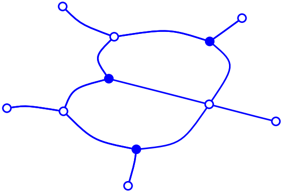
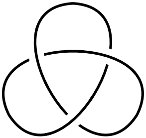
Positroids, knots, and q, t-Catalan numbers

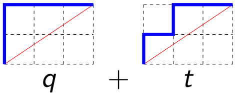
Pavel Galashin (UCLA)

Duke University Mathematics Department Colloquium
December 13, 2022

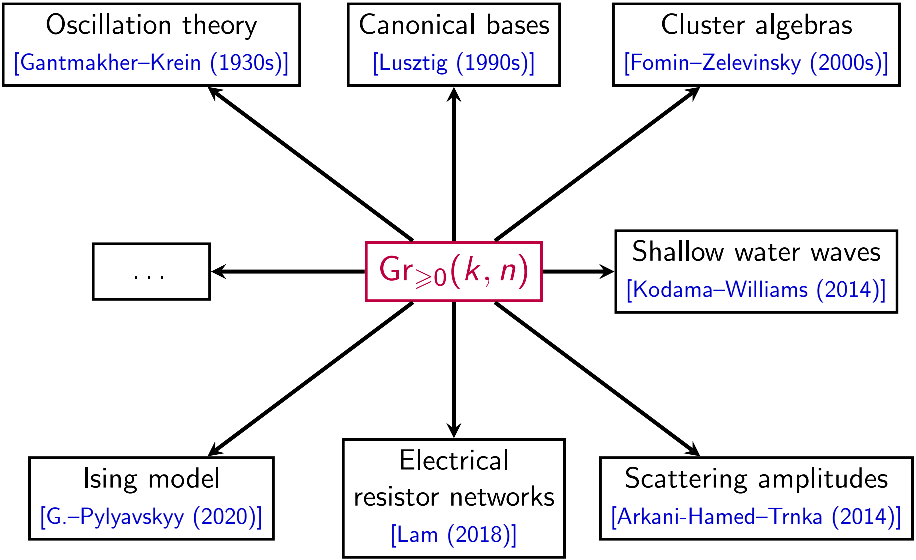
Joint work with Thomas Lam







Motivation: total positivity



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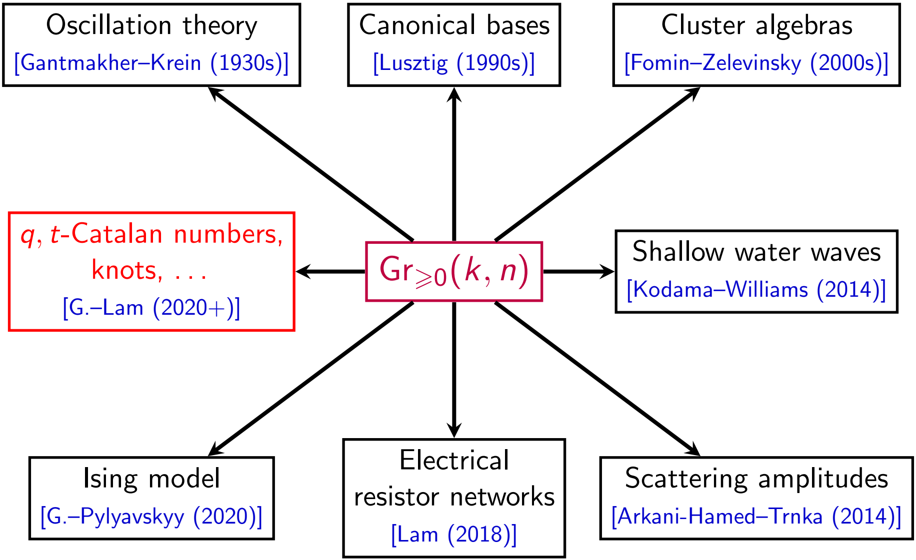
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This is a horrible stratification. [Mnëv (1988)]

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[Pos06] A. Postnikov. Total positivity, Grassmannians, and networks. arXiv:math/0609764.

