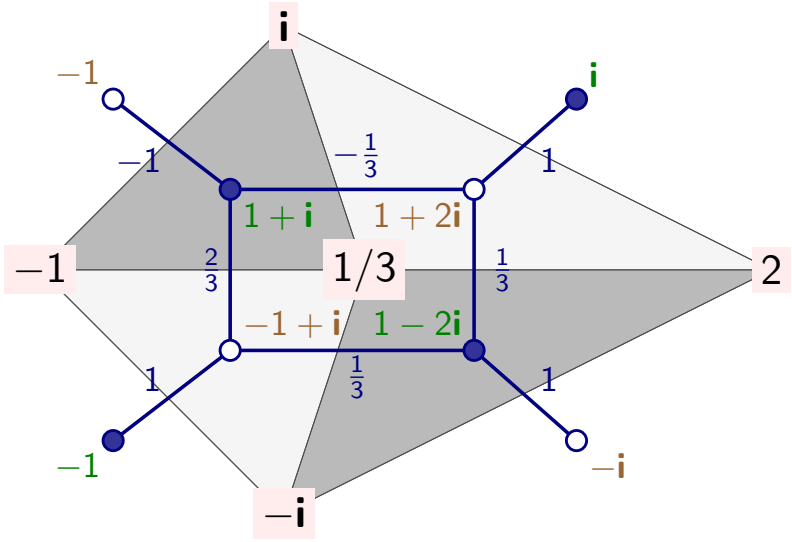

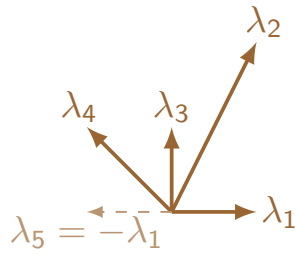

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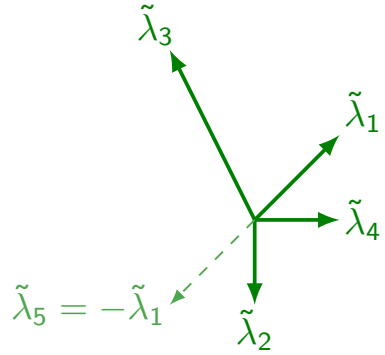

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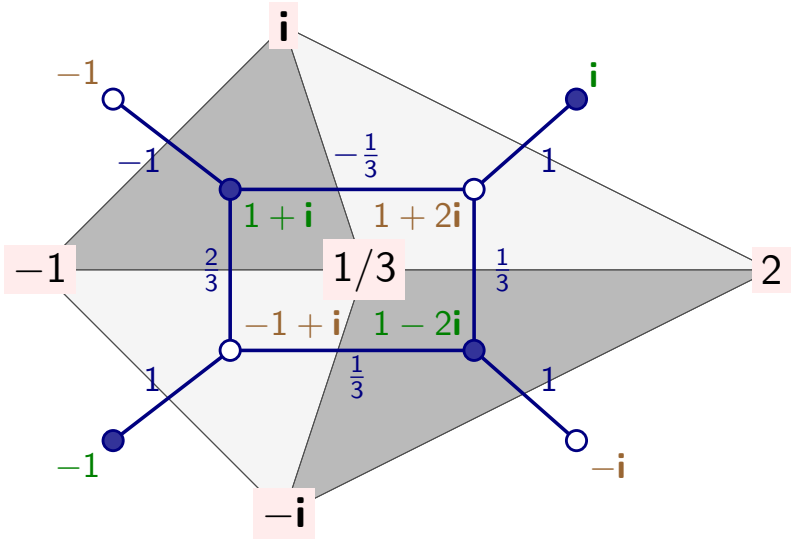

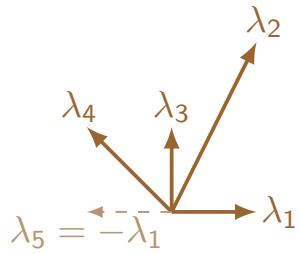

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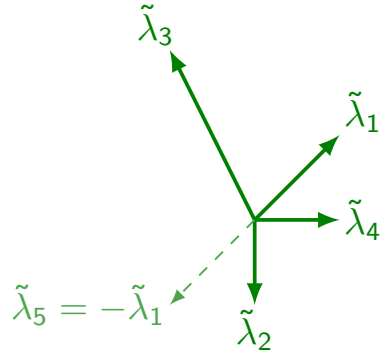

