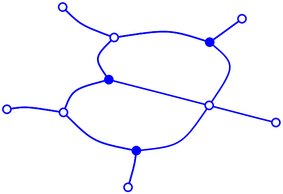
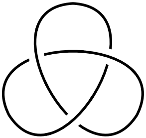
Positroids, knots, and q,t-Catalan numbers

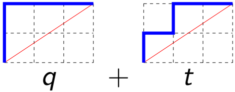
Pavel Galashin (UCLA)

Stanford Mathematics Department Colloquium
February 10, 2022

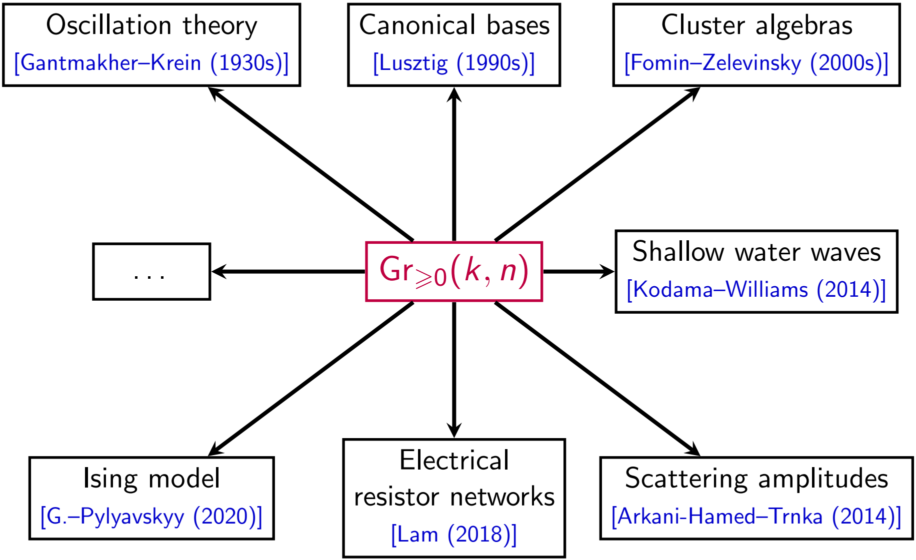
Joint work with Thomas Lam (arXiv:2012.09745)







Motivation: total positivity



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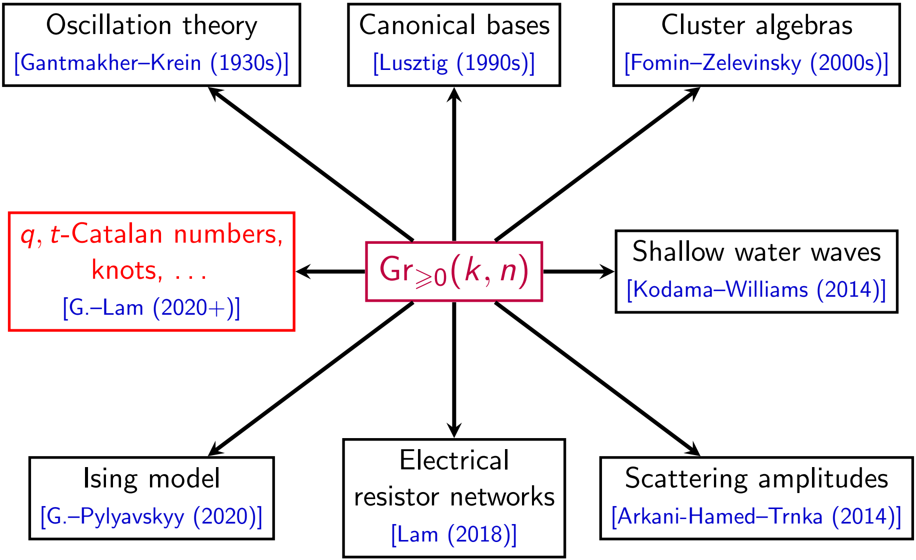

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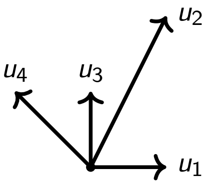

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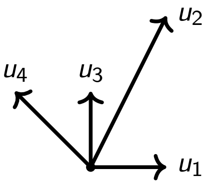

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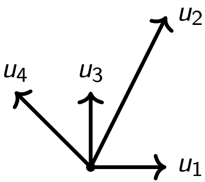

