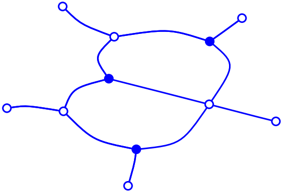
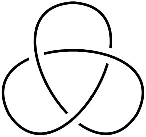
Positroids, knots, and q, t-Catalan numbers

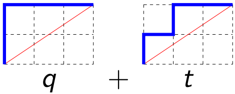
Pavel Galashin (UCLA)

MIT Applied Math Seminar
February 21, 2023

Joint work with Thomas Lam







Binomial coefficients

Binomial coefficients

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad [n]_q! := [1]_q[2]_q \cdots [n]_q$$

Binomial coefficients

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad [n]_q! := [1]_q[2]_q \cdots [n]_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$$

Binomial coefficients

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad [n]_q! := [1]_q[2]_q \cdots [n]_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$$

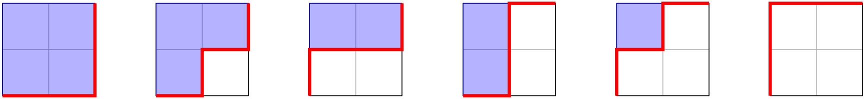
<u>Example:</u> $$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[1]_q[2]_q[3]_q[4]_q}{[1]_q[2]_q \cdot [1]_q[2]_q} = q^4 + q^3 + 2q^2 + q + 1.$$

Binomial coefficients

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad [n]_q! := [1]_q[2]_q \cdots [n]_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} = \sum_{P \in \mathrm{Paths}_{k,n-k}} q^{\mathrm{area}(P)}.$$

<u>Example:</u> $$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[1]_q[2]_q[3]_q[4]_q}{[1]_q[2]_q \cdot [1]_q[2]_q} = q^4 + q^3 + 2q^2 + q + 1.$$



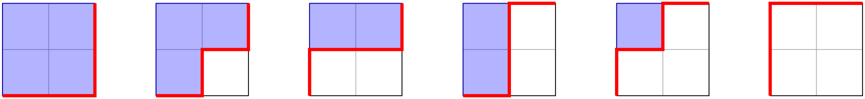
$$q^4 \quad + \quad q^3 \quad + \quad q^2 \quad + \quad q^2 \quad + \quad q^1 \quad + \quad q^0$$

Binomial coefficients

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad [n]_q! := [1]_q[2]_q \cdots [n]_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} = \sum_{P \in \mathrm{Paths}_{k,n-k}} q^{\mathrm{area}(P)}.$$

<u>Example:</u> $$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[1]_q[2]_q[3]_q[4]_q}{[1]_q[2]_q \cdot [1]_q[2]_q} = q^4 + q^3 + 2q^2 + q + 1.$$



$$q^4 \quad + \quad q^3 \quad + \quad q^2 \quad + \quad q^2 \quad + \quad q^1 \quad + \quad q^0$$

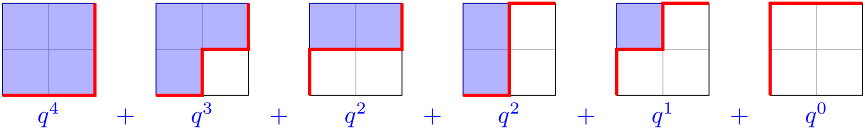
Properties of $\begin{bmatrix} n \\ k \end{bmatrix}_q$:

Binomial coefficients

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad [n]_q! := [1]_q[2]_q \cdots [n]_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} = \sum_{P \in \mathsf{Paths}_{k,n-k}} q^{\mathsf{area}(P)}.$$

<u>Example:</u> $$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[1]_q[2]_q[3]_q[4]_q}{[1]_q[2]_q \cdot [1]_q[2]_q} = q^4 + q^3 + 2q^2 + q + 1.$$



Properties of $\begin{bmatrix} n \\ k \end{bmatrix}_q$:

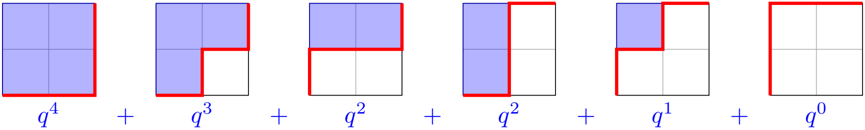
- symmetry: coefficients form a palindromic sequence;

Binomial coefficients

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad [n]_q! := [1]_q[2]_q \cdots [n]_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} = \sum_{P \in \mathrm{Paths}_{k,n-k}} q^{\mathrm{area}(P)}.$$

<u>Example:</u> $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \dfrac{[1]_q[2]_q[3]_q[4]_q}{[1]_q[2]_q \cdot [1]_q[2]_q} = q^4 + q^3 + 2q^2 + q + 1.$



Properties of $\begin{bmatrix} n \\ k \end{bmatrix}_q$:

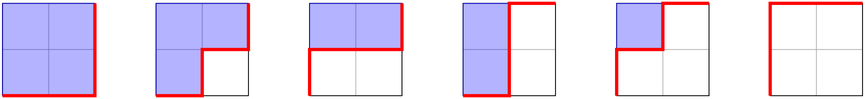
- symmetry: coefficients form a palindromic sequence;
- unimodality: coefficients increase up to the middle, then decrease;

Binomial coefficients

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad [n]_q! := [1]_q[2]_q \cdots [n]_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} = \sum_{P \in \mathrm{Paths}_{k,n-k}} q^{\mathrm{area}(P)}.$$

<u>Example:</u> $$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[1]_q[2]_q[3]_q[4]_q}{[1]_q[2]_q \cdot [1]_q[2]_q} = q^4 + q^3 + 2q^2 + q + 1.$$



$$q^4 \quad + \quad q^3 \quad + \quad q^2 \quad + \quad q^2 \quad + \quad q^1 \quad + \quad q^0$$

Properties of $\begin{bmatrix} n \\ k \end{bmatrix}_q$:

- symmetry: coefficients form a palindromic sequence;
- unimodality: coefficients increase up to the middle, then decrease;
- symmetry is easy, unimodality is hard.

Binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} = \sum_{P \in \mathsf{Paths}_{k,n-k}} q^{\mathsf{area}(P)}.$$

Properties of $\begin{bmatrix} n \\ k \end{bmatrix}_q$:

- symmetry: coefficients form a palindromic sequence;
- unimodality: coefficients increase up to the middle, then decrease;
- symmetry is easy, unimodality is hard.

Binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} = \sum_{P \in \mathrm{Paths}_{k,n-k}} q^{\mathrm{area}(P)}.$$

Properties of $\begin{bmatrix} n \\ k \end{bmatrix}_q$:

- symmetry: coefficients form a palindromic sequence;
- unimodality: coefficients increase up to the middle, then decrease;
- symmetry is easy, unimodality is hard.

Theorem (Sylvester (1878); conjectured by Cayley (1856))

The coefficients of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ form a unimodal sequence.

"I am about to demonstrate a theorem which has been waiting proof for the last quarter of a century and upwards. [...] I accomplished with scarcely an effort a task which I had believed lay outside the range of human power."

– J. J. Sylvester, 1878.

- Rational Catalan numbers: for $1 \leqslant k \leqslant n$ such that $\gcd(k, n) = 1$, let

$$C_{k,n-k} := \frac{1}{n}\binom{n}{k}.$$

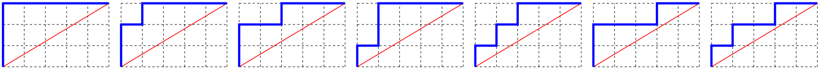
$$\boxed{C_{k,n-k} := \tfrac{1}{n}\tbinom{n}{k}}$$

- Rational Catalan numbers: for $1 \leqslant k \leqslant n$ such that $\gcd(k, n) = 1$, let
$$C_{k,n-k} := \frac{1}{n}\binom{n}{k}.$$
- Counts the number of Dyck paths inside a $k \times (n-k)$ rectangle.

$$\boxed{C_{k,n-k} := \tfrac{1}{n}\tbinom{n}{k} = \#\,\mathrm{Dyck}_{k,n-k}}$$

- Rational Catalan numbers: for $1 \leqslant k \leqslant n$ such that $\gcd(k, n) = 1$, let
$$C_{k,n-k} := \frac{1}{n}\binom{n}{k}.$$
- Counts the number of Dyck paths inside a $k \times (n-k)$ rectangle.

Example: $k = 3,\ n = 8,\ C_{k,n-k} = 7$:



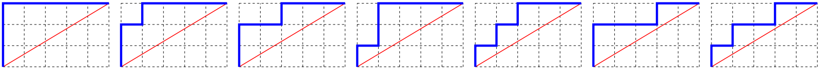
$$\boxed{C_{k,n-k} := \tfrac{1}{n}\tbinom{n}{k} = \#\,\mathrm{Dyck}_{k,n-k}}$$

- Rational Catalan numbers: for $1 \leqslant k \leqslant n$ such that $\gcd(k, n) = 1$, let

$$C_{k,n-k} := \frac{1}{n}\binom{n}{k}.$$

- Counts the number of Dyck paths inside a $k \times (n-k)$ rectangle.

<u>Example:</u> $k = 3,\ n = 8,\ C_{k,n-k} = 7$:



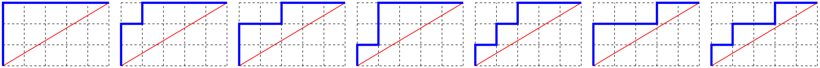
$$\boxed{C'_{k,n-k}(q) := \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q}$$

$q = 1$

$$\boxed{C_{k,n-k} := \frac{1}{n}\binom{n}{k} = \#\,\mathrm{Dyck}_{k,n-k}}$$

- Rational Catalan numbers: for $1 \leqslant k \leqslant n$ such that $\gcd(k,n) = 1$, let
$$C_{k,n-k} := \frac{1}{n}\binom{n}{k}.$$
- Counts the number of Dyck paths inside a $k \times (n-k)$ rectangle.

Example: $k = 3,\ n = 8,\ C_{k,n-k} = 7$:



$C'_{k,n-k}(q) = q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1.$

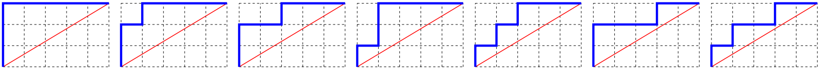
$$\boxed{C'_{k,n-k}(q) := \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q}$$

$q = 1$

$$\boxed{C_{k,n-k} := \frac{1}{n}\binom{n}{k} = \#\,\mathrm{Dyck}_{k,n-k}}$$

- Rational Catalan numbers: for $1 \leqslant k \leqslant n$ such that $\gcd(k, n) = 1$, let
$$C_{k,n-k} := \frac{1}{n}\binom{n}{k}.$$
- Counts the number of Dyck paths inside a $k \times (n-k)$ rectangle.

Example: $k = 3,\ n = 8,\ C_{k,n-k} = 7$:

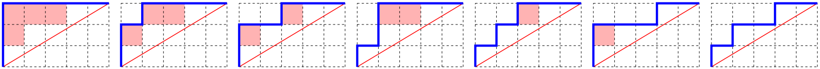


$C'_{k,n-k}(q) = q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1.$

$$\begin{tikzcd}
\boxed{C'_{k,n-k}(q) := \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q} \arrow[dr, "q=1"'] & & \boxed{C''_{k,n-k}(t) := \sum_{P \in \mathrm{Dyck}_{k,n-k}} t^{\mathrm{area}(P)}} \arrow[dl, "t=1"] \\
& \boxed{C_{k,n-k} := \frac{1}{n}\binom{n}{k} = \#\,\mathrm{Dyck}_{k,n-k}} &
\end{tikzcd}$$

- Rational Catalan numbers: for $1 \leqslant k \leqslant n$ such that $\gcd(k, n) = 1$, let
$$C_{k,n-k} := \frac{1}{n}\binom{n}{k}.$$
- Counts the number of Dyck paths inside a $k \times (n-k)$ rectangle.

<u>Example:</u> $k = 3,\ n = 8,\ C_{k,n-k} = 7$:



$C''_{k,n-k}(t) = t^4 + t^3 + t^2 + t^2 + t^1 + t^1 + t^0.$

$C'_{k,n-k}(q) = q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1.$

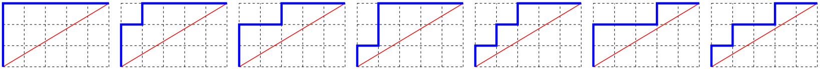
$$\boxed{C'_{k,n-k}(q) := \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q} \qquad \boxed{C''_{k,n-k}(t) := \sum_{P \in \mathrm{Dyck}_{k,n-k}} t^{\mathrm{area}(P)}}$$

$$q = 1 \qquad t = 1$$

$$\boxed{C_{k,n-k} := \frac{1}{n}\binom{n}{k} = \#\,\mathrm{Dyck}_{k,n-k}}$$

- Rational Catalan numbers: for $1 \leqslant k \leqslant n$ such that $\gcd(k, n) = 1$, let
$$C_{k,n-k} := \frac{1}{n}\binom{n}{k}.$$
- Counts the number of Dyck paths inside a $k \times (n-k)$ rectangle.

