R-systems

Pavel Galashin

MIT

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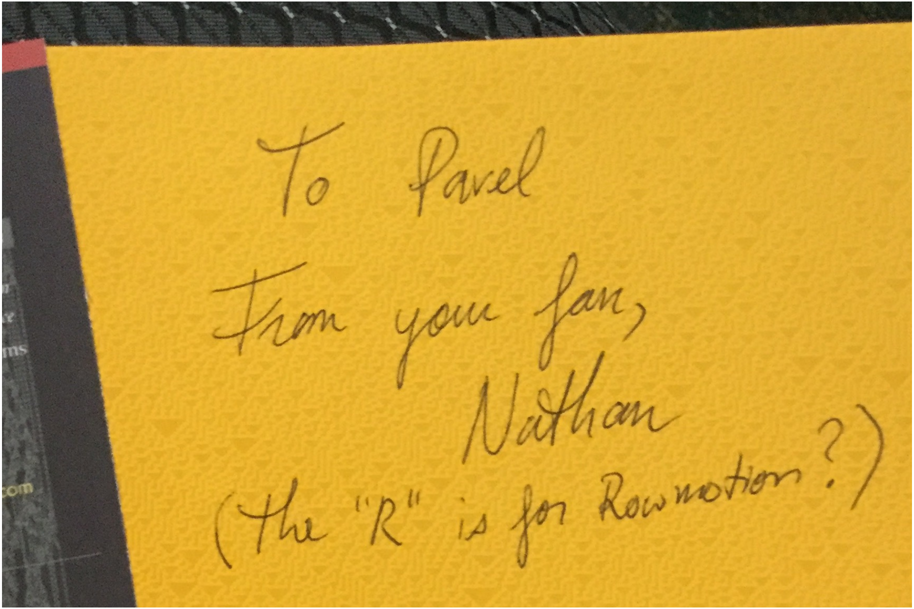
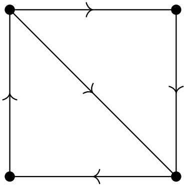
Joint work with Pavlo Pylyavskyy


To Pavel
From your fan,
Nathan
(the "R" is for Rowmotion?)

Part 1: Definition

A system of equations

Let $G = (V, E)$ be a *strongly connected digraph*.



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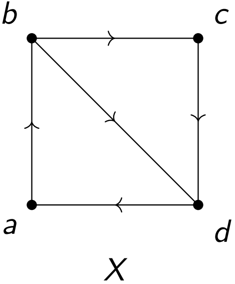

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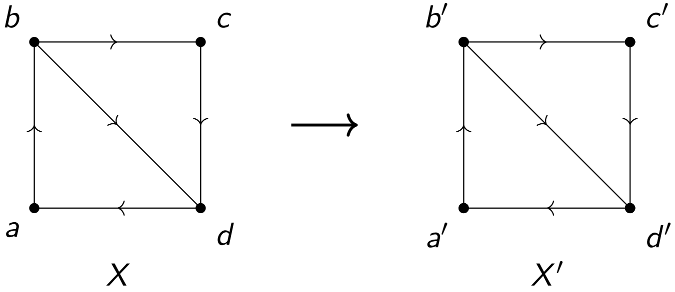

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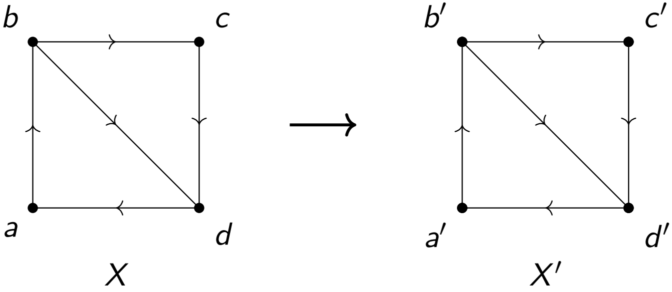

$$\forall\, v \in V, \quad X_v X'_v = \left(\sum_{v \to w} X_w \right) \left(\sum_{u \to v} \frac{1}{X'_u} \right)^{-1}$$

