Richardson varieties, cluster algebras, and knot homology

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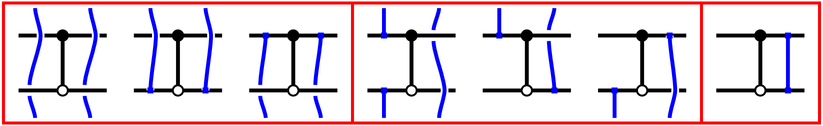
Joint work with T. Lam, M. Sherman-Bennett, and D. Speyer.



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Question: Which variety should we choose?

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- Euler characteristic: $\binom{n}{k}$.

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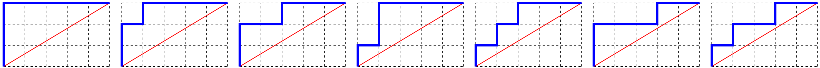

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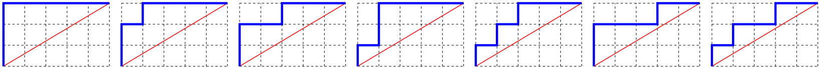

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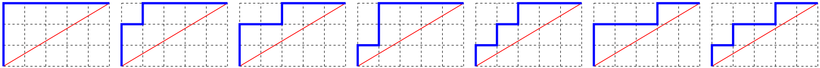

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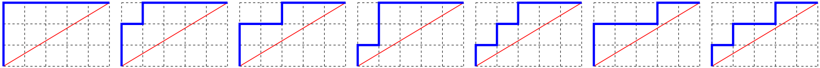

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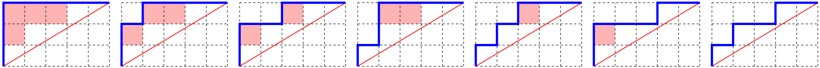

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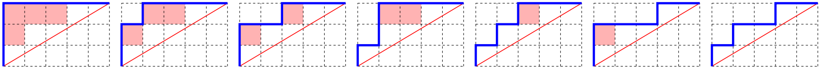

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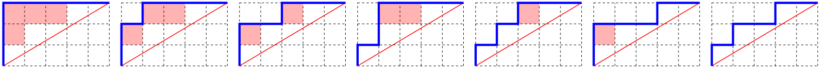

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For I of size k, let $\Delta_I(V)$ be the maximal minor of V with column set I.

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$$X^\circ_{2,5} = \left\{ \mathrm{RowSpan}\begin{pmatrix} 1 & 0 & a & b & c \\ 0 & 1 & d & e & f \end{pmatrix} \,\middle|\, \begin{array}{ll} -a = 1, & ae - bd = 1, \\ f = 1, & bf - ce = 1 \end{array} \right\}.$$

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Theorem (G.–Lam (2020))

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 & \boxed{\textit{Mixed Hodge polynomial } \mathcal{P}(X^\circ_{k,n}; q,t) = C_{k,n-k}(q,t)} \arrow[dl, "t^{\frac12} = -q^{-\frac12}"'] \arrow[dr, "q^{\frac12} = 1"] & \\
\boxed{\textit{Point count } \#X^\circ_{k,n}(\mathbb{F}_q) = C'_{k,n-k}(q)} \arrow[dr, "q = 1"'] & & \boxed{\textit{Poincaré polynomial } \mathcal{P}(X^\circ_{k,n}(\mathbb{C}); t) = C''_{k,n-k}(t)} \arrow[dl, "t^{\frac12} = -1"] \\
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General picture

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- Open positroid varieties inside $\mathrm{Gr}(k,n)$ are special cases of open Richardson varieties inside the flag variety SL_n / B.

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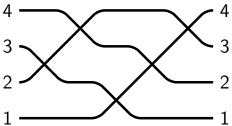
Theorem (G.–Lam (2020))

If $c(wv^{-1}) = 1$ *then*

$$\mathcal{P}(R^\circ_{v,w}/T; q, t) = \textcolor{red}{???}$$

- Given $u \in S_n$, choose a reduced word $u = s_{i_1} s_{i_2} \cdots s_{i_\ell}$.



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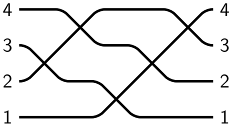
- Make each crossing into a positive braid crossing, get braid $\beta(u)$.