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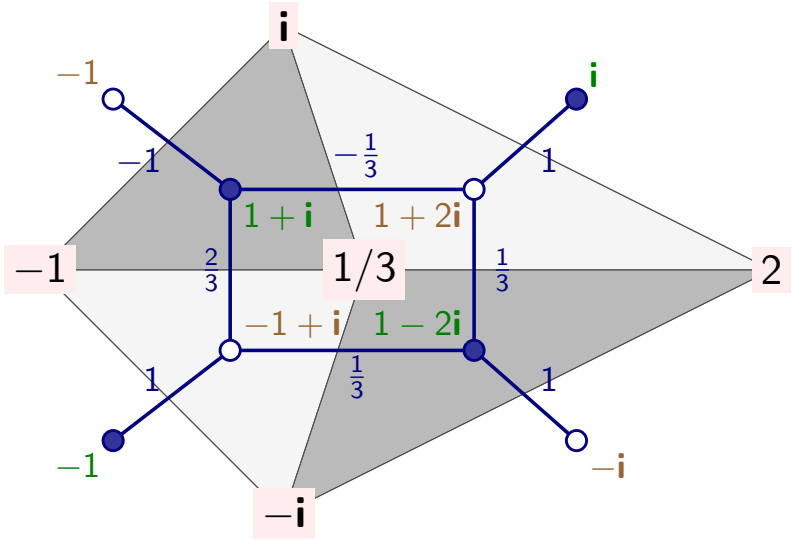

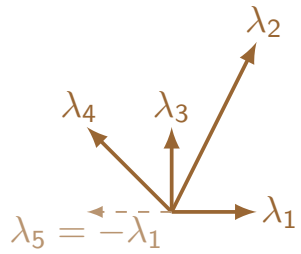

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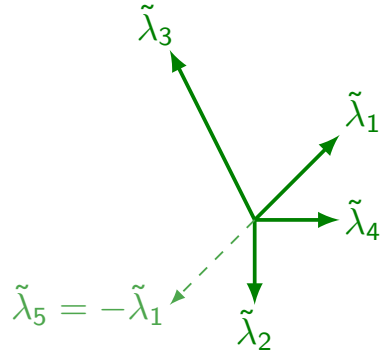

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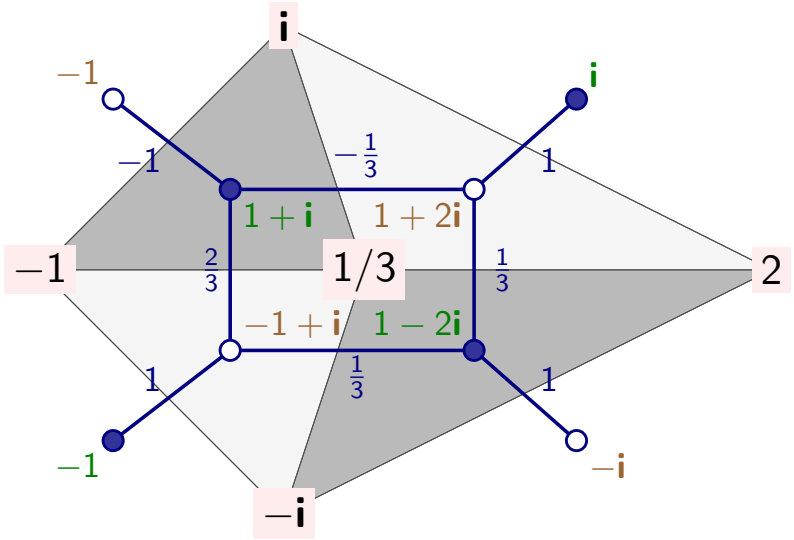

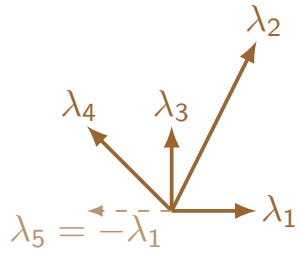

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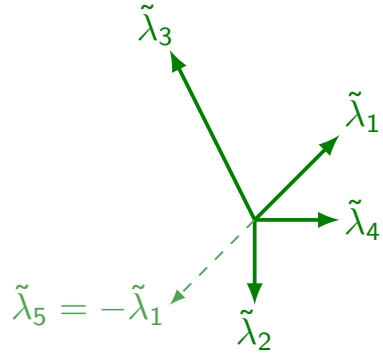