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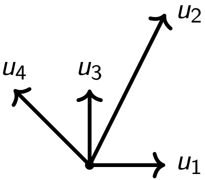

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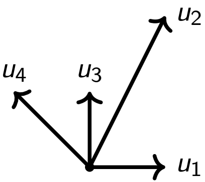

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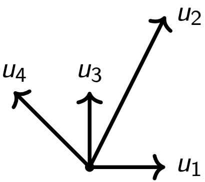

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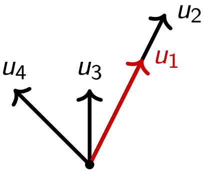

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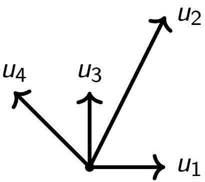

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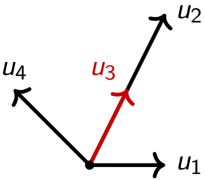

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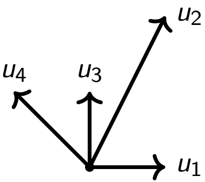

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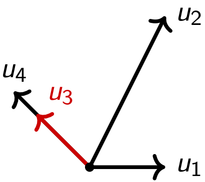

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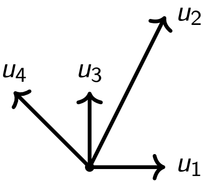

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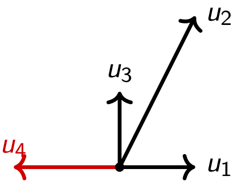

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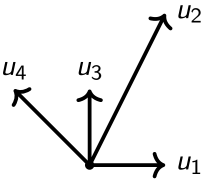
Codimension 1 cells: $\Delta_{12} = 0$, $\Delta_{23} = 0$, $\Delta_{34} = 0$, $\Delta_{14} = 0$.

Example: $\mathrm{Gr}_{\geqslant 0}(2,4)$

$$\mathrm{RowSpan}\begin{array}{c} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \\ \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \end{array} \in \mathrm{Gr}_{\geqslant 0}(2,4)$$



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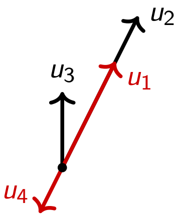

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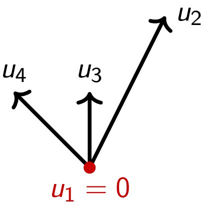

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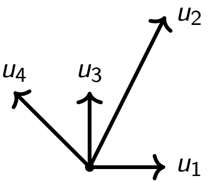

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Question

What is the topology of $\mathrm{Gr}_{\geqslant 0}(k,n)$ and of its boundary cells?

Topology of $\mathrm{Gr}_{\geqslant 0}(k, n)$

Theorem (Postnikov (2006))

Each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

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Theorem (G.–Karp–Lam)

[GKL1] $\mathrm{Gr}_{\geqslant 0}(k,n)$ *is homeomorphic to a closed ball.*

[GKL2] *The closure of each cell is homeomorphic to a ball.*

[GKL1] P. Galashin, S. Karp, and T. Lam. The totally nonnegative Grassmannian is a ball. *Adv. Math.*, 397: 108123, 2022.

[GKL2] P. Galashin, S. Karp, and T. Lam. Regularity theorem for totally nonnegative flag varieties. *J. Amer. Math. Soc.*, 35(2):513–579, 2021.