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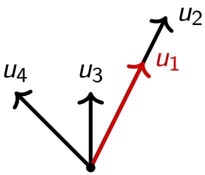

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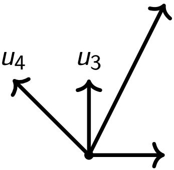

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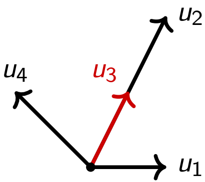

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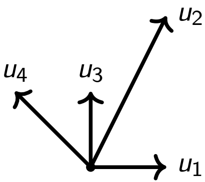

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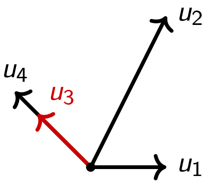

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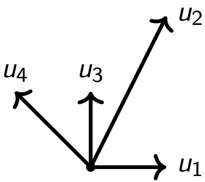

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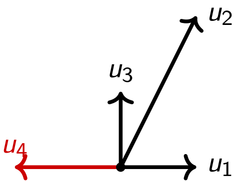

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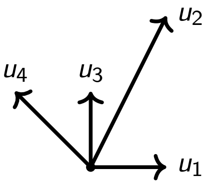

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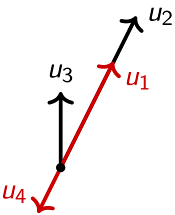
$\quad u_1 \quad u_2 \quad u_3 \quad u_4$



$$\Delta_{13} = 1, \quad \Delta_{24} = 3, \quad \Delta_{12} = 2, \quad \Delta_{34} = 1, \quad \Delta_{14} = 1, \quad \Delta_{23} = 1.$$

In $\mathrm{Gr}(2,4)$, we have a Plücker relation: $\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$.

Top cell: $\Delta_{13}, \Delta_{24}, \Delta_{12}, \Delta_{34}, \Delta_{14}, \Delta_{23} > 0$
Codimension 1 cells: $\Delta_{12} = 0$, $\Delta_{23} = 0$, $\Delta_{34} = 0$, $\Delta_{14} = 0$.
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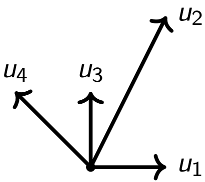

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Question

What is the topology of $\mathrm{Gr}_{\geqslant 0}(k,n)$ and of its boundary cells?

Topology of $\mathrm{Gr}_{\geqslant 0}(k, n)$

Theorem (Postnikov (2006))

Each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

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Conclusion

$\mathrm{Gr}_{\geqslant 0}(k,n)$ is not a polytope, but the “next best thing” to a polytope.

$$\mathsf{Gr}(k, n; \mathbb{F}) := \{W \subseteq \mathbb{F}^n \mid \dim(W) = k\} = \frac{\{k \times n \text{ matrices of rank } k\}}{(\text{row operations})}.$$

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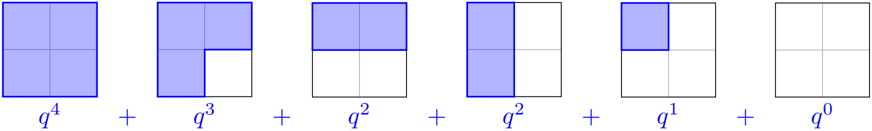

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- Reason: Schubert decomposition.

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Example ($k = 2$, $n = 4$)

$$\Pi^\circ_{2,4} \cong \left\{ \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \middle| a \neq 0, d \neq 0, ad \neq bc \right\}.$$

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Here's the top-dimensional one:

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Example ($k = 2,\ n = 4$)

$$\Pi^\circ_{2,4} \cong \left\{ \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \middle| a \neq 0, d \neq 0, ad \neq bc \right\}.$$

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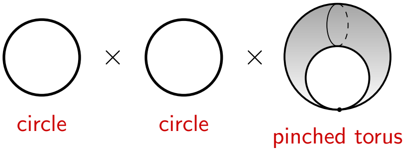

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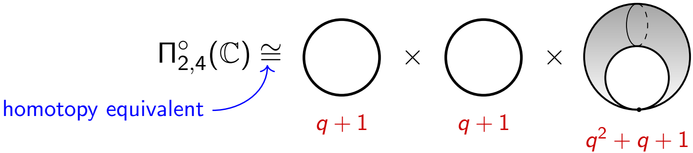