<u>Example:</u> $k = 3,\ n = 8,\ C_{k,n-k} = 7$:

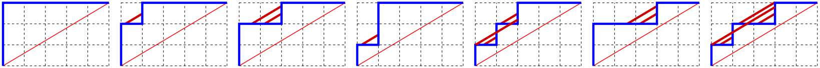


$$C''_{k,n-k}(t) \;=\; t^4 \;+\; t^3 \;+\; t^2 \;+\; t^2 \;+\; t^1 \;+\; t^1 \;+\; t^0.$$
$$C'_{k,n-k}(q) = q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1.$$

$$\begin{tikzcd}
 & \boxed{{\color{red}C_{k,n-k}(q,t)} := \sum_{P \in \mathrm{Dyck}_{k,n-k}} t^{\mathrm{area}(P)} q^{\mathrm{dinv}(P)}} \arrow[dl, "{\color{red}t = 1/q}"'] \arrow[dr, "{\color{red}q = 1}"] & \\
\boxed{C'_{k,n-k}(q) := \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q} \arrow[dr, "q = 1"'] & & \boxed{C''_{k,n-k}(t) := \sum_{P \in \mathrm{Dyck}_{k,n-k}} t^{\mathrm{area}(P)}} \arrow[dl, "t = 1"] \\
 & \boxed{C_{k,n-k} := \frac{1}{n}\binom{n}{k} = \#\,\mathrm{Dyck}_{k,n-k}} &
\end{tikzcd}$$

- Rational Catalan numbers: for $1 \leqslant k \leqslant n$ such that $\gcd(k, n) = 1$, let
$$C_{k,n-k} := \frac{1}{n}\binom{n}{k}.$$
- Counts the number of Dyck paths inside a $k \times (n-k)$ rectangle.

<u>Example:</u> $k = 3,\ n = 8,\ C_{k,n-k} = 7$:



$$C_{k,n-k}(q,t) = t^4q^0 + t^3q^1 + t^2q^2 + t^2q^1 + t^1q^3 + t^1q^2 + t^0q^4.$$
$$C''_{k,n-k}(t) = t^4 + t^3 + t^2 + t^2 + t^1 + t^1 + t^0.$$
$$C'_{k,n-k}(q) = q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1.$$

$$\begin{tikzcd}[column sep=small]
& \boxed{C_{k,n-k}(q,t) := \sum_{P \in \mathrm{Dyck}_{k,n-k}} t^{\mathrm{area}(P)} q^{\mathrm{dinv}(P)}} \arrow[dl, "t = 1/q"'] \arrow[dr, "q = 1"] & \\
\boxed{C'_{k,n-k}(q) := \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q} \arrow[dr, "q = 1"'] & & \boxed{C''_{k,n-k}(t) := \sum_{P \in \mathrm{Dyck}_{k,n-k}} t^{\mathrm{area}(P)}} \arrow[dl, "t = 1"] \\
& \boxed{C_{k,n-k} := \frac{1}{n}\binom{n}{k} = \#\,\mathrm{Dyck}_{k,n-k}} &
\end{tikzcd}$$

q, t-symmetry and unimodality

q, t-symmetry and unimodality

$C_{2,3}(q,t) = (q + t)$

q, t-symmetry and unimodality

$$\begin{aligned} C_{2,3}(q,t) &= (q+t) \\ C_{3,5}(q,t) &= (q^4 + q^3t + q^2t^2 + qt^3 + t^4) \\ &\quad + (q^2t + qt^2) \end{aligned}$$

q, t-symmetry and unimodality

$$
\begin{aligned}
C_{2,3}(q,t) &= (q+t) \\
C_{3,5}(q,t) &= (q^4 + q^3t + q^2t^2 + qt^3 + t^4) \\
&\quad + (q^2t + qt^2) \\
C_{5,6}(q,t) &= (q^{10} + q^9t + q^8t^2 + q^7t^3 + q^6t^4 + q^5t^5 + q^4t^6 + q^3t^7 + q^2t^8 + qt^9 + t^{10}) \\
&\quad + (q^8t + q^7t^2 + q^6t^3 + q^5t^4 + q^4t^5 + q^3t^6 + q^2t^7 + qt^8) \\
&\quad + (q^7t + 2\,q^6t^2 + 2\,q^5t^3 + 2\,q^4t^4 + 2\,q^3t^5 + 2\,q^2t^6 + qt^7) \\
&\quad + (q^6t + q^5t^2 + 2\,q^4t^3 + 2\,q^3t^4 + q^2t^5 + qt^6) \\
&\quad + (q^4t^2 + q^3t^3 + q^2t^4)
\end{aligned}
$$

q, t-symmetry and unimodality

$$\begin{aligned}
C_{2,3}(q,t) &= (q+t) \\
C_{3,5}(q,t) &= (q^4+q^3t+q^2t^2+qt^3+t^4) \\
&\quad + (q^2t+qt^2) \\
C_{5,6}(q,t) &= (q^{10}+q^9t+q^8t^2+q^7t^3+q^6t^4+q^5t^5+q^4t^6+q^3t^7+q^2t^8+qt^9+t^{10}) \\
&\quad + (q^8t+q^7t^2+q^6t^3+q^5t^4+q^4t^5+q^3t^6+q^2t^7+qt^8) \\
&\quad + (q^7t+2\,q^6t^2+2\,q^5t^3+2\,q^4t^4+2\,q^3t^5+2\,q^2t^6+qt^7) \\
&\quad + (q^6t+q^5t^2+2\,q^4t^3+2\,q^3t^4+q^2t^5+qt^6) \\
&\quad + (q^4t^2+q^3t^3+q^2t^4)
\end{aligned}$$

Corollary (G.–Lam)

Let $\gcd(k,n)=1$. *Then* $C_{k,n-k}(q,t)$ *satisfies:*

- *q, t-symmetry: coefficients in each row form a palindromic sequence;*

q, t-symmetry and unimodality

$$C_{2,3}(q,t) = (q+t)$$

$$C_{3,5}(q,t) = (q^4 + q^3t + q^2t^2 + qt^3 + t^4) + (q^2t + qt^2)$$

$$\begin{aligned} C_{5,6}(q,t) = {} & (q^{10} + q^9t + q^8t^2 + q^7t^3 + q^6t^4 + q^5t^5 + q^4t^6 + q^3t^7 + q^2t^8 + qt^9 + t^{10}) \\ & + (q^8t + q^7t^2 + q^6t^3 + q^5t^4 + q^4t^5 + q^3t^6 + q^2t^7 + qt^8) \\ & + (q^7t + 2\,q^6t^2 + 2\,q^5t^3 + 2\,q^4t^4 + 2\,q^3t^5 + 2\,q^2t^6 + qt^7) \\ & + (q^6t + q^5t^2 + 2\,q^4t^3 + 2\,q^3t^4 + q^2t^5 + qt^6) \\ & + (q^4t^2 + q^3t^3 + q^2t^4) \end{aligned}$$

Corollary (G.–Lam)

Let $\gcd(k,n) = 1$. *Then* $C_{k,n-k}(q,t)$ *satisfies:*

- *q, t-symmetry: coefficients in each row form a palindromic sequence;*
- *unimodality: coefficients in each row increase up to the middle, then decrease.*

q, t-symmetry and unimodality

$$C_{2,3}(q,t) = (q+t)$$

$$\begin{aligned} C_{3,5}(q,t) &= (q^4 + q^3 t + q^2 t^2 + q t^3 + t^4) \\ &\quad + (q^2 t + q t^2) \end{aligned}$$

$$\begin{aligned} C_{5,6}(q,t) &= (q^{10} + q^9 t + q^8 t^2 + q^7 t^3 + q^6 t^4 + q^5 t^5 + q^4 t^6 + q^3 t^7 + q^2 t^8 + q t^9 + t^{10}) \\ &\quad + (q^8 t + q^7 t^2 + q^6 t^3 + q^5 t^4 + q^4 t^5 + q^3 t^6 + q^2 t^7 + q t^8) \\ &\quad + (q^7 t + 2\,q^6 t^2 + 2\,q^5 t^3 + 2\,q^4 t^4 + 2\,q^3 t^5 + 2\,q^2 t^6 + q t^7) \\ &\quad + (q^6 t + q^5 t^2 + 2\,q^4 t^3 + 2\,q^3 t^4 + q^2 t^5 + q t^6) \\ &\quad + (q^4 t^2 + q^3 t^3 + q^2 t^4) \end{aligned}$$

Corollary (G.–Lam)

Let $\gcd(k,n)=1$. *Then* $C_{k,n-k}(q,t)$ *satisfies:*

- *q, t-symmetry: coefficients in each row form a palindromic sequence;*
- *unimodality: coefficients in each row increase up to the middle, then decrease.*

[Haiman '94], [Haiman '02], [Mellit '16], [Carlsson–Mellit '18], [Gorsky–Hogancamp–Mellit '21]

q, t-symmetry and unimodality

$$
\begin{aligned}
C_{2,3}(q,t) &= (q+t)\\
C_{3,5}(q,t) &= (q^4+q^3t+q^2t^2+qt^3+t^4)\\
&\quad+(q^2t+qt^2)\\
C_{5,6}(q,t) &= (q^{10}+q^9t+q^8t^2+q^7t^3+q^6t^4+q^5t^5+q^4t^6+q^3t^7+q^2t^8+qt^9+t^{10})\\
&\quad+(q^8t+q^7t^2+q^6t^3+q^5t^4+q^4t^5+q^3t^6+q^2t^7+qt^8)\\
&\quad+(q^7t+2\,q^6t^2+2\,q^5t^3+2\,q^4t^4+2\,q^3t^5+2\,q^2t^6+qt^7)\\
&\quad+(q^6t+q^5t^2+2\,q^4t^3+2\,q^3t^4+q^2t^5+qt^6)\\
&\quad+(q^4t^2+q^3t^3+q^2t^4)
\end{aligned}
$$

Corollary (G.–Lam)

Let $\gcd(k,n)=1$. *Then* $C_{k,n-k}(q,t)$ *satisfies:*

- q,t*-symmetry: coefficients in each row form a palindromic sequence;*
- *unimodality: coefficients in each row increase up to the middle, then decrease.*

[Haiman '94], [Haiman '02], [Mellit '16], [Carlsson–Mellit '18], [Gorsky–Hogancamp–Mellit '21]

Unimodality was not previously known.

Grassmannian

$$\mathrm{Gr}(k, n; \mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$

Grassmannian

$\mathrm{Gr}(k, n; \mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}$.
$\mathrm{Gr}(k, n; \mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations})$.

Grassmannian

$$\mathrm{Gr}(k, n; \mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k, n; \mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \in \mathrm{Gr}(2, 4; \mathbb{F})$$

Grassmannian

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k,n;\mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\text{RowSpan}\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \in \mathrm{Gr}(2,4;\mathbb{F})$$

Plücker coordinates: for $I \subseteq \{1,2,\ldots,n\}$ of size k,

Grassmannian

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k,n;\mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan}\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \in \mathrm{Gr}(2,4;\mathbb{F})$$

Plücker coordinates: for $I \subseteq \{1,2,\dots,n\}$ of size k,

$$\Delta_I := k \times k \text{ minor with column set } I.$$

Grassmannian

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k,n;\mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan}\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \in \mathrm{Gr}(2,4;\mathbb{F}) \qquad \begin{matrix} \Delta_{13} = 1 & \Delta_{12} = 2 & \Delta_{14} = 1 \\ \Delta_{24} = 3 & \Delta_{34} = 1 & \Delta_{23} = 1. \end{matrix}$$

Plücker coordinates: for $I \subseteq \{1, 2, \dots, n\}$ of size k,

$$\Delta_I := k \times k \text{ minor with column set } I.$$

Grassmannian

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k,n;\mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan}\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \in \mathrm{Gr}(2,4;\mathbb{F}) \qquad \begin{matrix} \Delta_{13} = 1 & \Delta_{12} = 2 & \Delta_{14} = 1 \\ \Delta_{24} = 3 & \Delta_{34} = 1 & \Delta_{23} = 1. \end{matrix}$$

Plücker coordinates: for $I \subseteq \{1, 2, \ldots, n\}$ of size k,

$$\Delta_I := k \times k \text{ minor with column set } I.$$

Grassmannian

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k,n;\mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan}\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \in \mathrm{Gr}(2,4;\mathbb{F}) \qquad \begin{array}{lll} \Delta_{13} = 1 & \Delta_{12} = 2 & \Delta_{14} = 1 \\ \Delta_{24} = 3 & \Delta_{34} = 1 & \Delta_{23} = 1. \end{array}$$

Plücker coordinates: for $I \subseteq \{1, 2, \ldots, n\}$ of size k,

$$\Delta_I := k \times k \text{ minor with column set } I.$$

The Δ_I's are defined up to common rescaling.