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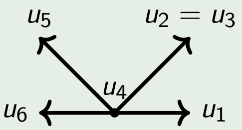
Example

$$\begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \\ \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \end{array} \quad \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \\ \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \end{array}$$



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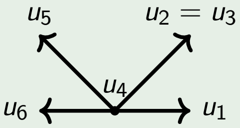

$$f_V = \begin{pmatrix} 1 & & & & & \\ 5 & & & & & \end{pmatrix}.$$

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Example

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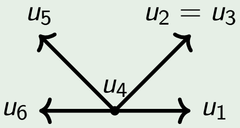

$$f_V = \begin{pmatrix} 1 & 2 & & & & \\ 5 & 3 & & & & \end{pmatrix}.$$

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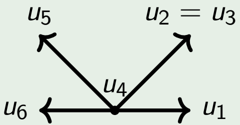

$$f_V = \begin{pmatrix} 1 & 2 & 3 & & & \\ 5 & 3 & 6 & & & \end{pmatrix}.$$

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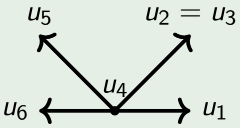
Example

$$\begin{array}{cccccc} \multicolumn{6}{c}{\begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \qquad \begin{array}{cccccc} \multicolumn{6}{c}{\begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array}$$



$$f_V = \begin{pmatrix} 1 & 2 & 3 & 4 & & \\ 5 & 3 & 6 & 4 & & \end{pmatrix}.$$

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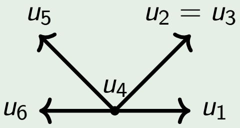

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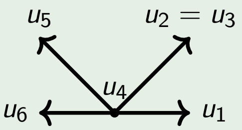

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Example (Generic case)

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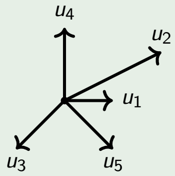

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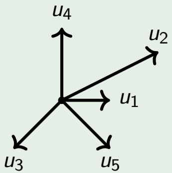

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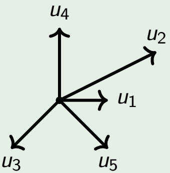

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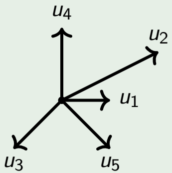

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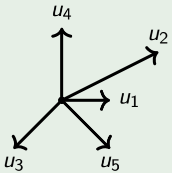

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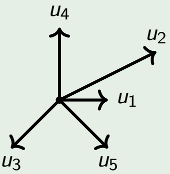

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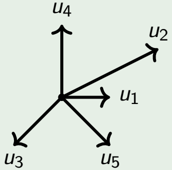

Let $f_{k,n}(i) \equiv i + k \pmod{n}$ for all $i = 1, 2, \dots, n$. Then

$$f_V = f_{k,n} \iff \Delta_{1,\dots,k}, \Delta_{2,\dots,k+1} \dots, \Delta_{n,1,\dots,k-1} \neq 0.$$

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- Both are defined for $\mathrm{Gr}(k,n;\mathbb{F})$ over arbitrary field $\mathbb{F}$.

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Question

What can we say about Π_f° over other fields $\mathbb{F}$?

Given a variety

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Given a variety [of “Hodge–Tate type”]

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Most interesting case: $\mathcal{P}(X; q, t) \in \mathbb{N}[q, t]$ (i.e., odd cohomology vanishes).

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Step 2. Compute Poincaré polynomial $\mathcal{P}(X(\mathbb{C}); t) := \sum_i t^{\frac{i}{2}} \dim H^i(X(\mathbb{C}))$.

Step 3. Compute the mixed Hodge polynomial $\mathcal{P}(X; q, t)$

[Deligne splitting / weight filtration $\to$ canonical second grading on $H^*(X)$]

$$\begin{tikzcd}
& \boxed{\text{Mixed Hodge polynomial } \mathcal{P}(X; q, t) \in \mathbb{N}\left[q^{\frac{1}{2}}, t^{\frac{1}{2}}\right]} \arrow[dl, "t^{\frac{1}{2}} = -q^{-\frac{1}{2}}"'] \arrow[dr, "q^{\frac{1}{2}} = 1"] & \\
\boxed{\text{Point count } \#X(\mathbb{F}_q)} \arrow[dr, "q = 1"'] & & \boxed{\text{Poincaré polynomial } \mathcal{P}(X(\mathbb{C}); t)} \arrow[dl, "t^{\frac{1}{2}} = -1"] \\
& \boxed{\text{Euler characteristic}} &
\end{tikzcd}$$

Most interesting case: $\mathcal{P}(X; q, t) \in \mathbb{N}[q, t]$ (i.e., odd cohomology vanishes).