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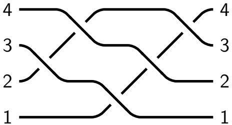

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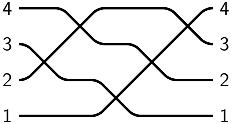

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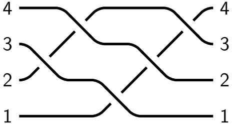

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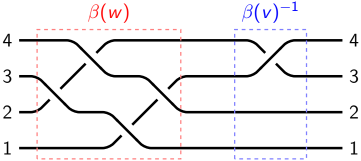

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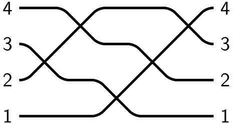

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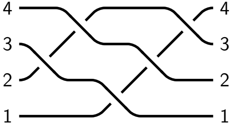

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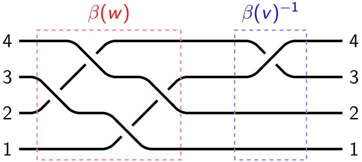

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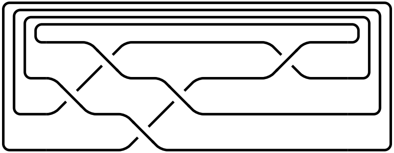

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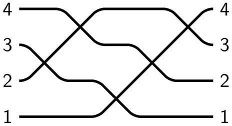
- When $c(wv^{-1}) = 1$, $\hat{\beta}_{v,w}$ is a knot, i.e., has a unique connected component.



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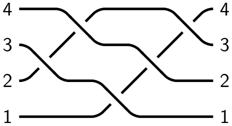

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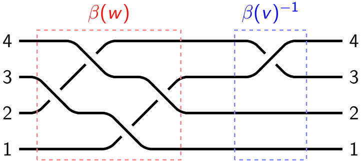

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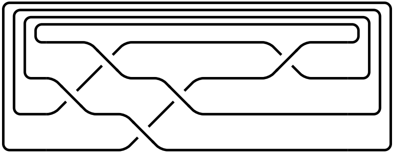

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Arbitrary $v \leqslant w$: LHS $=$ T-equivariant cohomology of $R^\circ_{v,w}$ with compact support.

What does it all have to do with cluster algebras?

Cluster algebras

[Fomin–Zelevinsky '02]

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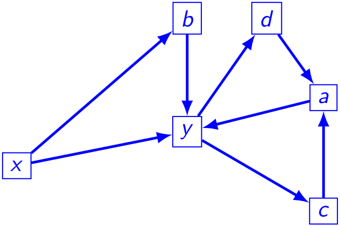
- Start with a quiver Q with some cluster variables labeling its vertices.



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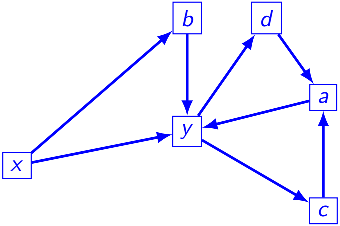

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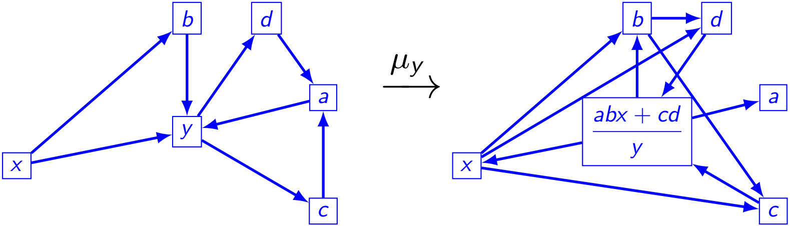

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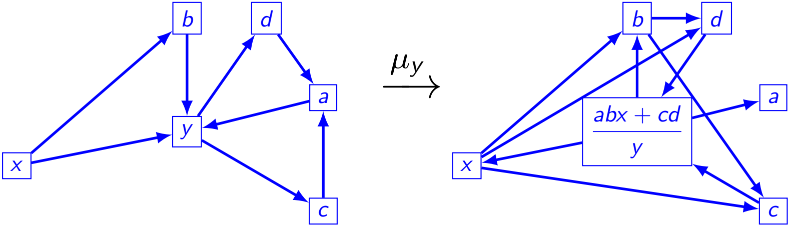

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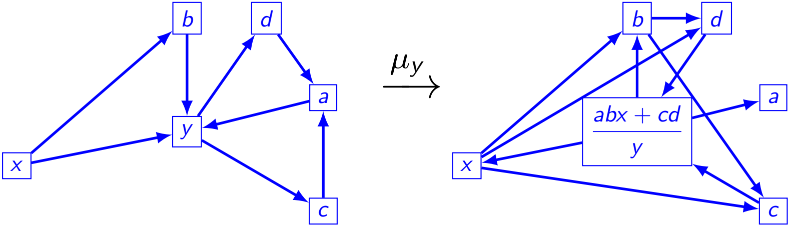