Grassmannian

$\mathrm{Gr}(k, n; \mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$
$\mathrm{Gr}(k, n; \mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$

Grassmannian

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k,n;\mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Question

- How many points in $\mathrm{Gr}(k,n;\mathbb{F}_q)$?

Grassmannian

$$\mathrm{Gr}(k, n; \mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k, n; \mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Question

- How many points in $\mathrm{Gr}(k, n; \mathbb{F}_q)$?
- What is the Poincaré polynomial of $\mathrm{Gr}(k, n; \mathbb{C})$?

Grassmannian

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k,n;\mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Question

- How many points in $\mathrm{Gr}(k,n;\mathbb{F}_q)$?
- What is the Poincaré polynomial of $\mathrm{Gr}(k,n;\mathbb{C})$?

- Point count: $$\#\,\mathrm{Gr}(k,n;\mathbb{F}_q) = \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

Grassmannian

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k,n;\mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Question

- How many points in $\mathrm{Gr}(k,n;\mathbb{F}_q)$?
- What is the Poincaré polynomial of $\mathrm{Gr}(k,n;\mathbb{C})$?

- Point count: $$\#\,\mathrm{Gr}(k,n;\mathbb{F}_q) = \begin{bmatrix} n \\ k \end{bmatrix}_q.$$
- Poincaré polynomial: $$\sum_i t^{\frac{i}{2}} \dim H^i(\mathrm{Gr}(k,n;\mathbb{C})) = \begin{bmatrix} n \\ k \end{bmatrix}_t.$$

Grassmannian

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k,n;\mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Question

- How many points in $\mathrm{Gr}(k,n;\mathbb{F}_q)$?
- What is the Poincaré polynomial of $\mathrm{Gr}(k,n;\mathbb{C})$?

- Point count: $$\#\,\mathrm{Gr}(k,n;\mathbb{F}_q) = \begin{bmatrix} n \\ k \end{bmatrix}_q.$$
- Poincaré polynomial: $$\sum_i t^{\frac{i}{2}} \dim H^i(\mathrm{Gr}(k,n;\mathbb{C})) = \begin{bmatrix} n \\ k \end{bmatrix}_t.$$
- Reason: Schubert decomposition.

Grassmannian

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k,n;\mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Question

- How many points in $\mathrm{Gr}(k,n;\mathbb{F}_q)$?
- What is the Poincaré polynomial of $\mathrm{Gr}(k,n;\mathbb{C})$?

- Point count: $$\#\,\mathrm{Gr}(k,n;\mathbb{F}_q) = \begin{bmatrix} n \\ k \end{bmatrix}_q .$$
- Poincaré polynomial: $$\sum_i t^{\frac{i}{2}} \dim H^i(\mathrm{Gr}(k,n;\mathbb{C})) = \begin{bmatrix} n \\ k \end{bmatrix}_t .$$
- Reason: Schubert decomposition.

Observation (Stanley (1980))

Symmetry and unimodality of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ *follow from the hard Lefschetz theorem for the cohomology of* $\mathrm{Gr}(k,n;\mathbb{C})$*.*

Given a variety

$$X(\mathbb{F}) = \{\mathbf{x} \in \mathbb{F}^r \mid P_1(\mathbf{x}) = \cdots = P_n(\mathbf{x}) = 0, \quad Q_1(\mathbf{x}) \neq 0, \ldots, Q_m(\mathbf{x}) \neq 0\}.$$

Given a variety

$$X(\mathbb{F}) = \{\mathbf{x} \in \mathbb{F}^r \mid P_1(\mathbf{x}) = \cdots = P_n(\mathbf{x}) = 0, \quad Q_1(\mathbf{x}) \neq 0, \ldots, Q_m(\mathbf{x}) \neq 0\}.$$

Step 1. Compute point count $\#X(\mathbb{F}_q)$ over $\mathbb{F}_q$.

$\boxed{\text{Point count } \#X(\mathbb{F}_q)}$

Given a variety

$$X(\mathbb{F}) = \{\mathbf{x} \in \mathbb{F}^r \mid P_1(\mathbf{x}) = \cdots = P_n(\mathbf{x}) = 0, \quad Q_1(\mathbf{x}) \neq 0, \ldots, Q_m(\mathbf{x}) \neq 0\}.$$

Step 1. Compute point count $\#X(\mathbb{F}_q)$ over $\mathbb{F}_q$.

Step 2. Compute Poincaré polynomial $\mathcal{P}(X(\mathbb{C}); t) := \sum_i t^{\frac{i}{2}} \dim H^i(X(\mathbb{C}))$.

$\boxed{\text{Point count } \#X(\mathbb{F}_q)}$ $\boxed{\text{Poincaré polynomial } \mathcal{P}(X(\mathbb{C}); t)}$

Given a variety

$$X(\mathbb{F}) = \{\mathbf{x} \in \mathbb{F}^r \mid P_1(\mathbf{x}) = \cdots = P_n(\mathbf{x}) = 0, \quad Q_1(\mathbf{x}) \neq 0, \ldots, Q_m(\mathbf{x}) \neq 0\}.$$

Step 1. Compute point count $\#X(\mathbb{F}_q)$ over $\mathbb{F}_q$.

Step 2. Compute Poincaré polynomial $\mathcal{P}(X(\mathbb{C}); t) := \sum_i t^{\frac{i}{2}} \dim H^i(X(\mathbb{C}))$.

$$\begin{tikzcd}
 & \boxed{\text{Mixed Hodge polynomial } \mathcal{P}(X; q, t) \in \mathbb{N}\left[q^{\frac{1}{2}}, t^{\frac{1}{2}}\right]} \arrow[dl, "t^{\frac{1}{2}} = -q^{-\frac{1}{2}}"'] \arrow[dr, "q^{\frac{1}{2}} = 1"] & \\
\boxed{\text{Point count } \#X(\mathbb{F}_q)} & & \boxed{\text{Poincaré polynomial } \mathcal{P}(X(\mathbb{C}); t)}
\end{tikzcd}$$

Given a variety [of "Hodge–Tate type"]

$$X(\mathbb{F}) = \{\mathbf{x}\in\mathbb{F}^r \mid P_1(\mathbf{x})=\cdots=P_n(\mathbf{x})=0,\quad Q_1(\mathbf{x})\neq 0,\ldots,Q_m(\mathbf{x})\neq 0\}.$$

Step 1. Compute point count $\#X(\mathbb{F}_q)$ over $\mathbb{F}_q$.

Step 2. Compute Poincaré polynomial $\mathcal{P}(X(\mathbb{C});t) := \sum_i t^{\frac{i}{2}} \dim H^i(X(\mathbb{C}))$.

Step 3. Compute the mixed Hodge polynomial $\mathcal{P}(X;q,t)$

[Deligne splitting / weight filtration $\to$ canonical second grading on $H^*(X)$]

$$\begin{tikzcd}[column sep=small]
& \boxed{\text{Mixed Hodge polynomial } \mathcal{P}(X;q,t)\in\mathbb{N}\left[q^{\frac{1}{2}},t^{\frac{1}{2}}\right]} \arrow[dl, "t^{\frac{1}{2}}=-q^{-\frac{1}{2}}"'] \arrow[dr, "q^{\frac{1}{2}}=1"] & \\
\boxed{\text{Point count } \#X(\mathbb{F}_q)} & & \boxed{\text{Poincaré polynomial } \mathcal{P}(X(\mathbb{C});t)}
\end{tikzcd}$$

Given a variety [of "Hodge–Tate type"]

$$X(\mathbb{F}) = \{\mathbf{x} \in \mathbb{F}^r \mid P_1(\mathbf{x}) = \cdots = P_n(\mathbf{x}) = 0, \quad Q_1(\mathbf{x}) \neq 0, \ldots, Q_m(\mathbf{x}) \neq 0\}.$$

<u>Step 1.</u> Compute point count $\#X(\mathbb{F}_q)$ over $\mathbb{F}_q$.

<u>Step 2.</u> Compute Poincaré polynomial $\mathcal{P}(X(\mathbb{C}); t) := \sum_i t^{\frac{i}{2}} \dim H^i(X(\mathbb{C}))$.

<u>Step 3.</u> Compute the mixed Hodge polynomial $\mathcal{P}(X; q, t)$

[Deligne splitting / weight filtration $\to$ canonical second grading on $H^*(X)$]

$$\begin{tikzcd}
& \boxed{\text{Mixed Hodge polynomial } \mathcal{P}(X;q,t) \in \mathbb{N}\left[q^{\frac{1}{2}}, t^{\frac{1}{2}}\right]} \arrow[dl, "t^{\frac{1}{2}} = -q^{-\frac{1}{2}}"'] \arrow[dr, "q^{\frac{1}{2}} = 1"] & \\
\boxed{\text{Point count } \#X(\mathbb{F}_q)} \arrow[dr, "q = 1"'] & & \boxed{\text{Poincaré polynomial } \mathcal{P}(X(\mathbb{C}); t)} \arrow[dl, "t^{\frac{1}{2}} = -1"] \\
& \boxed{\text{Euler characteristic}} &
\end{tikzcd}$$

Given a variety [of "Hodge–Tate type"]

$$X(\mathbb{F}) = \{\mathbf{x} \in \mathbb{F}^r \mid P_1(\mathbf{x}) = \cdots = P_n(\mathbf{x}) = 0, \quad Q_1(\mathbf{x}) \neq 0, \ldots, Q_m(\mathbf{x}) \neq 0\}.$$

Step 1. Compute point count $\#X(\mathbb{F}_q)$ over $\mathbb{F}_q$.

Step 2. Compute Poincaré polynomial $\mathcal{P}(X(\mathbb{C}); t) := \sum_i t^{\frac{i}{2}} \dim H^i(X(\mathbb{C}))$.

Step 3. Compute the mixed Hodge polynomial $\mathcal{P}(X; q, t)$

[Deligne splitting / weight filtration $\rightarrow$ canonical second grading on $H^*(X)$]

$$\begin{tikzcd}
 & \boxed{\text{Mixed Hodge polynomial } \mathcal{P}(X; q, t) \in \mathbb{N}\left[q^{\frac{1}{2}}, t^{\frac{1}{2}}\right]} \arrow[dl, "t^{\frac{1}{2}} = -q^{-\frac{1}{2}}"'] \arrow[dr, "q^{\frac{1}{2}} = 1"] & \\
\boxed{\text{Point count } \#X(\mathbb{F}_q)} \arrow[dr, "q = 1"'] & & \boxed{\text{Poincaré polynomial } \mathcal{P}(X(\mathbb{C}); t)} \arrow[dl, "t^{\frac{1}{2}} = -1"] \\
 & \boxed{\text{Euler characteristic}} &
\end{tikzcd}$$

Question: Which variety should we choose?

Definition (G.–Lam)

Let $\gcd(k,n)=1$. The Catalan variety is given by

$$X^\circ_{k,n} := \{V \in \mathrm{Gr}(k,n) \mid \Delta_{1,\dots,k}(V) = \Delta_{2,\dots,k+1}(V) = \cdots = \Delta_{n,1,\dots,k-1}(V) = 1\}.$$

Definition (G.–Lam)

Let $\gcd(k, n) = 1$. The Catalan variety is given by

$$X^\circ_{k,n} := \{V \in \mathrm{Gr}(k, n) \mid \Delta_{1,\dots,k}(V) = \Delta_{2,\dots,k+1}(V) = \cdots = \Delta_{n,1,\dots,k-1}(V) = 1\}.$$

Example:

$$X^\circ_{2,5} = \left\{ \mathrm{RowSpan} \begin{pmatrix} 1 & 0 & a & b & c \\ 0 & 1 & d & e & f \end{pmatrix} \middle| \begin{array}{cc} -a = 1, & ae - bd = 1, \\ f = 1, & bf - ce = 1 \end{array} \right\}.$$

Definition (G.–Lam)

Let $\gcd(k,n)=1$. The Catalan variety is given by

$$X^\circ_{k,n} := \{V \in \mathrm{Gr}(k,n) \mid \Delta_{1,\dots,k}(V) = \Delta_{2,\dots,k+1}(V) = \cdots = \Delta_{n,1,\dots,k-1}(V) = 1\}.$$

Example:

$$X^\circ_{2,5} = \left\{ \mathrm{RowSpan}\begin{pmatrix} 1 & 0 & a & b & c \\ 0 & 1 & d & e & f \end{pmatrix} \middle| \begin{array}{ll} -a = 1, & ae - bd = 1, \\ f = 1, & bf - ce = 1 \end{array} \right\}.$$

Theorem (G.–Lam)

Definition (G.–Lam)

Let $\gcd(k, n) = 1$. The Catalan variety is given by

$$X^\circ_{k,n} := \{V \in \mathrm{Gr}(k,n) \mid \Delta_{1,\dots,k}(V) = \Delta_{2,\dots,k+1}(V) = \cdots = \Delta_{n,1,\dots,k-1}(V) = 1\}.$$