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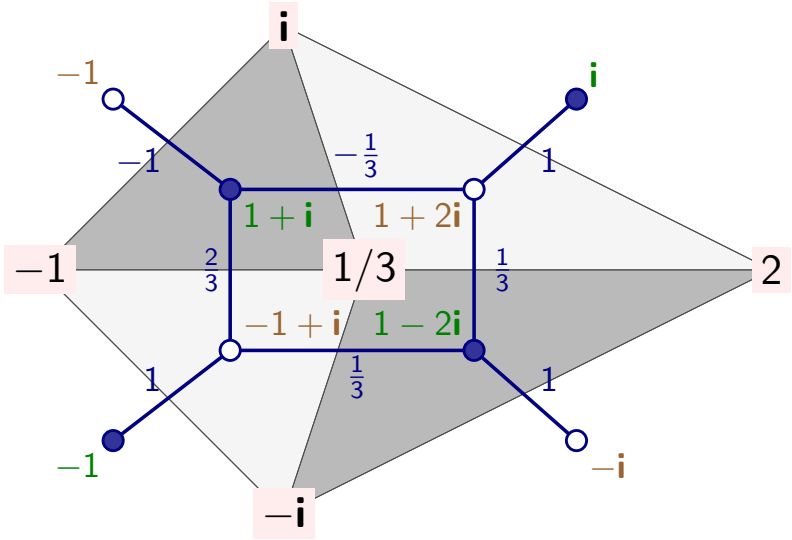

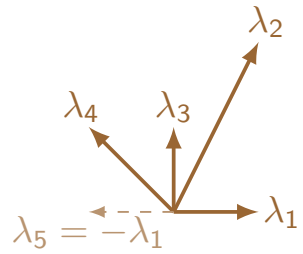

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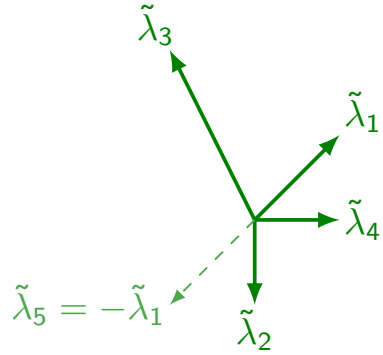

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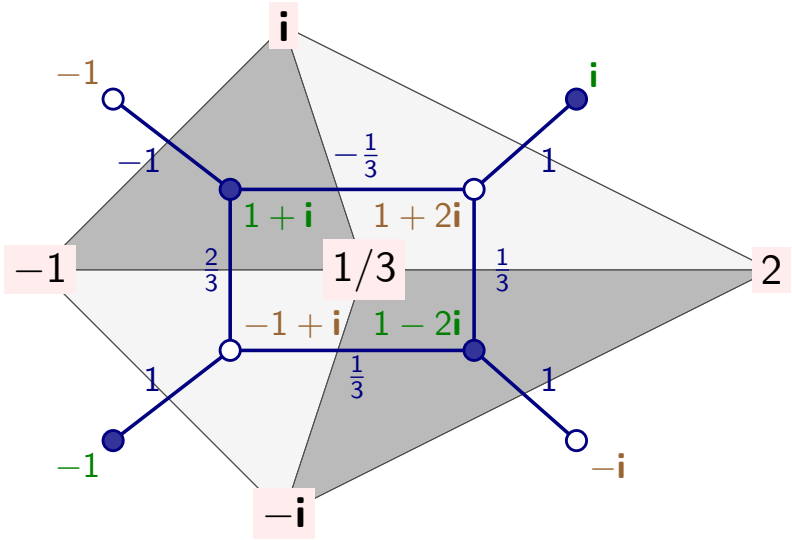