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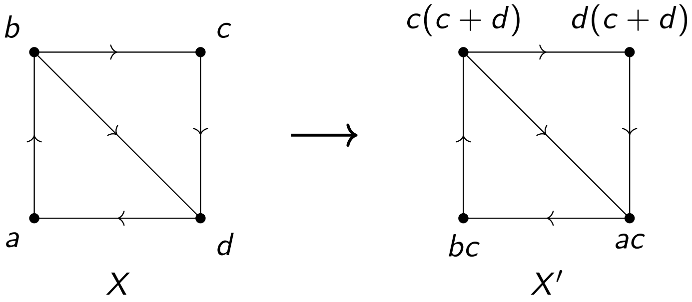

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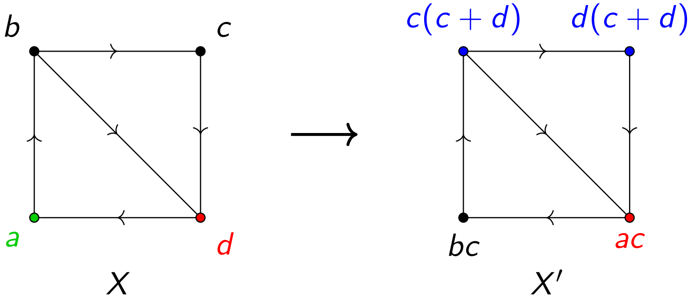

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$$d \cdot ac = (a)\left(\frac{1}{c(c+d)} + \frac{1}{d(c+d)}\right)^{-1}$$

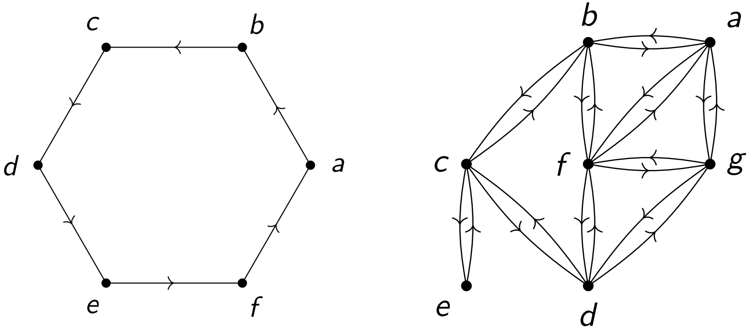
Solution

Theorem (G.-Pylyavskyy, 2017)

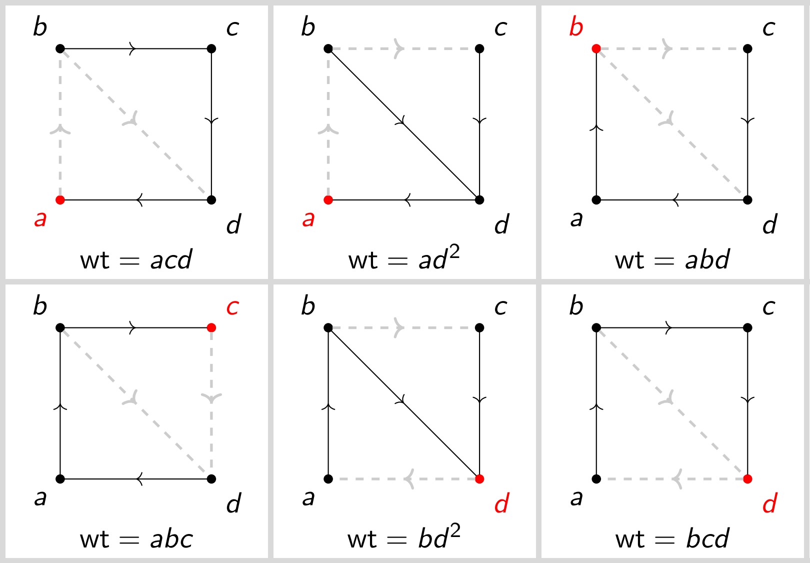
Let $G = (V, E)$ be a strongly connected digraph. Then there exists a birational map $\phi : \mathbb{P}^V \dashrightarrow \mathbb{P}^V$ such that

$$X, X' \in \mathbb{P}^V \textit{ give a solution} \quad \Longleftrightarrow \quad X' = \phi(X).$$