Example:

$$X^\circ_{2,5} = \left\{ \mathrm{RowSpan} \begin{pmatrix} 1 & 0 & a & b & c \\ 0 & 1 & d & e & f \end{pmatrix} \middle| \begin{array}{ll} -a = 1, & ae - bd = 1, \\ f = 1, & bf - ce = 1 \end{array} \right\}.$$

Theorem (G.–Lam)

$$\begin{tikzcd}
& \boxed{\textit{Mixed Hodge polynomial } \mathcal{P}(X^\circ_{k,n}; q, t) = C_{k,n-k}(q,t)} \arrow[dl, "t^{\frac{1}{2}} = -q^{-\frac{1}{2}}"'] \arrow[dr, "q^{\frac{1}{2}} = 1"] & \\
\boxed{\textit{Point count } \# X^\circ_{k,n}(\mathbb{F}_q) = C'_{k,n-k}(q)} \arrow[dr, "q = 1"'] & & \boxed{\textit{Poincaré polynomial } \mathcal{P}(X^\circ_{k,n}(\mathbb{C}); t) = C''_{k,n-k}(t)} \arrow[dl, "t^{\frac{1}{2}} = -1"] \\
& \boxed{\textit{Euler characteristic} = C_{k,n-k}} &
\end{tikzcd}$$

Definition (G.–Lam)

Let $\gcd(k,n)=1$. The Catalan variety is given by

$$X^\circ_{k,n} := \{V \in \mathrm{Gr}(k,n) \mid \Delta_{1,\dots,k}(V) = \Delta_{2,\dots,k+1}(V) = \cdots = \Delta_{n,1,\dots,k-1}(V) = 1\}.$$

Example:

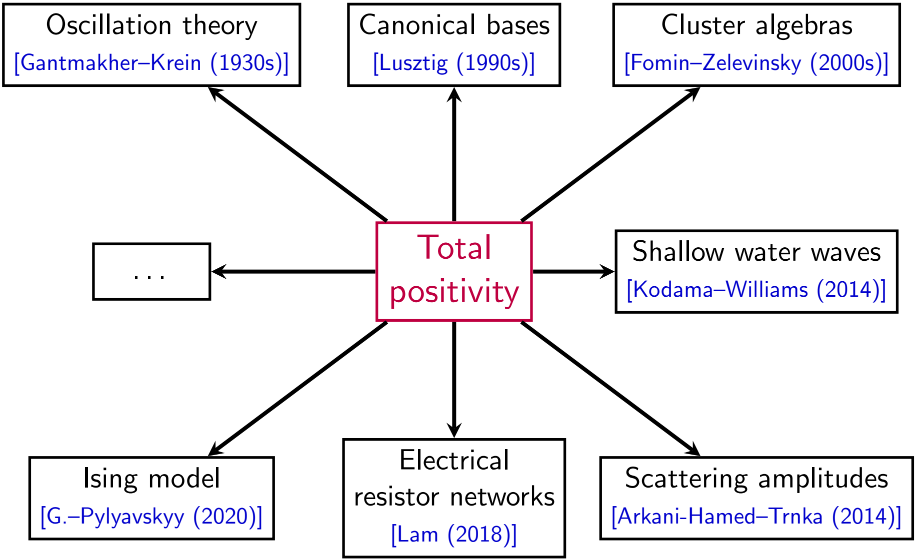
$$X^\circ_{2,5} = \left\{ \mathrm{RowSpan}\begin{pmatrix} 1 & 0 & a & b & c \\ 0 & 1 & d & e & f \end{pmatrix} \middle| \begin{array}{ll} -a = 1, & ae-bd=1, \\ f=1, & bf-ce=1 \end{array} \right\}.$$

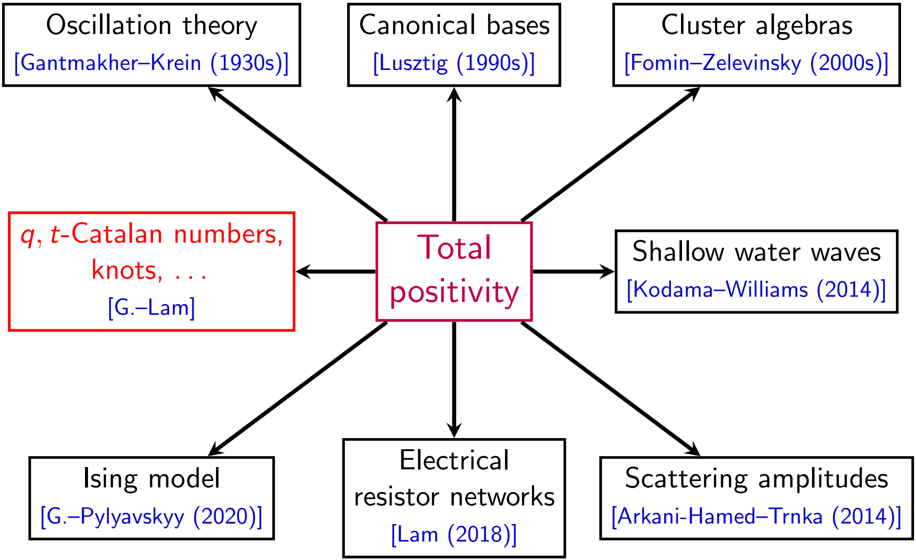
$$\# X^\circ_{2,5}(\mathbb{F}_q) = q^2+1, \quad \mathcal{P}(X^\circ_{2,5}(\mathbb{C});t) = 1+t, \quad \mathcal{P}(X^\circ_{2,5};q,t) = q+t.$$

Theorem (G.–Lam)

$$\begin{tikzcd}
 & \boxed{\textit{Mixed Hodge polynomial } \mathcal{P}(X^\circ_{k,n};q,t) = C_{k,n-k}(q,t)} \arrow[dl, "t^{\frac12} = -q^{-\frac12}"'] \arrow[dr, "q^{\frac12} = 1"] & \\
\boxed{\textit{Point count } \# X^\circ_{k,n}(\mathbb{F}_q) = C'_{k,n-k}(q)} \arrow[dr, "q=1"'] & & \boxed{\textit{Poincaré polynomial } \mathcal{P}(X^\circ_{k,n}(\mathbb{C});t) = C''_{k,n-k}(t)} \arrow[dl, "t^{\frac12} = -1"] \\
 & \boxed{\textit{Euler characteristic } = C_{k,n-k}} &
\end{tikzcd}$$

Motivation: total positivity


Oscillation theory
[Gantmakher–Krein (1930s)]
Canonical bases
[Lusztig (1990s)]
Cluster algebras
[Fomin–Zelevinsky (2000s)]
...
Total positivity
Shallow water waves
[Kodama–Williams (2014)]
Ising model
[G.–Pylyavskyy (2020)]
Electrical resistor networks
[Lam (2018)]
Scattering amplitudes
[Arkani-Hamed–Trnka (2014)]



Oscillation theory
[Gantmakher–Krein (1930s)]
Canonical bases
[Lusztig (1990s)]
Cluster algebras
[Fomin–Zelevinsky (2000s)]
q, t-Catalan numbers,
knots, . . .
[G.–Lam]
Total
positivity
Shallow water waves
[Kodama–Williams (2014)]
Ising model
[G.–Pylyavskyy (2020)]
Electrical
resistor networks
[Lam (2018)]
Scattering amplitudes
[Arkani-Hamed–Trnka (2014)]

The Grassmannian

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k,n;\mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan}\begin{bmatrix}1 & 1 & 0 & -1\\ 0 & 2 & 1 & 1\end{bmatrix} \in \mathrm{Gr}(2,4;\mathbb{F}) \qquad \begin{matrix}\Delta_{13} = 1 & \Delta_{12} = 2 & \Delta_{14} = 1\\ \Delta_{24} = 3 & \Delta_{34} = 1 & \Delta_{23} = 1.\end{matrix}$$

Plücker coordinates: for $I \subseteq \{1, 2, \ldots, n\}$ of size k,

$$\Delta_I := k \times k \text{ minor with column set } I.$$

The Δ_I's are defined up to common rescaling.

The Grassmannian

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k,n;\mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan}\begin{bmatrix}1 & 1 & 0 & -1\\ 0 & 2 & 1 & 1\end{bmatrix} \in \mathrm{Gr}(2,4;\mathbb{F}) \qquad \begin{matrix}\Delta_{13} = 1 & \Delta_{12} = 2 & \Delta_{14} = 1\\ \Delta_{24} = 3 & \Delta_{34} = 1 & \Delta_{23} = 1.\end{matrix}$$

Plücker coordinates: for $I \subseteq \{1, 2, \ldots, n\}$ of size k,

$$\Delta_I := k \times k \text{ minor with column set } I.$$

The Δ_I's are defined up to common rescaling.

Definition (Gelfand–Goresky–MacPherson–Serganova (1987))

- $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$.

The Grassmannian

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k,n;\mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan}\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \in \mathrm{Gr}(2,4;\mathbb{F}) \qquad \begin{matrix} \Delta_{13} = 1 & \Delta_{12} = 2 & \Delta_{14} = 1 \\ \Delta_{24} = 3 & \Delta_{34} = 1 & \Delta_{23} = 1. \end{matrix}$$

Plücker coordinates: for $I \subseteq \{1, 2, \ldots, n\}$ of size k,

$$\Delta_I := k \times k \text{ minor with column set } I.$$

The Δ_I's are defined up to common rescaling.

Definition (Gelfand–Goresky–MacPherson–Serganova (1987))

- $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$.
- Matroid stratification: $\mathrm{Gr}(k,n;\mathbb{F}) = \bigsqcup_{\mathcal{M}} S_{\mathcal{M}}$, where $S_{\mathcal{M}} = \{V \in \mathrm{Gr}(k,n;\mathbb{F}) \mid \mathcal{M}_V = \mathcal{M}\}$.

The Grassmannian

$$\mathrm{Gr}(k, n; \mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k, n; \mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \in \mathrm{Gr}(2, 4; \mathbb{F}) \qquad \begin{matrix} \Delta_{13} = 1 & \Delta_{12} = 2 & \Delta_{14} = 1 \\ \Delta_{24} = 3 & \Delta_{34} = 1 & \Delta_{23} = 1. \end{matrix}$$

Plücker coordinates: for $I \subseteq \{1, 2, \ldots, n\}$ of size k,

$$\Delta_I := k \times k \text{ minor with column set } I.$$

The Δ_I's are defined up to common rescaling.

Definition (Gelfand–Goresky–MacPherson–Serganova (1987))

- $V \in \mathrm{Gr}(k, n; \mathbb{F}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$.
- Matroid stratification: $\mathrm{Gr}(k, n; \mathbb{F}) = \bigsqcup_{\mathcal{M}} S_{\mathcal{M}}$, where $S_{\mathcal{M}} = \{V \in \mathrm{Gr}(k, n; \mathbb{F}) \mid \mathcal{M}_V = \mathcal{M}\}$.

This is a horrible stratification. [Mnëv (1988)]

The Grassmannian

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k,n;\mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan}\begin{bmatrix}1 & 1 & 0 & -1\\ 0 & 2 & 1 & 1\end{bmatrix} \in \mathrm{Gr}(2,4;\mathbb{F}) \qquad \begin{array}{lll}\Delta_{13} = 1 & \Delta_{12} = 2 & \Delta_{14} = 1\\ \Delta_{24} = 3 & \Delta_{34} = 1 & \Delta_{23} = 1.\end{array}$$

Plücker coordinates: for $I \subseteq \{1, 2, \ldots, n\}$ of size k,

$$\Delta_I := k \times k \text{ minor with column set } I.$$

The Δ_I's are defined up to common rescaling.

The Grassmannian

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k,n;\mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan}\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \in \mathrm{Gr}(2,4;\mathbb{F}) \qquad \begin{matrix} \Delta_{13} = 1 & \Delta_{12} = 2 & \Delta_{14} = 1 \\ \Delta_{24} = 3 & \Delta_{34} = 1 & \Delta_{23} = 1. \end{matrix}$$

Plücker coordinates: for $I \subseteq \{1,2,\dots,n\}$ of size k,

$$\Delta_I := k \times k \text{ minor with column set } I.$$

The Δ_I's are defined up to common rescaling.

Definition (Lusztig (1994), Postnikov (2006))

The nonnegative Grassmannian is

$$\mathrm{Gr}_{\geqslant 0}(k,n) := \{V \in \mathrm{Gr}(k,n;\mathbb{R}) \mid \text{all } \Delta_I(V) \geqslant 0 \text{ or all } \Delta_I(V) \leqslant 0\}.$$

[Lus94] G. Lusztig. Total positivity in reductive groups. In *Lie theory and geometry*, volume 123 of *Progr. Math.*, pp. 531–568, Birkhäuser Boston, Boston, MA, 1994.

[Pos06] A. Postnikov. Total positivity, Grassmannians, and networks. arXiv:math/0609764.