Question: Which variety should we choose?

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{W \subseteq \mathbb{F}^n \mid \dim(W) = k\} = \frac{\{k \times n \text{ matrices of rank } k\}}{(\text{row operations})}.$$

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<u>Example:</u> $$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[1]_q[2]_q[3]_q[4]_q}{[1]_q[2]_q \cdot [1]_q[2]_q} = q^4 + q^3 + 2q^2 + q + 1.$$

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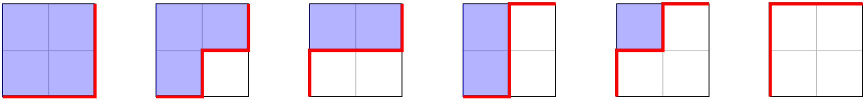

$$q^4 \quad + \quad q^3 \quad + \quad q^2 \quad + \quad q^2 \quad + \quad q^1 \quad + \quad q^0$$

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- Reason: Schubert decomposition.

- Rational Catalan numbers: for $1 \leqslant k \leqslant n$ such that $\gcd(k,n)=1$, let

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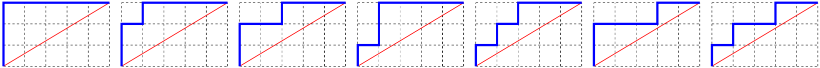
Example: $k = 3,\ n = 8,\ C_{k,n-k} = 7$:



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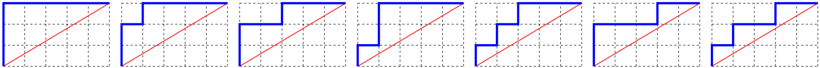

$$\boxed{\color{red}{C'_{k,n-k}(q)} := \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q}$$

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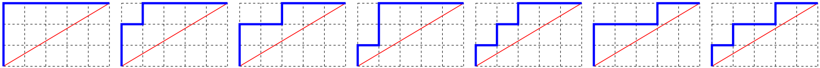

$C'_{k,n-k}(q) = q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1.$

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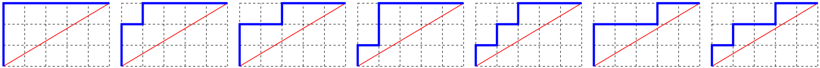

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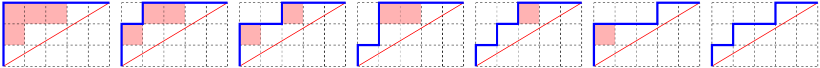

$C''_{k,n-k}(t) \;=\; t^4 + t^3 + t^2 + t^2 + t^1 + t^1 + t^0.$

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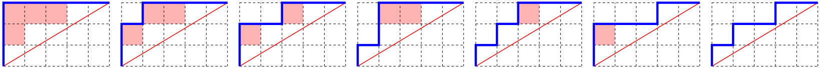

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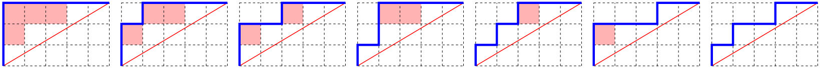

$$C_{k,n-k}(q,t) = t^4q^0 + t^3q^1 + t^2q^2 + t^2q^1 + t^1q^3 + t^1q^2 + t^0q^4.$$

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Definition (G.–Lam)

Let $\gcd(k, n) = 1$. The Catalan variety is given by

$$X^{\circ}_{k,n} := \{V \in \mathrm{Gr}(k, n) \mid \Delta_{1,\dots,k}(V) = \Delta_{2,\dots,k+1}(V) = \cdots = \Delta_{n,1,\dots,k-1}(V) = 1\}.$$