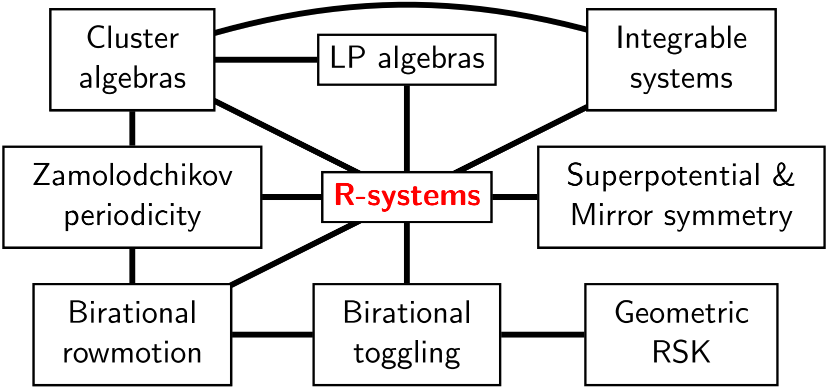
Periodic examples (exercise)



Arborescence formula

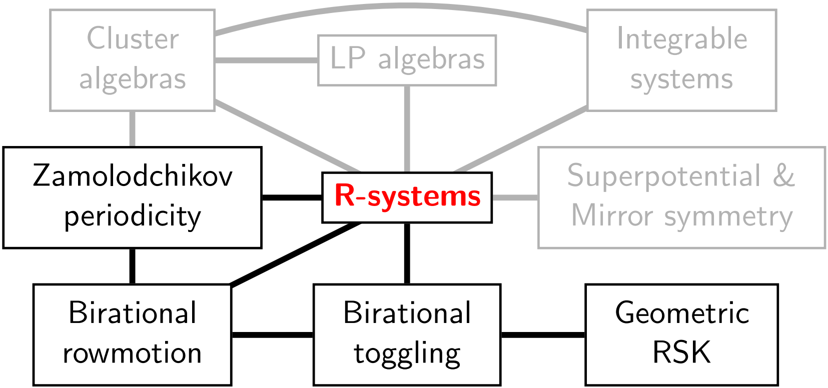


Map



Cluster algebras
LP algebras
Integrable systems
Zamolodchikov periodicity
R-systems
Superpotential & Mirror symmetry
Birational rowmotion
Birational toggling
Geometric RSK

Map



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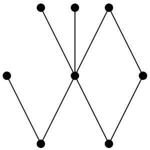
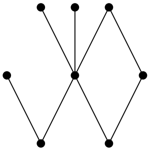
Birational rowmotion $\subseteq$ R-systems



P

Birational rowmotion $\subseteq$ R-systems



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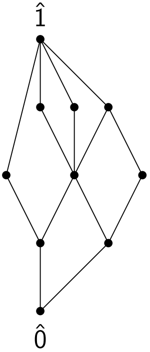
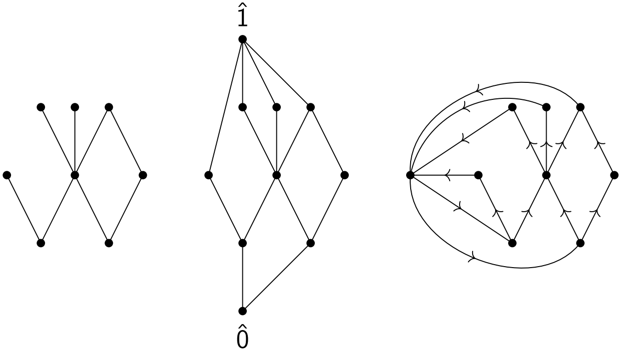