The Grassmannian

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k,n;\mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan}\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \in \mathrm{Gr}_{\geqslant 0}(2,4) \qquad \begin{matrix} \Delta_{13} = 1 & \Delta_{12} = 2 & \Delta_{14} = 1 \\ \Delta_{24} = 3 & \Delta_{34} = 1 & \Delta_{23} = 1. \end{matrix}$$

Plücker coordinates: for $I \subseteq \{1,2,\ldots,n\}$ of size k,

$$\Delta_I := k \times k \text{ minor with column set } I.$$

The Δ_I's are defined up to common rescaling.

Definition (Lusztig (1994), Postnikov (2006))

The nonnegative Grassmannian is

$$\mathrm{Gr}_{\geqslant 0}(k,n) := \{V \in \mathrm{Gr}(k,n;\mathbb{R}) \mid \text{all } \Delta_I(V) \geqslant 0 \text{ or all } \Delta_I(V) \leqslant 0\}.$$

[Lus94] G. Lusztig. Total positivity in reductive groups. In *Lie theory and geometry*, volume 123 of *Progr. Math.*, pp. 531–568, Birkhäuser Boston, Boston, MA, 1994.

[Pos06] A. Postnikov. Total positivity, Grassmannians, and networks. `arXiv:math/0609764`.

The Grassmannian

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\}.$$
$$\mathrm{Gr}(k,n;\mathbb{F}) := \{k \times n \text{ matrices of rank } k\}/(\text{row operations}).$$

Example:

$$\mathrm{RowSpan}\begin{bmatrix}1 & 1 & 0 & -1\\ 0 & 2 & 1 & 1\end{bmatrix} \in \mathrm{Gr}_{>0}(2,4) \qquad \begin{matrix}\Delta_{13}=1 & \Delta_{12}=2 & \Delta_{14}=1\\ \Delta_{24}=3 & \Delta_{34}=1 & \Delta_{23}=1.\end{matrix}$$

Plücker coordinates: for $I \subseteq \{1,2,\dots,n\}$ of size k,

$$\Delta_I := k \times k \text{ minor with column set } I.$$

The Δ_I's are defined up to common rescaling.

Definition (Lusztig (1994), Postnikov (2006))

The nonnegative Grassmannian is

$$\mathrm{Gr}_{\geqslant 0}(k,n) := \{V \in \mathrm{Gr}(k,n;\mathbb{R}) \mid \text{all } \Delta_I(V) \geqslant 0 \text{ or all } \Delta_I(V) \leqslant 0\}.$$

[Lus94] G. Lusztig. Total positivity in reductive groups. In *Lie theory and geometry*, volume 123 of *Progr. Math.*, pp. 531–568, Birkhäuser Boston, Boston, MA, 1994.

[Pos06] A. Postnikov. Total positivity, Grassmannians, and networks. `arXiv:math/0609764`.

Topology of $\mathrm{Gr}_{\geqslant 0}(k, n)$

Topology of $\mathrm{Gr}_{\geqslant 0}(k, n)$

- [Lusztig (1998)]: $\mathrm{Gr}_{\geqslant 0}(k, n)$ is contractible.

Topology of $\mathrm{Gr}_{\geqslant 0}(k, n)$

- [Lusztig (1998)]: $\mathrm{Gr}_{\geqslant 0}(k, n)$ is contractible.
- [Rietsch (1999)], [Postnikov (2006)]:
  each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

Topology of $\mathrm{Gr}_{\geqslant 0}(k,n)$

- [Lusztig (1998)]: $\mathrm{Gr}_{\geqslant 0}(k,n)$ is contractible.
- [Rietsch (1999)], [Postnikov (2006)]:
  each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

Conjecture (Lusztig (1998) / Fomin–Shapiro (2000) / Postnikov (2006))

- the closure of each boundary cell is homeomorphic to a closed ball;

Topology of $\mathrm{Gr}_{\geqslant 0}(k,n)$

- [Lusztig (1998)]: $\mathrm{Gr}_{\geqslant 0}(k,n)$ is contractible.
- [Rietsch (1999)], [Postnikov (2006)]:
  each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

Conjecture (Lusztig (1998) / Fomin–Shapiro (2000) / Postnikov (2006))

- the closure of each boundary cell is homeomorphic to a closed ball;
- in particular, $\mathrm{Gr}_{\geqslant 0}(k,n)$ is homeomorphic to a closed ball.

Topology of $\mathrm{Gr}_{\geqslant 0}(k, n)$

- [Lusztig (1998)]: $\mathrm{Gr}_{\geqslant 0}(k, n)$ is contractible.
- [Rietsch (1999)], [Postnikov (2006)]:
  each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

Conjecture (Lusztig (1998) / Fomin–Shapiro (2000) / Postnikov (2006))

- the closure of each boundary cell is homeomorphic to a closed ball;
- in particular, $\mathrm{Gr}_{\geqslant 0}(k, n)$ is homeomorphic to a closed ball.

[Williams (2007)], [Rietsch–Williams (2010)], [Hersh (2014)], ...

Topology of $\mathrm{Gr}_{\geqslant 0}(k, n)$

- [Lusztig (1998)]: $\mathrm{Gr}_{\geqslant 0}(k, n)$ is contractible.
- [Rietsch (1999)], [Postnikov (2006)]:
  each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

Conjecture (Lusztig (1998) / Fomin–Shapiro (2000) / Postnikov (2006))

- the closure of each boundary cell is homeomorphic to a closed ball;
- in particular, $\mathrm{Gr}_{\geqslant 0}(k, n)$ is homeomorphic to a closed ball.

[Williams (2007)], [Rietsch–Williams (2010)], [Hersh (2014)], ...

Theorem (G.–Karp–Lam)

[GKL1] $\mathrm{Gr}_{\geqslant 0}(k, n)$ *is homeomorphic to a closed ball.*

[GKL2] *The closure of each cell is homeomorphic to a closed ball.*

[GKL1] P. Galashin, S. Karp, and T. Lam. The totally nonnegative Grassmannian is a ball. *Adv. Math.*, 397: 108123, 2022.

[GKL2] P. Galashin, S. Karp, and T. Lam. Regularity theorem for totally nonnegative flag varieties. *J. Amer. Math. Soc.*, 35(2):513–579, 2021.

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k, n; \mathbb{F}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).
Better: $V \in \mathrm{Gr}(k, n; \mathbb{F}) \longrightarrow$ Permutation $f_V : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).
Better: $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Permutation $f_V : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \dots, u_n \in \mathbb{F}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1},\dots,u_j)\} \pmod{n}.$$

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).
Better: $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Permutation $f_V : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \dots, u_n \in \mathbb{F}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1},\dots,u_j)\} \pmod{n}.$$

Example

$$\begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \\ \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \end{array} \quad \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \\ \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \end{array} \quad \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \\ \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \end{array}$$

$$f_V = \left(\qquad \qquad \qquad \right).$$

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k, n; \mathbb{F}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).
Better: $V \in \mathrm{Gr}(k, n; \mathbb{F}) \longrightarrow$ Permutation $f_V : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \ldots, u_n \in \mathbb{F}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1}, \ldots, u_j)\} \pmod{n}.$$

Example

$$\begin{array}{cccccc} \begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \\ u_1 \; u_2 \; u_3 \; u_4 \; u_5 \; u_6 & u_1 \; u_2 \; u_3 \; u_4 \; u_5 \; u_6 & u_1 \; u_2 \; u_3 \; u_4 \; u_5 \; u_6 \end{array}$$

$$f_V = \begin{pmatrix} 1 & \\ 5 & \end{pmatrix}.$$

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).
Better: $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Permutation $f_V : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \dots, u_n \in \mathbb{F}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1},\dots,u_j)\} \pmod{n}.$$

Example

$$\begin{array}{cccccc} \multicolumn{6}{c}{\begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \begin{array}{cccccc} \multicolumn{6}{c}{\begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \begin{array}{cccccc} \multicolumn{6}{c}{\begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array}$$

$$f_V = \begin{pmatrix} 1 & 2 & & & & \\ 5 & 3 & & & & \end{pmatrix}.$$

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k, n; \mathbb{F}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).
Better: $V \in \mathrm{Gr}(k, n; \mathbb{F}) \longrightarrow$ Permutation $f_V : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \dots, u_n \in \mathbb{F}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1}, \dots, u_j)\} \pmod{n}.$$

Example

$$\begin{array}{cccccc} & & & & & \\ \end{array}\begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{cccccccccccccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array}$$

$$f_V = \begin{pmatrix} 1 & 2 & 3 & & & \\ 5 & 3 & 6 & & & \end{pmatrix}.$$

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).
Better: $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Permutation $f_V : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \dots, u_n \in \mathbb{F}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1},\dots,u_j)\} \pmod{n}.$$

Example

$$\begin{array}{cccccc} \begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \\ u_1\ u_2\ u_3\ u_4\ u_5\ u_6 & u_1\ u_2\ u_3\ u_4\ u_5\ u_6 & u_1\ u_2\ u_3\ u_4\ u_5\ u_6 \end{array}$$

$$f_V = \begin{pmatrix} 1 & 2 & 3 & 4 & & \\ 5 & 3 & 6 & 4 & & \end{pmatrix}.$$

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k, n; \mathbb{F}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).
Better: $V \in \mathrm{Gr}(k, n; \mathbb{F}) \longrightarrow$ Permutation $f_V : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \ldots, u_n \in \mathbb{F}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1}, \ldots, u_j)\} \pmod{n}.$$

Example

$$\begin{array}{cccccc} \multicolumn{6}{c}{\begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \quad \begin{array}{cccccc} \multicolumn{6}{c}{\begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \quad \begin{array}{cccccc} \multicolumn{6}{c}{\begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array}$$

$$f_V = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \\ 5 & 3 & 6 & 4 & 2 & \end{pmatrix}.$$

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).
Better: $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Permutation $f_V : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \dots, u_n \in \mathbb{F}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1},\dots,u_j)\} \pmod{n}.$$

Example

$$\begin{array}{cccccc|cccccc|cccccc} 1 & 1 & 1 & 0 & -1 & -1 & 1 & 1 & 1 & 0 & -1 & -1 & 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array}$$

$$f_V = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 6 & 4 & 2 & 1 \end{pmatrix}.$$

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k, n; \mathbb{R}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).
Better: $V \in \mathrm{Gr}(k, n; \mathbb{R}) \longrightarrow$ Permutation $f_V : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \dots, u_n \in \mathbb{R}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1}, \dots, u_j)\} \pmod{n}.$$

Example (Generic case)

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k,n;\mathbb{R}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).

Better: $V \in \mathrm{Gr}(k,n;\mathbb{R}) \longrightarrow$ Permutation $f_V : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \dots, u_n \in \mathbb{R}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1},\dots,u_j)\} \pmod{n}.$$

Example (Generic case)

$$\begin{array}{ccccc} \multicolumn{5}{c}{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \quad \begin{array}{ccccc} \multicolumn{5}{c}{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \quad \begin{array}{ccccc} \multicolumn{5}{c}{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 \end{array}$$

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k,n;\mathbb{R}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).
Better: $V \in \mathrm{Gr}(k,n;\mathbb{R}) \longrightarrow$ Permutation $f_V : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \dots, u_n \in \mathbb{R}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1}, \dots, u_j)\} \pmod{n}.$$

Example (Generic case)

$$\begin{array}{ccccc} \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \\ u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 & u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 & u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \end{array}$$

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k, n; \mathbb{R}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).
Better: $V \in \mathrm{Gr}(k, n; \mathbb{R}) \longrightarrow$ Permutation $f_V : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \ldots, u_n \in \mathbb{R}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1}, \ldots, u_j)\} \pmod{n}.$$

Example (Generic case)

$$\begin{array}{ccccc} \multicolumn{5}{c}{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \quad \begin{array}{ccccc} \multicolumn{5}{c}{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \quad \begin{array}{ccccc} \multicolumn{5}{c}{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 \end{array}$$

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k, n; \mathbb{R}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).

Better: $V \in \mathrm{Gr}(k, n; \mathbb{R}) \longrightarrow$ Permutation $f_V : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \ldots, u_n \in \mathbb{R}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1}, \ldots, u_j)\} \pmod{n}.$$

Example (Generic case)

$$\begin{array}{c} \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \\ \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{matrix} \end{array} \quad \begin{array}{c} \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \\ \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{matrix} \end{array} \quad \begin{array}{c} \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \\ \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{matrix} \end{array}$$

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k,n;\mathbb{R}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).