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Example:

$$X^\circ_{2,5} = \left\{ \mathrm{RowSpan} \begin{pmatrix} 1 & 0 & a & b & c \\ 0 & 1 & d & e & f \end{pmatrix} \middle| \begin{array}{cc} -a = 1, & ae - bd = 1, \\ f = 1, & bf - ce = 1 \end{array} \right\}.$$

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& \boxed{\textit{Mixed Hodge polynomial } \mathcal{P}(X^\circ_{k,n}; q,t) = C_{k,n-k}(q,t)} \arrow[dl, "t^{\frac{1}{2}} = -q^{-\frac{1}{2}}"'] \arrow[dr, "q^{\frac{1}{2}} = 1"] & \\
\boxed{\textit{Point count } \#X^\circ_{k,n}(\mathbb{F}_q) = C'_{k,n-k}(q)} \arrow[dr, "q = 1"'] & & \boxed{\textit{Poincaré polynomial } \mathcal{P}(X^\circ_{k,n}(\mathbb{C}); t) = C''_{k,n-k}(t)} \arrow[dl, "t^{\frac{1}{2}} = -1"] \\
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$$\#X^\circ_{2,5}(\mathbb{F}_q) = q^2 + 1, \quad \mathcal{P}(X^\circ_{2,5}(\mathbb{C}); t) = 1 + t, \quad \mathcal{P}(X^\circ_{2,5}; q, t) = q + t.$$

Theorem (G.–Lam)

$$\begin{tikzcd}
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\end{tikzcd}$$

Definition

Let $\gcd(k,n)=1$. Recall:

$\Pi^{\circ}_{k,n} := \{V \in \mathrm{Gr}(k,n) \mid \Delta_{1,\dots,k}(V), \Delta_{2,\dots,k+1}(V), \cdots, \Delta_{n,1,\dots,k-1}(V) \neq 0\}$.

Example:

$$\Pi^{\circ}_{2,5} = \left\{ \mathrm{RowSpan}\begin{pmatrix} 1 & 0 & a & b & c \\ 0 & 1 & d & e & f \end{pmatrix} \;\middle|\; \begin{matrix} -a \neq 0, & ae-bd \neq 0, \\ f \neq 0, & bf-ce \neq 0 \end{matrix} \right\}.$$

Theorem (G.–Lam)

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& \boxed{\mathcal{P}(\Pi^{\circ}_{k,n}; q,t) = (q^{\frac{1}{2}} + t^{\frac{1}{2}})^{n-1} C_{k,n-k}(q,t)} \arrow[dl, "t^{\frac{1}{2}} = -q^{-\frac{1}{2}}"'] \arrow[dr, "q^{\frac{1}{2}} = 1"] & \\
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& \boxed{\textit{Euler characteristic} = 0} &
\end{tikzcd}$$

Summary so far

- $V \in \mathrm{Gr}(k, n; \mathbb{F}) \longrightarrow$ Permutation $f_V : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.
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What can we say about Π_f° over other fields $\mathbb{F}$?

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What about arbitrary Π_f°?

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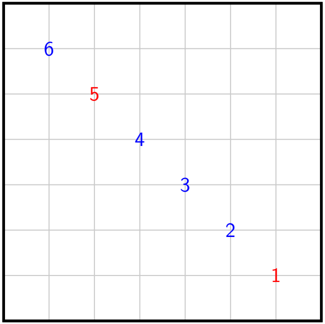

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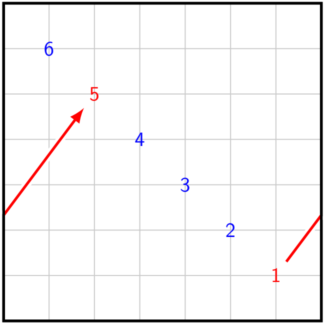