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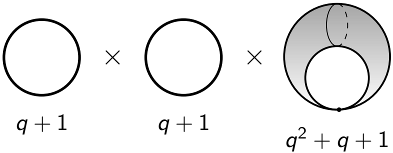

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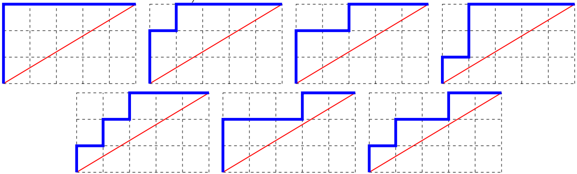

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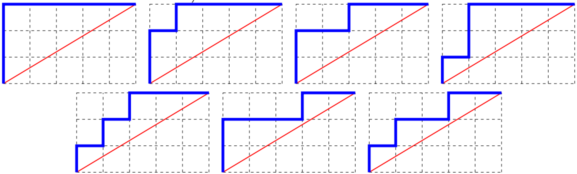

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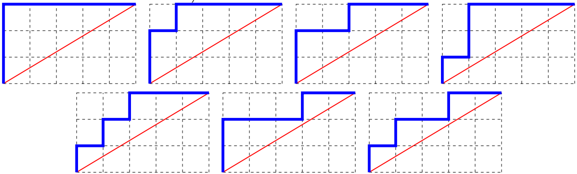

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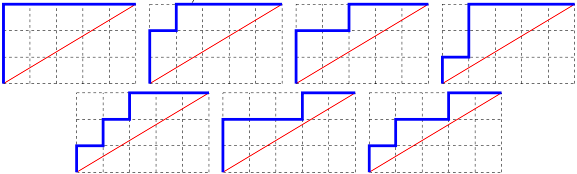

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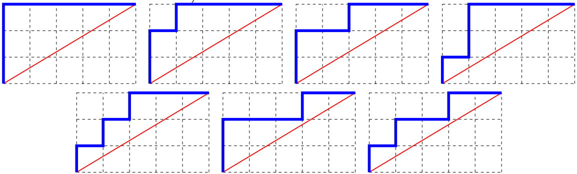

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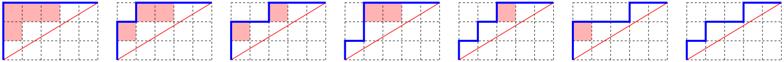

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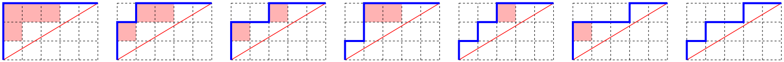

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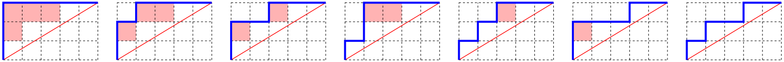

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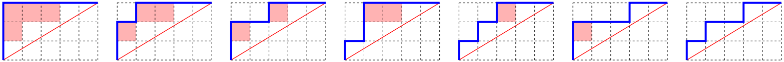

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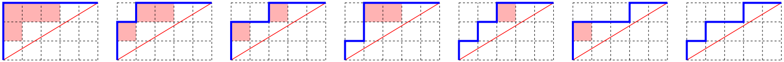

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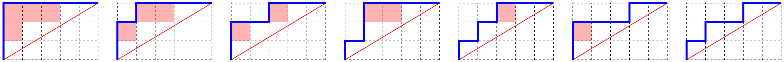

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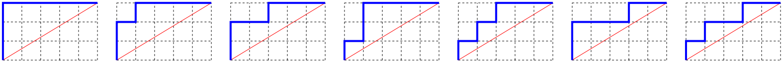

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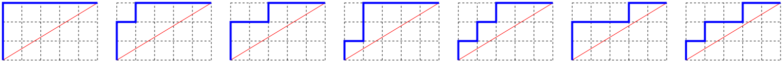

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Arbitrary positroid varieties

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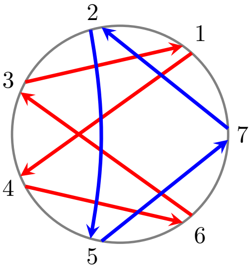