Better: $V \in \mathrm{Gr}(k,n;\mathbb{R}) \longrightarrow$ Permutation $f_V : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \dots, u_n \in \mathbb{R}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1},\dots,u_j)\} \pmod{n}.$$

Example (Generic case)

$$\begin{array}{ccc} \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \\ u_1 \;\; u_2 \;\; u_3 \;\; u_4 \;\; u_5 & u_1 \;\; u_2 \;\; u_3 \;\; u_4 \;\; u_5 & u_1 \;\; u_2 \;\; u_3 \;\; u_4 \;\; u_5 \end{array}$$

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k,n;\mathbb{R}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).
Better: $V \in \mathrm{Gr}(k,n;\mathbb{R}) \longrightarrow$ Permutation $f_V : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \dots, u_n \in \mathbb{R}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1},\dots,u_j)\} \pmod{n}.$$

Example (Generic case)

$$\begin{array}{ccccc} \multicolumn{5}{c}{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \begin{array}{ccccc} \multicolumn{5}{c}{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \begin{array}{ccccc} \multicolumn{5}{c}{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 \end{array}$$

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k,n;\mathbb{R}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).

Better: $V \in \mathrm{Gr}(k,n;\mathbb{R}) \longrightarrow$ Permutation $f_V : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \dots, u_n \in \mathbb{R}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1},\dots,u_j)\} \pmod{n}.$$

Example (Generic case)

$$\begin{array}{c} \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \\ \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{matrix} \end{array} \begin{array}{c} \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \\ \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{matrix} \end{array} \begin{array}{c} \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \\ \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{matrix} \end{array}$$

Let $f_{k,n}(i) \equiv i + k \pmod{n}$ for all $i = 1,2,\dots,n$. Then

$$f_V = f_{k,n} \iff \Delta_{1,\dots,k}, \Delta_{2,\dots,k+1}\dots, \Delta_{n,1,\dots,k-1} \neq 0.$$

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).
Better: $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Permutation $f_V : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \dots, u_n \in \mathbb{F}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1}, \dots, u_j)\} \pmod{n}.$$

- Generic permutation: $f_{k,n}(i) \equiv i + k \pmod{n}$,

$$\begin{aligned}\Pi^\circ_{k,n} &= \{V \in \mathrm{Gr}(k,n;\mathbb{F}) \mid f_V = f_{k,n}\}\\ &= \{\Delta_{1,\dots,k}, \Delta_{2,\dots,k+1}, \dots, \Delta_{n,1,\dots,k-1} \neq 0\}.\end{aligned}$$

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k, n; \mathbb{F}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).
Better: $V \in \mathrm{Gr}(k, n; \mathbb{F}) \longrightarrow$ Permutation $f_V : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \dots, u_n \in \mathbb{F}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1}, \dots, u_j)\} \pmod{n}.$$

- Generic permutation: $f_{k,n}(i) \equiv i + k \pmod{n}$,

$$\begin{aligned}\Pi^\circ_{k,n} &= \{V \in \mathrm{Gr}(k, n; \mathbb{F}) \mid f_V = f_{k,n}\} \\ &= \{\Delta_{1,\dots,k}, \Delta_{2,\dots,k+1}, \dots, \Delta_{n,1,\dots,k-1} \neq 0\}.\end{aligned}$$

- For all $V \in \mathrm{Gr}(k, n; \mathbb{F})$, f_V is a permutation.

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).
Better: $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Permutation $f_V : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \dots, u_n \in \mathbb{F}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1}, \dots, u_j)\} \pmod{n}.$$

- Generic permutation: $f_{k,n}(i) \equiv i + k \pmod{n}$,

$$\begin{aligned}\Pi^\circ_{k,n} &= \{V \in \mathrm{Gr}(k,n;\mathbb{F}) \mid f_V = f_{k,n}\}\\ &= \{\Delta_{1,\dots,k}, \Delta_{2,\dots,k+1}, \dots, \Delta_{n,1,\dots,k-1} \neq 0\}.\end{aligned}$$

- For all $V \in \mathrm{Gr}(k,n;\mathbb{F})$, f_V is a permutation.
- **Positroid stratification**: [Knutson–Lam–Speyer (2013)]

$$\mathrm{Gr}(k,n;\mathbb{F}) = \bigsqcup_f \Pi^\circ_f, \quad \text{where} \quad \Pi^\circ_f := \{V \in \mathrm{Gr}(k,n;\mathbb{F}) \mid f_V = f\}.$$

Combinatorics of positroids

Recall: $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Matroid $\mathcal{M}_V := \{I \mid \Delta_I(V) \neq 0\}$ (horrible).
Better: $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Permutation $f_V : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$.

Definition (Knutson–Lam–Speyer (2013))

Label the columns of V by $u_1, u_2, \dots, u_n \in \mathbb{F}^k$. Set

$$f_V(i) \equiv \min\{j \geqslant i \mid u_i \in \mathrm{Span}(u_{i+1}, \dots, u_j)\} \pmod{n}.$$

- Generic permutation: $f_{k,n}(i) \equiv i + k \pmod{n}$,

$$\begin{aligned}\Pi^\circ_{k,n} &= \{V \in \mathrm{Gr}(k,n;\mathbb{F}) \mid f_V = f_{k,n}\}\\ &= \{\Delta_{1,\dots,k}, \Delta_{2,\dots,k+1}, \dots, \Delta_{n,1,\dots,k-1} \neq 0\}.\end{aligned}$$

- For all $V \in \mathrm{Gr}(k,n;\mathbb{F})$, f_V is a permutation.
- **Positroid stratification**: [Knutson–Lam–Speyer (2013)]

$$\mathrm{Gr}(k,n;\mathbb{F}) = \bigsqcup_f \Pi^\circ_f, \quad \text{where} \quad \Pi^\circ_f := \{V \in \mathrm{Gr}(k,n;\mathbb{F}) \mid f_V = f\}.$$

- Motivated by total positivity [Postnikov (2006)].

Definition

Let $\gcd(k, n) = 1$.
$X^\circ_{k,n} := \{V \in \mathrm{Gr}(k, n) \mid \Delta_{1,\dots,k}(V) = \Delta_{2,\dots,k+1}(V) = \cdots = \Delta_{n,1,\dots,k-1}(V) = 1\}$.

Example:

$$X^\circ_{2,5} = \left\{ \mathrm{RowSpan} \begin{pmatrix} 1 & 0 & a & b & c \\ 0 & 1 & d & e & f \end{pmatrix} \middle| \begin{array}{ll} -a = 1, & ae - bd = 1, \\ f = 1, & bf - ce = 1 \end{array} \right\}.$$

Theorem (G.–Lam)

$$\begin{tikzcd}
 & \boxed{\mathcal{P}(X^\circ_{k,n}; q, t) = C_{k,n-k}(q, t)} \arrow[dl, "t^{\frac{1}{2}} = -q^{-\frac{1}{2}}"'] \arrow[dr, "q^{\frac{1}{2}} = 1"] & \\
\boxed{\# X^\circ_{k,n}(\mathbb{F}_q) = C'_{k,n-k}(q)} \arrow[dr, "q = 1"'] & & \boxed{\mathcal{P}(X^\circ_{k,n}(\mathbb{C}); t) = C''_{k,n-k}(t)} \arrow[dl, "t^{\frac{1}{2}} = -1"] \\
 & \boxed{\textit{Euler characteristic} = C_{k,n-k}} &
\end{tikzcd}$$

Definition

Let $\gcd(k,n)=1$.
$\Pi^\circ_{k,n} := \{V \in \mathrm{Gr}(k,n) \mid \Delta_{1,\dots,k}(V), \Delta_{2,\dots,k+1}(V), \cdots, \Delta_{n,1,\dots,k-1}(V) \neq 0\}$.

Example:

$$\Pi^\circ_{2,5} = \left\{ \mathrm{RowSpan}\begin{pmatrix} 1 & 0 & a & b & c \\ 0 & 1 & d & e & f \end{pmatrix} \;\middle|\; \begin{matrix} -a \neq 0, & ae-bd \neq 0, \\ f \neq 0, & bf-ce \neq 0 \end{matrix} \right\}.$$

Theorem (G.–Lam)

$$\begin{tikzcd}
 & \boxed{\mathcal{P}(\Pi^\circ_{k,n}; q,t) = (q^{\frac{1}{2}}+t^{\frac{1}{2}})^{n-1} C_{k,n-k}(q,t)} \arrow[dl, "t^{\frac{1}{2}} = -q^{-\frac{1}{2}}"'] \arrow[dr, "q^{\frac{1}{2}} = 1"] & \\
\boxed{\#\Pi^\circ_{k,n}(\mathbb{F}_q) = (q-1)^{n-1} C'_{k,n-k}(q)} \arrow[dr, "q = 1"'] & & \boxed{\mathcal{P}(\Pi^\circ_{k,n}(\mathbb{C}); t) = (1+t^{\frac{1}{2}})^{n-1} C''_{k,n-k}(t)} \arrow[dl, "t^{\frac{1}{2}} = -1"] \\
 & \boxed{\textit{Euler characteristic} = 0} &
\end{tikzcd}$$

Summary so far

- $V \in \mathrm{Gr}(k, n; \mathbb{F}) \longrightarrow$ Permutation $f_V : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Summary so far

- $V \in \mathrm{Gr}(k, n; \mathbb{F}) \longrightarrow$ Permutation $f_V : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.
- Positroid stratification:

$$\mathrm{Gr}(k, n; \mathbb{F}) = \bigsqcup_{f} \Pi_f^\circ, \quad \text{where} \quad \Pi_f^\circ := \{V \in \mathrm{Gr}(k, n; \mathbb{F}) \mid f_V = f\}.$$

Summary so far

- $V \in \mathrm{Gr}(k,n;\mathbb{F}) \longrightarrow$ Permutation $f_V : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$.
- Positroid stratification:
$$\mathrm{Gr}(k,n;\mathbb{F}) = \bigsqcup_f \Pi_f^\circ, \quad \text{where} \quad \Pi_f^\circ := \{V \in \mathrm{Gr}(k,n;\mathbb{F}) \mid f_V = f\}.$$
- Top-dimensional piece: $f_{k,n}(i) \equiv i + k \pmod{n}$,
$$\begin{aligned} \Pi_{k,n}^\circ &= \{V \in \mathrm{Gr}(k,n;\mathbb{F}) \mid f_V = f_{k,n}\} \\ &= \{\Delta_{1,\dots,k}, \Delta_{2,\dots,k+1}, \dots, \Delta_{n,1,\dots,k-1} \neq 0\}. \end{aligned}$$

Summary so far

- $V \in \mathrm{Gr}(k, n; \mathbb{F}) \longrightarrow$ Permutation $f_V : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.
- Positroid stratification:
$$\mathrm{Gr}(k, n; \mathbb{F}) = \bigsqcup_f \Pi_f^\circ, \quad \text{where} \quad \Pi_f^\circ := \{V \in \mathrm{Gr}(k, n; \mathbb{F}) \mid f_V = f\}.$$
- Top-dimensional piece: $f_{k,n}(i) \equiv i + k \pmod{n}$,
$$\begin{aligned}\Pi_{k,n}^\circ &= \{V \in \mathrm{Gr}(k, n; \mathbb{F}) \mid f_V = f_{k,n}\} \\ &= \{\Delta_{1,\ldots,k}, \Delta_{2,\ldots,k+1}, \ldots, \Delta_{n,1,\ldots,k-1} \neq 0\}.\end{aligned}$$

Theorem (G.-Lam)

Cohomology/point count of $\Pi_{k,n}^\circ$ is given by q, t-Catalan numbers.

Summary so far

- $V \in \mathrm{Gr}(k, n; \mathbb{F}) \longrightarrow$ Permutation $f_V : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$.
- Positroid stratification:

$$\mathrm{Gr}(k, n; \mathbb{F}) = \bigsqcup_f \Pi_f^\circ, \quad \text{where} \quad \Pi_f^\circ := \{V \in \mathrm{Gr}(k, n; \mathbb{F}) \mid f_V = f\}.$$

- Top-dimensional piece: $f_{k,n}(i) \equiv i + k \pmod{n}$,

$$\begin{aligned} \Pi_{k,n}^\circ &= \{V \in \mathrm{Gr}(k, n; \mathbb{F}) \mid f_V = f_{k,n}\} \\ &= \{\Delta_{1,\dots,k}, \Delta_{2,\dots,k+1}, \dots, \Delta_{n,1,\dots,k-1} \neq 0\}. \end{aligned}$$

Theorem (G.-Lam)

Cohomology/point count of $\Pi_{k,n}^\circ$ is given by q, t-Catalan numbers.

Question

What about arbitrary Π_f°?

Positroid links

- A knot is an embedding of an oriented circle into $\mathbb{R}^3$.

Positroid links

- A knot is an embedding of an oriented circle into $\mathbb{R}^3$.
- A link is an embedding of several oriented circles into $\mathbb{R}^3$.

Positroid links

- A knot is an embedding of an oriented circle into $\mathbb{R}^3$.
- A link is an embedding of several oriented circles into $\mathbb{R}^3$.
- Knots/links are considered up to ambient isotopy.

Positroid links

- A knot is an embedding of an oriented circle into $\mathbb{R}^3$.
- A link is an embedding of several oriented circles into $\mathbb{R}^3$.
- Knots/links are considered up to ambient isotopy.