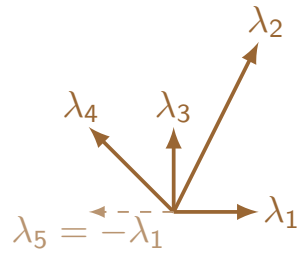

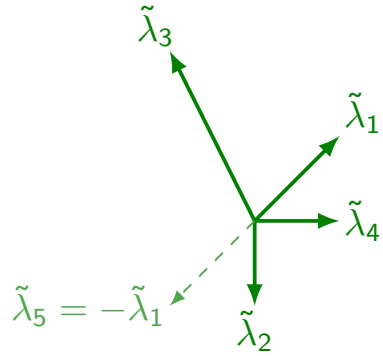

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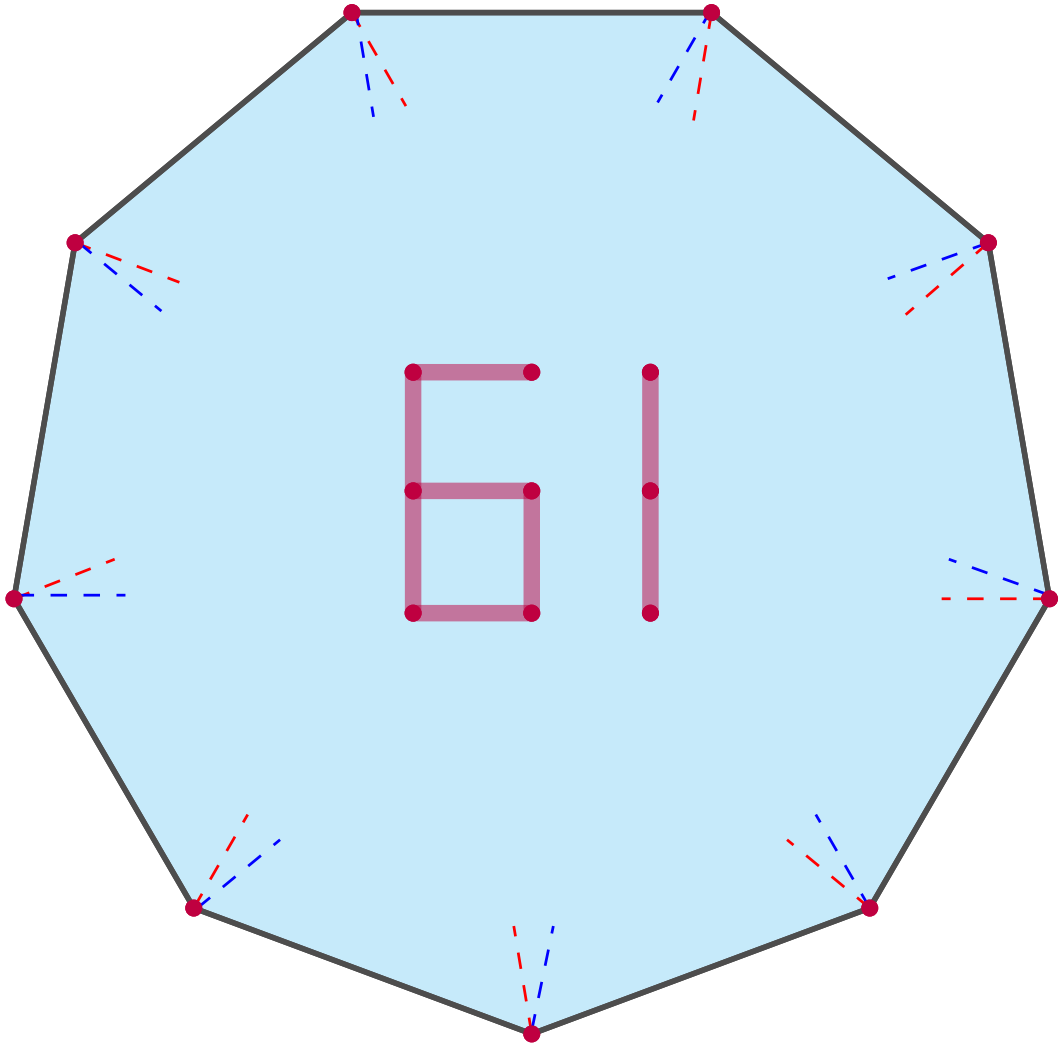
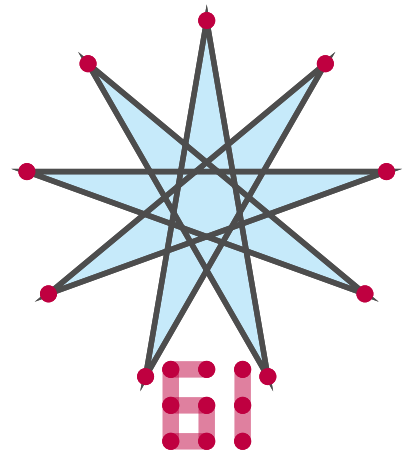
$\int \lambda^{\circ}\tilde{\lambda}^{\bullet}\,dz$ is a valid, embedded OCP $\iff$ *$(\lambda, \tilde{\lambda})$ belongs to positive kinematic space:* $\lambda \cdot \tilde{\lambda}^T = 0_{2\times 2},\ \langle j\,j+1\rangle > 0,\ [j\,j-1] < 0,\quad \mathrm{wind}(\lambda) = (k-1)\pi,\ \mathrm{wind}(\tilde{\lambda}) = -(k+1)\pi.$

Proof.

- $C := \{\text{boundary restrictions of discrete holomorphic functions } \mathbf{V}^{\circ} \to \mathbb{R}\} \subset \mathbb{R}^n$.
- $\dim \lambda = \dim \tilde{\lambda} = 2$, $\dim C = k$, and $\lambda \subset C \subset \tilde{\lambda}^{\perp}$ as linear subspaces of $\mathbb{R}^n$.
- Γ has positive edge weights $\iff$ all $k \times k$ minors of a $k \times n$ matrix representing C are nonnegative [Postnikov '06].
- Momentum space $\xrightarrow{\textit{shift}}$ momentum-twistor space, get a $(k-2) \times n$ matrix $Q_\lambda(C)$.
- All white faces of the OCP are embedded with correct orientation $\iff$ all $(k-2) \times (k-2)$ minors of $Q_\lambda(C)$ are nonnegative. $\square$





THANKS!