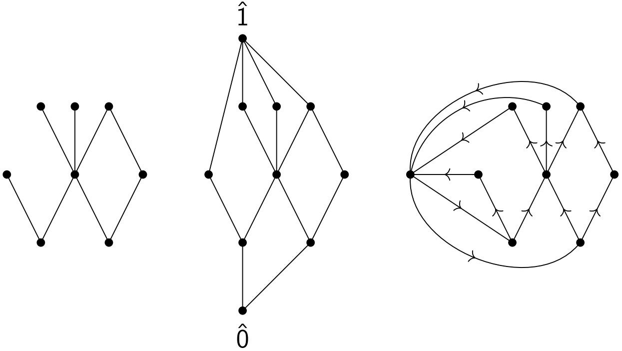
$\hat{P}$

Birational rowmotion ⊆ R-systems



P $\hat{P}$ $G(P)$

Birational rowmotion $\subseteq$ R-systems



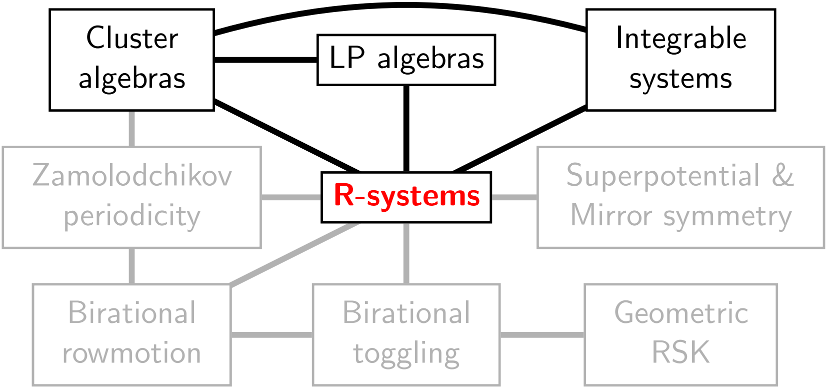
P $\hat{P}$ $G(P)$

Proposition (G.-Pylyavskyy, 2017)

Birational rowmotion on P $=$ R-system associated with $G(P)$.

Part 2: Singularity confinement

Map



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The Laurent phenomenon

Somos-4 sequence: $\tau_{n+4} = \frac{\alpha \tau_{n+1}\tau_{n+3} + \beta \tau_{n+2}^2}{\tau_n}$.

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Theorem (Fomin-Zelevinsky, 2002)

For each $n > 4$, τ_n is a Laurent polynomial in $\alpha, \beta, \tau_1, \tau_2, \tau_3, \tau_4$.

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A006720 Somos-4 sequence: a(0)=a(1)=a(2)=a(3)=1; for n >= 4, a(n) = (a(n-1) * a(n-3) + a(n-2)^2) / a(n-4). [78]
(Formerly M0857)

1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, 620297, 7869898, 126742987, 1687054711, 47301104551, 1123424582771, 32606721084786, 1662315215971057, 61958046554226593, 4257998884448335457, 334806306946199122193, 23385756731869683322514, 3416372868727801226636179 (list; graph; refs; listen; history; text; internal format)

Singularity confinement

Consider a mapping of the plane $(x_{n-1}, x_n) \mapsto (x_n, x_{n+1})$ given by

$$x_{n+1} = \frac{\alpha x_n + \beta}{x_{n-1} x_n^2}.$$

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substitute $x_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}$

$$x_3 = \frac{\alpha x_2 + \beta}{x_1 x_2^2}$$

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$$\tau_4 = \alpha x_2 + \beta$$

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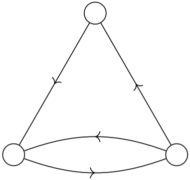
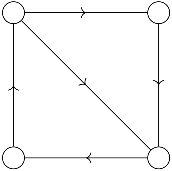
$$\tau_9 = \alpha^3\beta^3 x_1^6 x_2^8 + \cdots + \alpha\beta^8$$

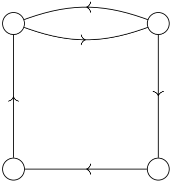
Examples: Somos and Gale-Robinson sequences



Somos-4



Somos-5



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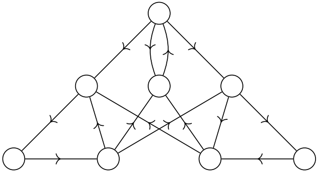
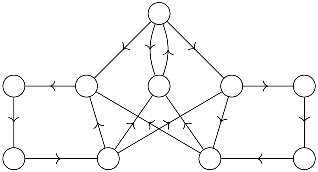
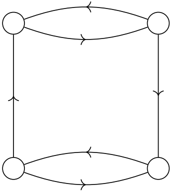


Somos-6 = GR(1 + 2 + 3)