Positroid stratification: $\mathrm{Gr}(k,n) = \bigsqcup_f \Pi_f^\circ$, where each f is a permutation.

Positroid links

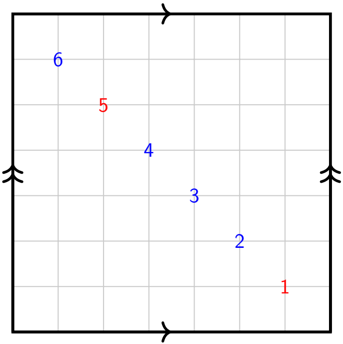
Positroid stratification: $\mathrm{Gr}(k,n) = \bigsqcup_f \Pi_f^\circ$, where each f is a permutation.

Positroid links

Positroid stratification: $\mathrm{Gr}(k,n) = \bigsqcup_f \Pi_f^\circ$, where each f is a permutation.

Associate a link L_f (on a torus) to each permutation $f \in S_n$ as follows:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 3 & 1 & 2 \end{pmatrix} \longrightarrow$$



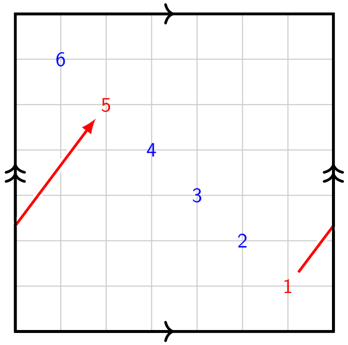
Positroid links

Positroid stratification: $\mathrm{Gr}(k, n) = \bigsqcup_f \Pi_f^\circ$, where each f is a permutation.

Associate a link L_f (on a torus) to each permutation $f \in S_n$ as follows:

- Draw an arrow $i \to f(i)$ in the NE direction for each $i = 1, 2, \ldots, n$.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 3 & 1 & 2 \end{pmatrix} \longrightarrow$$



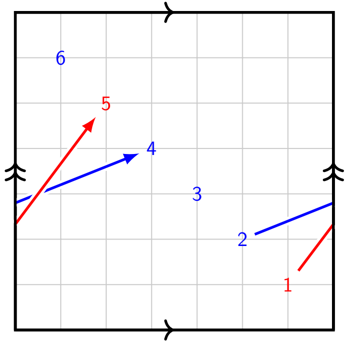
Positroid links

Positroid stratification: $\quad \mathrm{Gr}(k,n) = \bigsqcup_f \Pi_f^\circ$, where each f is a permutation.

Associate a link L_f (on a torus) to each permutation $f \in S_n$ as follows:

- Draw an arrow $i \to f(i)$ in the NE direction for each $i = 1, 2, \ldots, n$.
- Arrows with higher slope are drawn above arrows with lower slope.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 3 & 1 & 2 \end{pmatrix} \longrightarrow$$



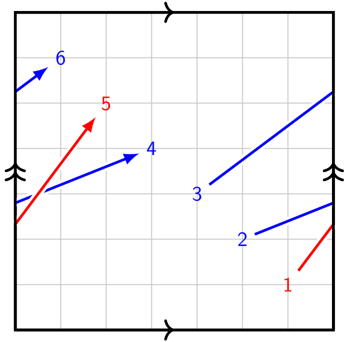
Positroid links

Positroid stratification: $\quad \mathrm{Gr}(k,n) = \bigsqcup_f \Pi_f^\circ$, where each f is a permutation.

Associate a link L_f (on a torus) to each permutation $f \in S_n$ as follows:

- Draw an arrow $i \to f(i)$ in the NE direction for each $i = 1, 2, \ldots, n$.
- Arrows with higher slope are drawn above arrows with lower slope.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 3 & 1 & 2 \end{pmatrix} \longrightarrow$$



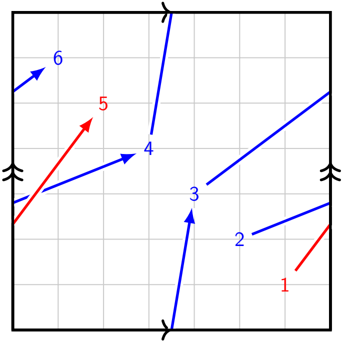
Positroid links

Positroid stratification: $\mathrm{Gr}(k,n) = \bigsqcup_f \Pi_f^\circ$, where each f is a permutation.

Associate a link L_f (on a torus) to each permutation $f \in S_n$ as follows:

- Draw an arrow $i \to f(i)$ in the NE direction for each $i = 1, 2, \ldots, n$.
- Arrows with higher slope are drawn above arrows with lower slope.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 3 & 1 & 2 \end{pmatrix} \longrightarrow$$



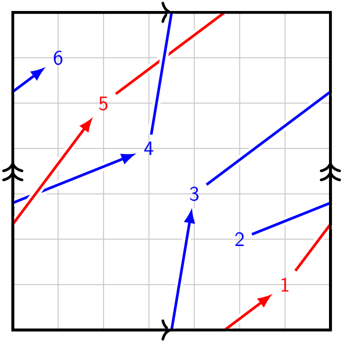
Positroid links

Positroid stratification: $\mathrm{Gr}(k,n) = \bigsqcup_f \Pi_f^\circ$, where each f is a permutation.

Associate a link L_f (on a torus) to each permutation $f \in S_n$ as follows:

- Draw an arrow $i \to f(i)$ in the NE direction for each $i = 1, 2, \ldots, n$.
- Arrows with higher slope are drawn above arrows with lower slope.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 3 & 1 & 2 \end{pmatrix} \longrightarrow$$



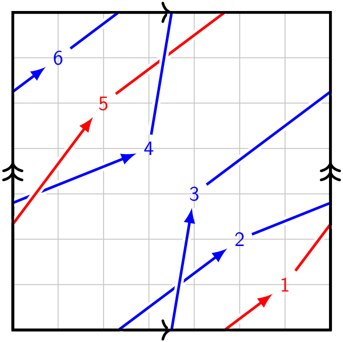
Positroid links

Positroid stratification: $\quad \mathrm{Gr}(k,n) = \bigsqcup_f \Pi_f^\circ$, where each f is a permutation.

Associate a link L_f (on a torus) to each permutation $f \in S_n$ as follows:

- Draw an arrow $i \to f(i)$ in the NE direction for each $i = 1, 2, \ldots, n$.
- Arrows with higher slope are drawn above arrows with lower slope.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 3 & 1 & 2 \end{pmatrix} \longrightarrow$$



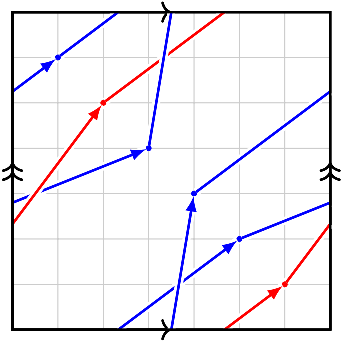
Positroid links

Positroid stratification: $\mathrm{Gr}(k,n) = \bigsqcup_f \Pi_f^\circ$, where each f is a permutation.

Associate a link L_f (on a torus) to each permutation $f \in S_n$ as follows:

- Draw an arrow $i \to f(i)$ in the NE direction for each $i = 1, 2, \ldots, n$.
- Arrows with higher slope are drawn above arrows with lower slope.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 3 & 1 & 2 \end{pmatrix} \longrightarrow$$



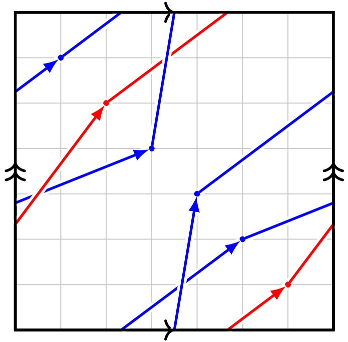
Positroid links

Positroid stratification: $\mathrm{Gr}(k,n) = \bigsqcup_f \Pi_f^\circ$, where each f is a permutation.

Associate a link L_f (on a torus) to each permutation $f \in S_n$ as follows:

- Draw an arrow $i \to f(i)$ in the NE direction for each $i = 1, 2, \ldots, n$.
- Arrows with higher slope are drawn above arrows with lower slope.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 3 & 1 & 2 \end{pmatrix} \longrightarrow$$



This construction: [G.–Lam '22]. Related constructions: [G.–Lam '20], [Shende–Treumann–Williams–Zaslow '15], [Fomin–Pylyavskyy–Shustin–Thurston '17], [Casals–Gorsky–Gorsky–Simental '21]

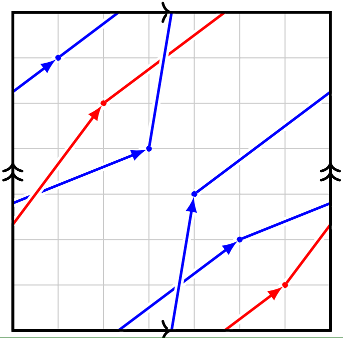
Positroid links

Positroid stratification: $\mathrm{Gr}(k,n) = \bigsqcup_f \Pi_f^\circ$, where each f is a permutation.

Associate a link L_f (on a torus) to each permutation $f \in S_n$ as follows:

- Draw an arrow $i \to f(i)$ in the NE direction for each $i = 1, 2, \ldots, n$.
- Arrows with higher slope are drawn above arrows with lower slope.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 3 & 1 & 2 \end{pmatrix} \longrightarrow$$

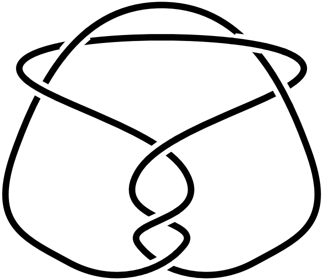


Conclusion

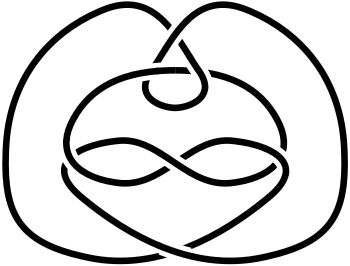
For each permutation $f \in S_n$, get a variety Π_f° and a link $L_f \subseteq \mathbb{R}^3$.

How to tell if two knots/links are isotopic?

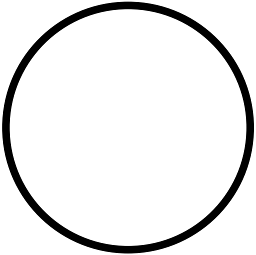
One of these knots is not like the others



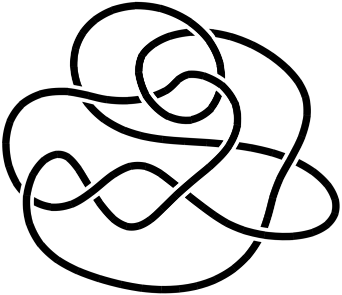
(A)



(B)



(C)



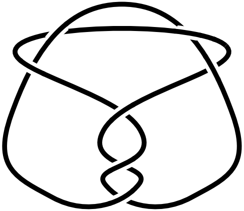
(D)

Given a link L, the HOMFLY polynomial $P(L; a, q)$ is defined by

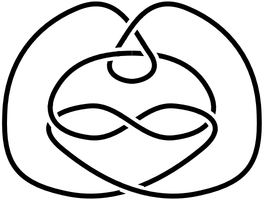
$$P(\bigcirc) = 1 \quad \text{and} \quad aP(L_+) - a^{-1}P(L_-) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) P(L_0), \quad \text{where}$$

L_+ $\quad$ L_- $\quad$ L_0

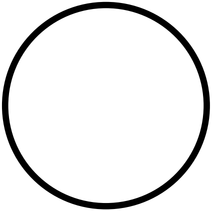
One of these knots is not like the others



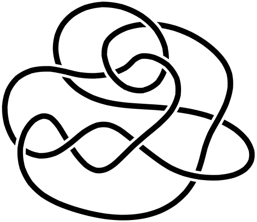
(A) $P(L;a,q) = \frac{q^4-q^3+q^2-q+1}{a^4q^2} + \frac{q^4-q^3+2q^2-q+1}{a^6q^2} - \frac{q^2+1}{a^8q}$



(B) $P(L;a,q) = 1$



(C) $P(L;a,q) = 1$



(D) $P(L;a,q) = 1$

Given a link L, the HOMFLY polynomial $P(L; a, q)$ is defined by

$$P(\bigcirc) = 1 \quad \text{and} \quad aP(L_+) - a^{-1}P(L_-) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) P(L_0), \quad \text{where}$$

L_+ $\quad$ L_- $\quad$ L_0

Given a link L, the HOMFLY polynomial $P(L; a, q)$ is defined by

$$P(\bigcirc) = 1 \quad \text{and} \quad aP(L_+) - a^{-1}P(L_-) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) P(L_0), \quad \text{where}$$

L_+ L_- L_0

Theorem (G.–Lam)

Let $f \in S_n$. Then the point count of Π_f° is given by

$$\#\Pi_f^\circ(\mathbb{F}_q) = (q-1)^{n-1} \cdot (\textit{top } a\textit{-degree coefficient of } P(L_f; a, q)).$$

Given a link L, the HOMFLY polynomial $P(L; a, q)$ is defined by

$$P(\bigcirc) = 1 \quad \text{and} \quad aP(L_+) - a^{-1}P(L_-) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) P(L_0), \quad \text{where}$$

L_+ $\quad$ L_- $\quad$ L_0

Theorem (G.–Lam)

Let $f \in S_n$. Then the point count of Π_f° is given by

$$\#\Pi_f^\circ(\mathbb{F}_q) = (q-1)^{n-1} \cdot (\textit{top } a\textit{-degree coefficient of } P(L_f; a, q)).$$

The group $T \subseteq \mathrm{SL}_n$ of diagonal $n \times n$ matrices acts on Π_f° by rescaling columns.