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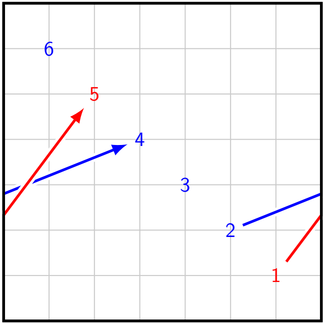

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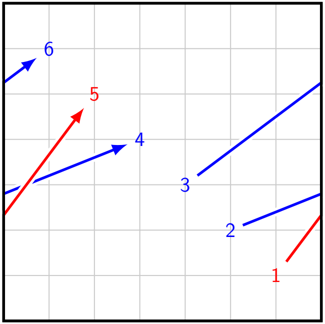

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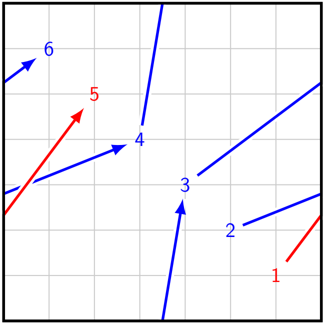

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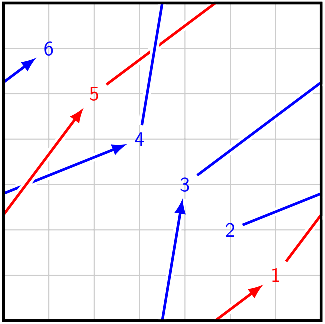

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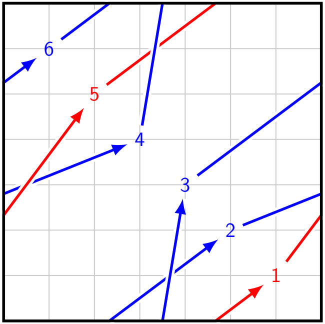

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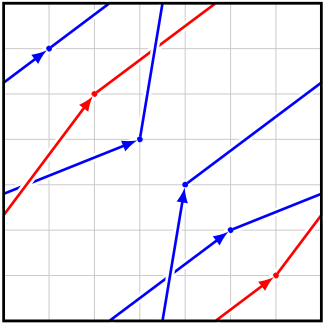

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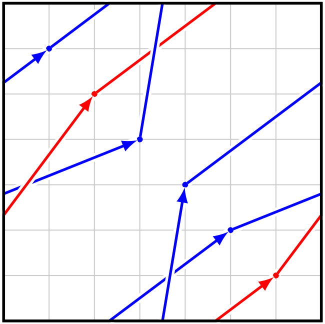

This construction: [G.–Lam '22+]. Related constructions: [G.–Lam '20], [Shende–Treumann–Williams–Zaslow '15], [Fomin–Pylyavskyy–Shustin–Thurston '17], [Casals–Gorsky–Gorsky–Simental '21]