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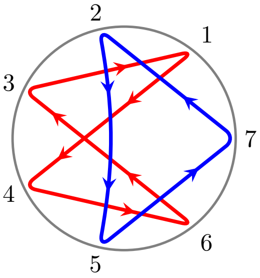

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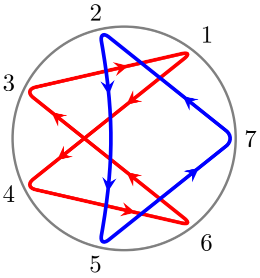

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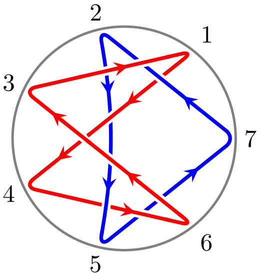

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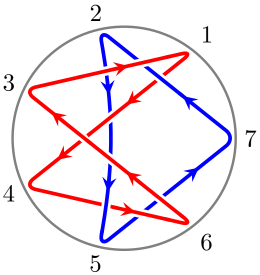

[FPST17] S. Fomin, P. Pylyavskyy, E. Shustin, and D. Thurston. Morsifications and mutations. arXiv:1711.10598.

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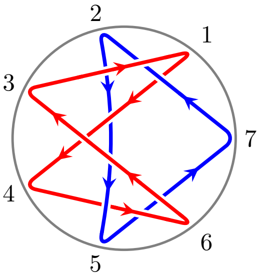

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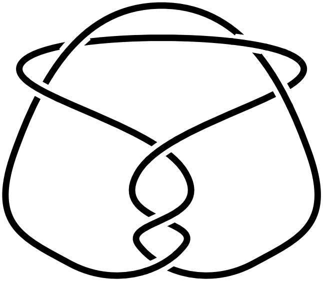
Conclusion

For each permutation $f \in S_n$, get a variety Π_f° and a link L_f.

How to tell if two links are isotopic?

One of these knots is not like the others



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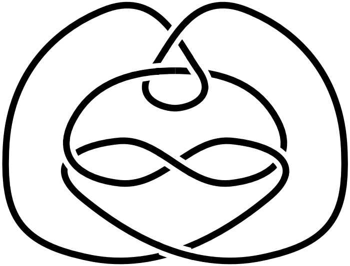

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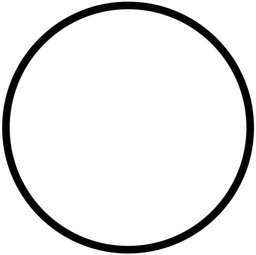
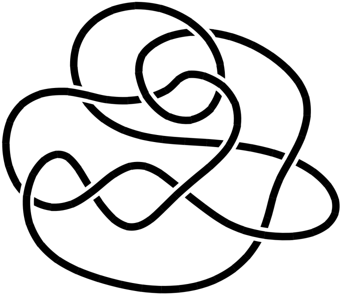


(C)



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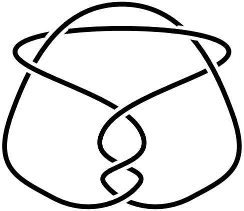

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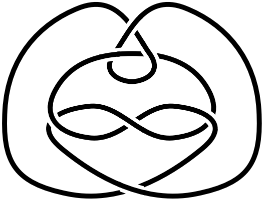

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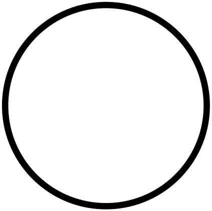

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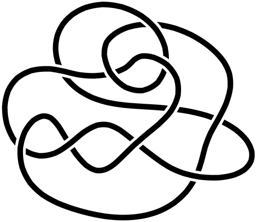

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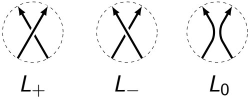

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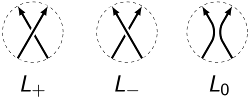

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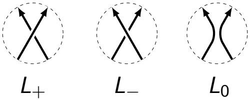

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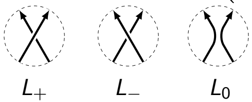

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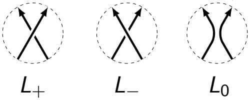

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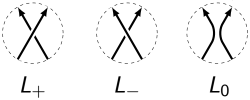

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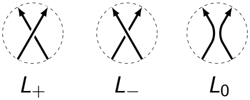

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