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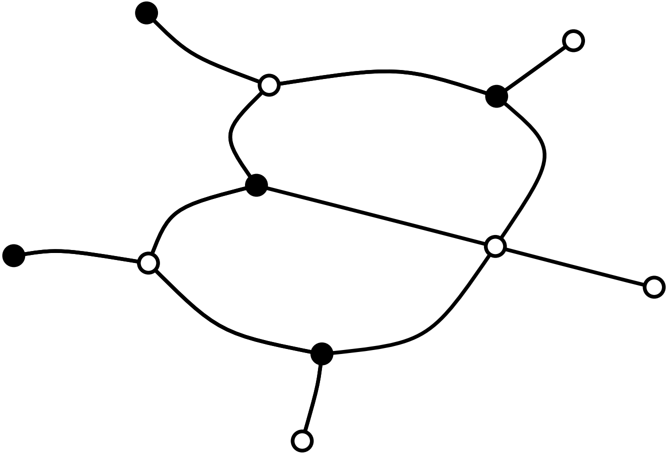

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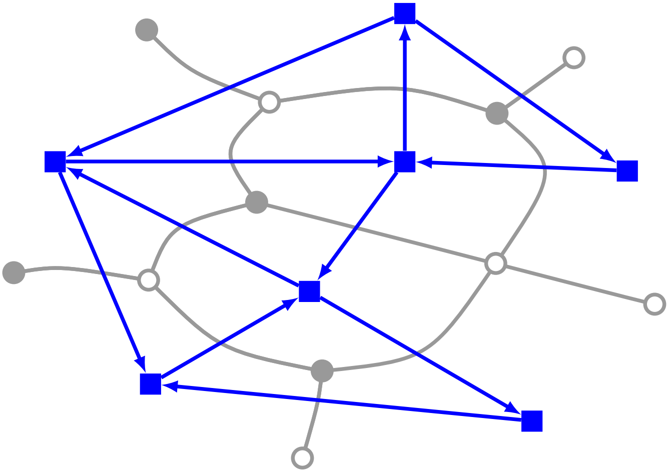

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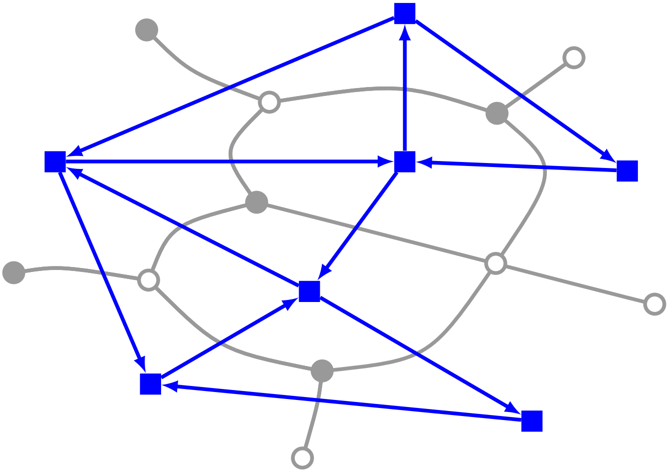

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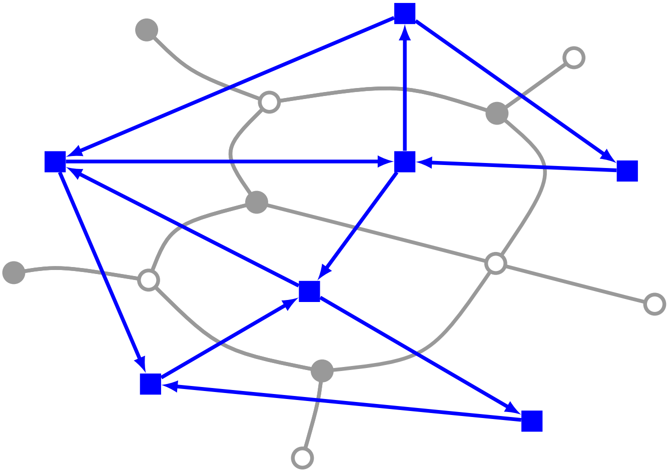