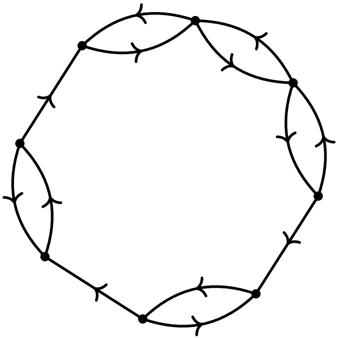


Somos-7 = GR(1 + 2 + 4)



dP_3

Examples: subgraphs of a bidirected cycle



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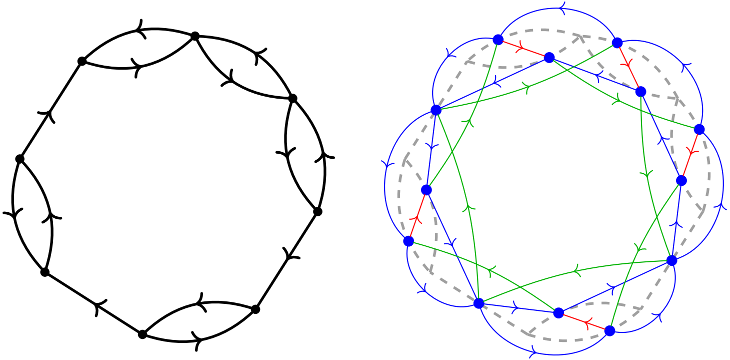
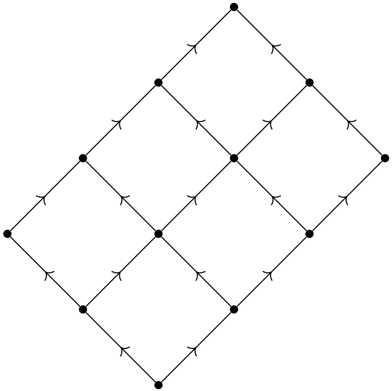


Controlled by a cluster algebra

Examples: rectangle posets (Grinberg-Roby)



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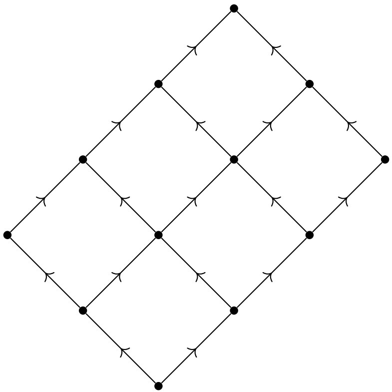
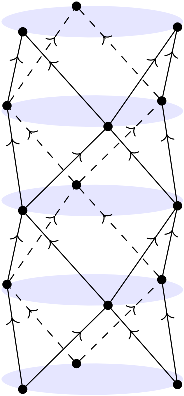


Controlled by a Y-system

Examples: cylindric posets



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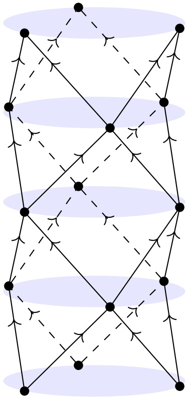
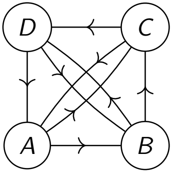
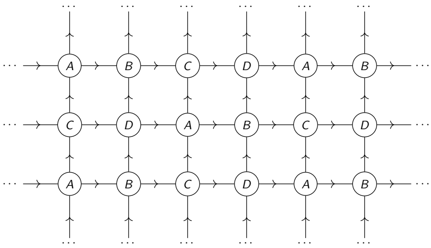


Controlled by an LP algebra

Examples: toric digraphs





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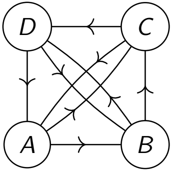
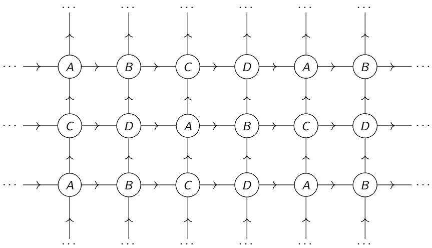
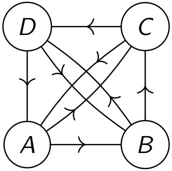
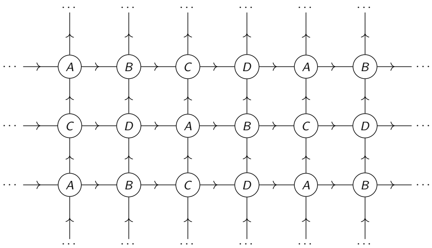




Controlled by ???