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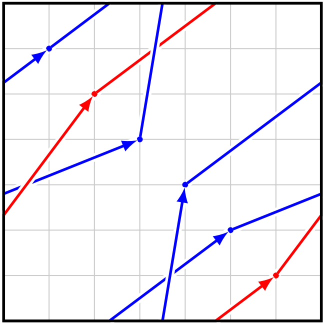
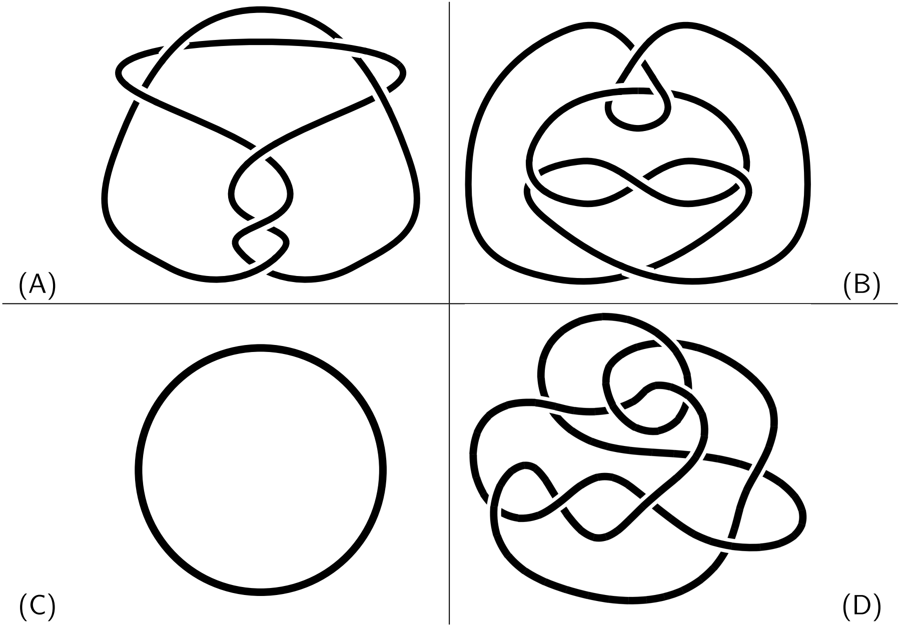


Conclusion

For each permutation $f \in S_n$, get a variety Π_f° and a link $L_f \subseteq \mathbb{R}^3$.

How to tell if two knots/links are isotopic?

One of these knots is not like the others



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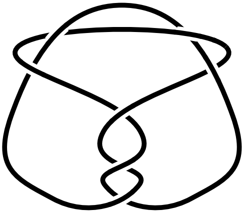
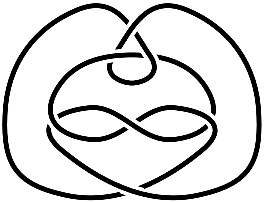
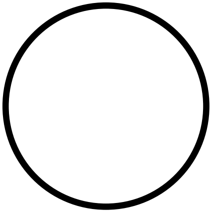


(A) $P(L;a,q) = \frac{q^4-q^3+q^2-q+1}{a^4q^2} + \frac{q^4-q^3+2q^2-q+1}{a^6q^2} - \frac{q^2+1}{a^8q}$



(B) $P(L;a,q) = 1$



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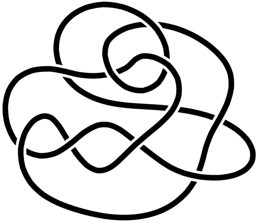

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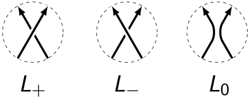

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In this case, Π_f°/T is smooth and $\mathcal{P}(\Pi_f^\circ;q,t)=\left(q^{\frac{1}{2}}+t^{\frac{1}{2}}\right)^{n-1}\cdot\mathcal{P}(\Pi_f^\circ/T;q,t)$.

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Arbitrary $f \in S_n$: LHS = T-equivariant cohomology of Π_f° with compact support.

Thanks!

Theorem (G.–Lam)

Let $f \in S_n$. Then the point count of Π_f° is given by

$$\#\Pi_f^\circ(\mathbb{F}_q) = (q-1)^{n-1} \cdot (\textit{top } a\textit{-degree coefficient of } P(L_f; a, q)).$$

T acts freely on $\Pi_f^\circ \iff f$ is a single cycle $\iff L_f$ is a knot.

Khovanov–Rozansky link homology yields a polynomial $\mathcal{P}_{\mathrm{KR}}(L; a, q, t)$ which specializes to the HOMFLY polynomial $P(L; a, q)$.

Theorem (G.–Lam)

Let $f \in S_n$ be a single cycle. Then

$$\mathcal{P}(\Pi_f^\circ / T; q, t) = \textit{top } a\textit{-degree coefficient of } \mathcal{P}_{\mathrm{KR}}(L_f; a, q, t).$$

Arbitrary $f \in S_n$: LHS $=$ T-equivariant cohomology of Π_f° with compact support.