Given a link L, the HOMFLY polynomial $P(L; a, q)$ is defined by

$$P(\bigcirc) = 1 \quad \text{and} \quad aP(L_+) - a^{-1}P(L_-) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) P(L_0), \quad \text{where}$$

L_+ L_- L_0

Theorem (G.–Lam)

Let $f \in S_n$. Then the point count of Π_f° is given by

$$\#\Pi_f^\circ(\mathbb{F}_q) = (q-1)^{n-1} \cdot (\textit{top a-degree coefficient of } P(L_f; a, q)).$$

The group $T \subseteq \mathrm{SL}_n$ of diagonal $n \times n$ matrices acts on Π_f° by rescaling columns.

Lemma

Let $f \in S_n$. The following are equivalent:

Given a link L, the HOMFLY polynomial $P(L;a,q)$ is defined by

$$P(\bigcirc) = 1 \quad \text{and} \quad aP(L_+) - a^{-1}P(L_-) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) P(L_0), \quad \text{where}$$

L_+ $\quad$ L_- $\quad$ L_0

Theorem (G.–Lam)

Let $f \in S_n$. Then the point count of Π_f° is given by

$$\#\Pi_f^\circ(\mathbb{F}_q) = (q-1)^{n-1} \cdot (\textit{top a-degree coefficient of } P(L_f;a,q)).$$

The group $T \subseteq \mathrm{SL}_n$ of diagonal $n \times n$ matrices acts on Π_f° by rescaling columns.

Lemma

Let $f \in S_n$. The following are equivalent:

- *The T-action on Π_f° is free.*

Given a link L, the HOMFLY polynomial $P(L; a, q)$ is defined by

$$P(\bigcirc) = 1 \quad \text{and} \quad aP(L_+) - a^{-1}P(L_-) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) P(L_0), \quad \text{where}$$

L_+ L_- L_0

Theorem (G.–Lam)

Let $f \in S_n$. Then the point count of Π_f° is given by

$$\#\Pi_f^\circ(\mathbb{F}_q) = (q-1)^{n-1} \cdot (\textit{top a-degree coefficient of } P(L_f; a, q)).$$

The group $T \subseteq \mathrm{SL}_n$ of diagonal $n \times n$ matrices acts on Π_f° by rescaling columns.

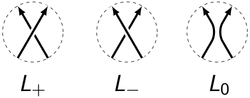
Lemma

Let $f \in S_n$. The following are equivalent:

- *The T-action on Π_f° is free.*
- *f is a single cycle.*

Given a link L, the HOMFLY polynomial $P(L; a, q)$ is defined by

$$P(\bigcirc) = 1 \quad \text{and} \quad aP(L_+) - a^{-1}P(L_-) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) P(L_0), \quad \text{where}$$



Theorem (G.–Lam)

Let $f \in S_n$. Then the point count of Π_f° is given by

$$\#\Pi_f^\circ(\mathbb{F}_q) = (q-1)^{n-1} \cdot (\textit{top } a\textit{-degree coefficient of } P(L_f; a, q)).$$

The group $T \subseteq \mathrm{SL}_n$ of diagonal $n \times n$ matrices acts on Π_f° by rescaling columns.

Lemma

Let $f \in S_n$. The following are equivalent:

- *The T-action on Π_f° is free.*
- *f is a single cycle.*
- *The link L_f is a knot.*

Given a link L, the HOMFLY polynomial $P(L;a,q)$ is defined by

$$P(\bigcirc) = 1 \quad \text{and} \quad aP(L_+) - a^{-1}P(L_-) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) P(L_0), \quad \text{where}$$

L_+ L_- L_0

Theorem (G.–Lam)

Let $f \in S_n$. Then the point count of Π_f° is given by

$$\#\Pi_f^\circ(\mathbb{F}_q) = (q-1)^{n-1} \cdot (\textit{top } a\textit{-degree coefficient of } P(L_f;a,q)).$$

The group $T \subseteq \mathrm{SL}_n$ of diagonal $n \times n$ matrices acts on Π_f° by rescaling columns.

Lemma

Let $f \in S_n$. The following are equivalent:

- *The T-action on Π_f° is free.*
- *f is a single cycle.*
- *The link L_f is a knot.*

In this case, Π_f°/T is smooth and $\mathcal{P}(\Pi_f^\circ;q,t) = \left(q^{\frac{1}{2}} + t^{\frac{1}{2}}\right)^{n-1} \cdot \mathcal{P}(\Pi_f^\circ/T;q,t)$.

Given a link L, the HOMFLY polynomial $P(L; a, q)$ is defined by

$$P(\bigcirc) = 1 \quad \text{and} \quad aP(L_+) - a^{-1}P(L_-) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) P(L_0), \quad \text{where}$$

L_+ $\quad$ L_- $\quad$ L_0

Theorem (G.–Lam)

Let $f \in S_n$. Then the point count of Π_f° is given by

$$\#\Pi_f^\circ(\mathbb{F}_q) = (q-1)^{n-1} \cdot (\textit{top a-degree coefficient of } P(L_f; a, q)).$$

T acts freely on Π_f° $\iff$ f is a single cycle $\iff$ L_f is a knot.

Given a link L, the HOMFLY polynomial $P(L;a,q)$ is defined by

$$P(\bigcirc) = 1 \quad \text{and} \quad aP(L_+) - a^{-1}P(L_-) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)P(L_0), \quad \text{where}$$

L_+ L_- L_0

Theorem (G.–Lam)

Let $f \in S_n$. Then the point count of Π_f° is given by

$$\#\Pi_f^\circ(\mathbb{F}_q) = (q-1)^{n-1} \cdot (\textit{top a-degree coefficient of } P(L_f;a,q)).$$

$$T \text{ acts freely on } \Pi_f^\circ \iff f \text{ is a single cycle} \iff L_f \text{ is a knot.}$$

Khovanov–Rozansky link homology yields a polynomial $\mathcal{P}_{\mathrm{KR}}(L;a,q,t)$ which specializes to the HOMFLY polynomial $P(L;a,q)$.

Given a link L, the HOMFLY polynomial $P(L; a, q)$ is defined by

$$P(\bigcirc) = 1 \quad \text{and} \quad aP(L_+) - a^{-1}P(L_-) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) P(L_0), \quad \text{where}$$

L_+ L_- L_0

Theorem (G.–Lam)

Let $f \in S_n$. Then the point count of Π_f° is given by

$$\#\Pi_f^\circ(\mathbb{F}_q) = (q-1)^{n-1} \cdot (\textit{top } a\textit{-degree coefficient of } P(L_f; a, q)).$$

$$T \text{ acts freely on } \Pi_f^\circ \iff f \text{ is a single cycle} \iff L_f \text{ is a knot.}$$

Khovanov–Rozansky link homology yields a polynomial $\mathcal{P}_{\mathrm{KR}}(L; a, q, t)$ which specializes to the HOMFLY polynomial $P(L; a, q)$.

Theorem (G.–Lam)

Let $f \in S_n$ be a single cycle. Then

$$\mathcal{P}(\Pi_f^\circ / T; q, t) = \textit{top } a\textit{-degree coefficient of } \mathcal{P}_{\mathrm{KR}}(L_f; a, q, t).$$

Given a link L, the HOMFLY polynomial $P(L; a, q)$ is defined by

$$P(\bigcirc) = 1 \quad \text{and} \quad aP(L_+) - a^{-1}P(L_-) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) P(L_0), \quad \text{where}$$

Theorem (G.–Lam)

Let $f \in S_n$. Then the point count of Π_f° is given by

$$\#\Pi_f^\circ(\mathbb{F}_q) = (q-1)^{n-1} \cdot (\text{top } a\text{-degree coefficient of } P(L_f; a, q)).$$

T acts freely on $\Pi_f^\circ \iff f$ is a single cycle $\iff L_f$ is a knot.

Khovanov–Rozansky link homology yields a polynomial $\mathcal{P}_{\mathrm{KR}}(L; a, q, t)$ which specializes to the HOMFLY polynomial $P(L; a, q)$.

Theorem (G.–Lam)

Let $f \in S_n$ be a single cycle. Then

$$\mathcal{P}(\Pi_f^\circ/T; q, t) = \text{top } a\text{-degree coefficient of } \mathcal{P}_{\mathrm{KR}}(L_f; a, q, t).$$

Arbitrary $f \in S_n$: LHS $=$ T-equivariant cohomology of Π_f° with compact support.

Cluster structure on positroid varieties

Theorem (G.–Lam)

For each $f \in S_n$, the coordinate ring $\mathbb{C}[\Pi_f^\circ]$ is a cluster algebra.

Cluster structure on positroid varieties

Theorem (G.–Lam)

For each $f \in S_n$, the coordinate ring $\mathbb{C}[\Pi_f^\circ]$ is a cluster algebra.

[Fomin–Zelevinsky (2002)], [Scott (2006)], [Muller–Speyer (2014)], [Leclerc (2014)], [Serhiyenko–Sherman-Bennett–Williams (2019)]

Cluster structure on positroid varieties

Theorem (G.–Lam)

For each $f \in S_n$, the coordinate ring $\mathbb{C}[\Pi_f^\circ]$ is a cluster algebra.

[Fomin–Zelevinsky (2002)], [Scott (2006)], [Muller–Speyer (2014)], [Leclerc (2014)], [Serhiyenko–Sherman-Bennett–Williams (2019)]

Theorem (Lam–Speyer (2016))

$\mathbb{C}[X]$ *is a cluster algebra* $\Longrightarrow \mathcal{P}(X; q, t)$ *is q, t-symmetric and unimodal.*

Cluster structure on positroid varieties

Theorem (G.–Lam)

For each $f \in S_n$, the coordinate ring $\mathbb{C}[\Pi_f^\circ]$ is a cluster algebra.

[Fomin–Zelevinsky (2002)], [Scott (2006)], [Muller–Speyer (2014)], [Leclerc (2014)], [Serhiyenko–Sherman-Bennett–Williams (2019)]

Theorem (Lam–Speyer (2016))

$\mathbb{C}[X]$ *is a cluster algebra* $\Longrightarrow \mathcal{P}(X; q, t)$ *is* q, t*-symmetric and unimodal.*

Corollary (G.–Lam)

For $f \in S_n$, $\mathcal{P}(\Pi_f^\circ; q, t)$ is q, t-symmetric and unimodal.

Cluster structure on positroid varieties

Theorem (G.–Lam)

For each $f \in S_n$, the coordinate ring $\mathbb{C}[\Pi_f^\circ]$ is a cluster algebra.

[Fomin–Zelevinsky (2002)], [Scott (2006)], [Muller–Speyer (2014)], [Leclerc (2014)], [Serhiyenko–Sherman-Bennett–Williams (2019)]

Theorem (Lam–Speyer (2016))

$\mathbb{C}[X]$ *is a cluster algebra* $\Longrightarrow \mathcal{P}(X; q, t)$ *is* q, t*-symmetric and unimodal.*

Corollary (G.–Lam)

For $f \in S_n$, $\mathcal{P}(\Pi_f^\circ; q, t)$ is q, t-symmetric and unimodal.
In particular, for $\gcd(k, n) = 1$, the q, t-Catalan numbers $C_{k,n-k}(q, t)$ are q, t-symmetric and unimodal.

Cluster structure on positroid varieties

Theorem (G.–Lam)

For each $f \in S_n$, the coordinate ring $\mathbb{C}[\Pi_f^\circ]$ is a cluster algebra.

[Fomin–Zelevinsky (2002)], [Scott (2006)], [Muller–Speyer (2014)], [Leclerc (2014)], [Serhiyenko–Sherman-Bennett–Williams (2019)]

Theorem (Lam–Speyer (2016))

$\mathbb{C}[X]$ *is a cluster algebra* $\Longrightarrow \mathcal{P}(X; q, t)$ *is* q, t*-symmetric and unimodal.*

Corollary (G.–Lam)

For $f \in S_n$, $\mathcal{P}(\Pi_f^\circ; q, t)$ is q, t-symmetric and unimodal.
In particular, for $\gcd(k, n) = 1$, the q, t-Catalan numbers $C_{k,n-k}(q, t)$ are q, t-symmetric and unimodal.

Thanks!