Examples: toric digraphs





Controlled by ??? $R_v(t) = \frac{\tau_v(t-1)}{\tau_v(t)};$

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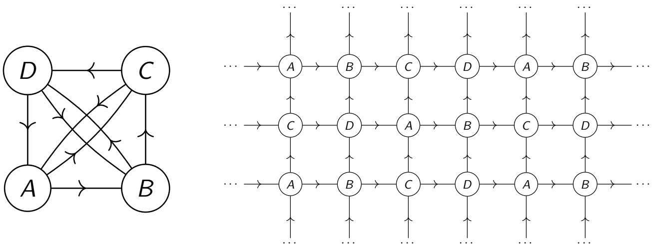

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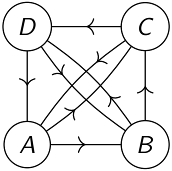

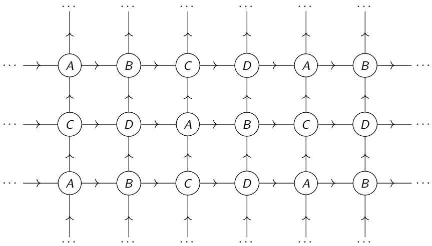

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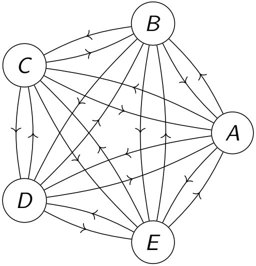
Example: the universal R-system



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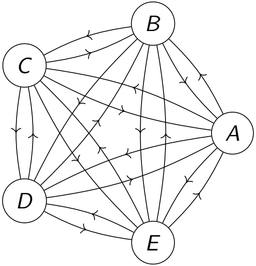

Controlled by ???

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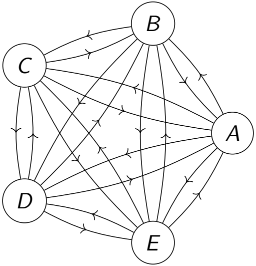

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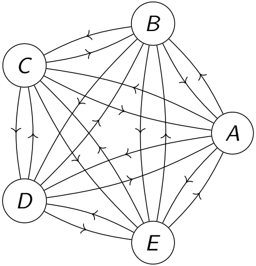

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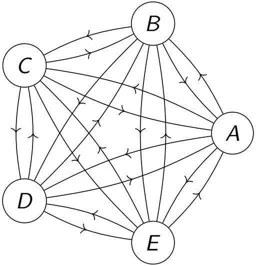

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Bibliography

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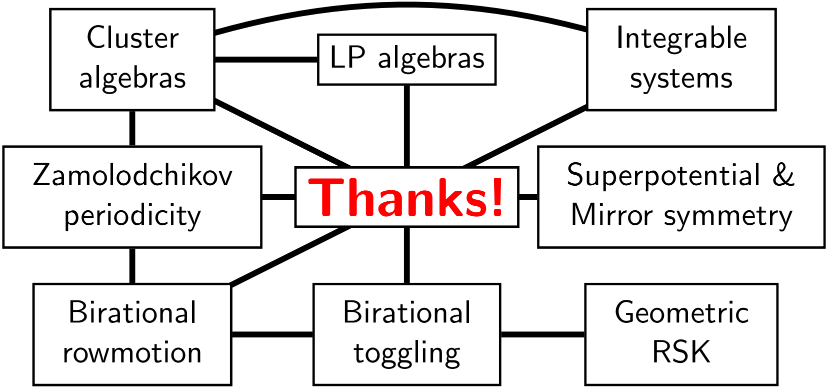
Cluster algebras
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Integrable systems
Zamolodchikov periodicity
Thanks!
Superpotential & Mirror symmetry
Birational rowmotion
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Geometric RSK

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