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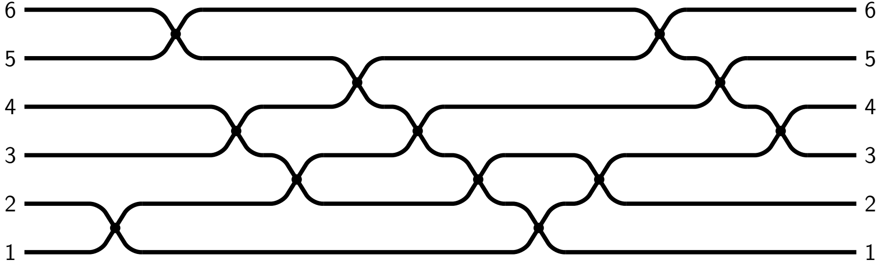

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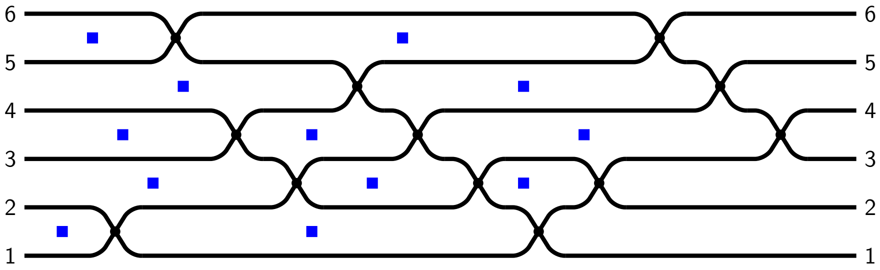

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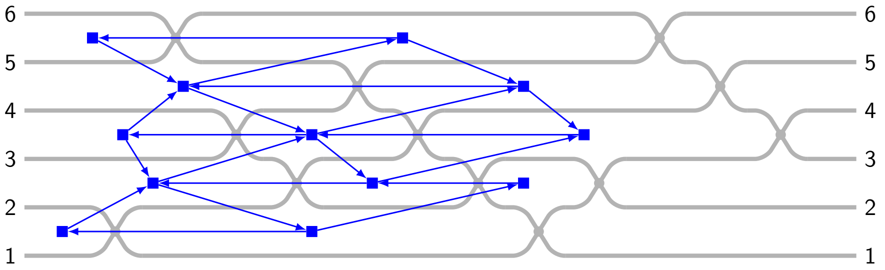

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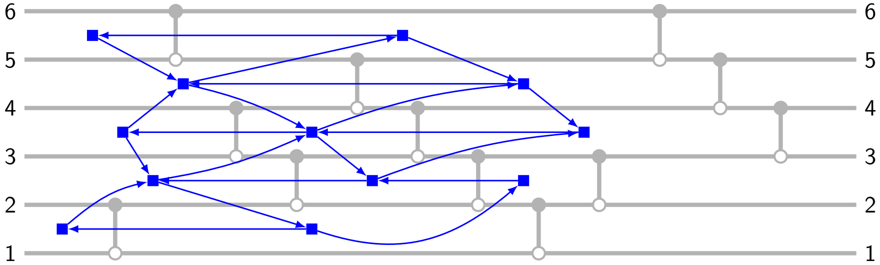

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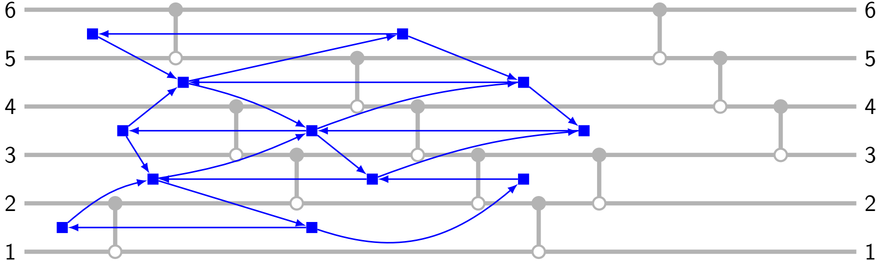

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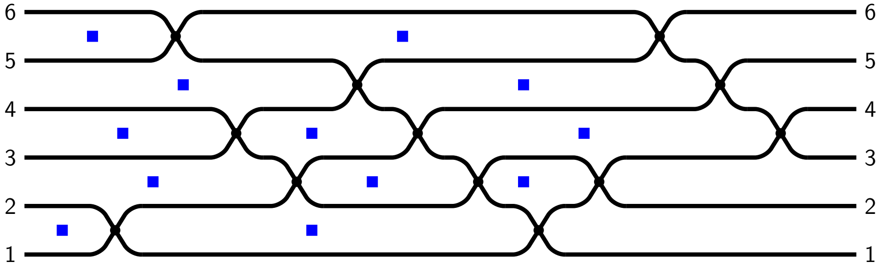

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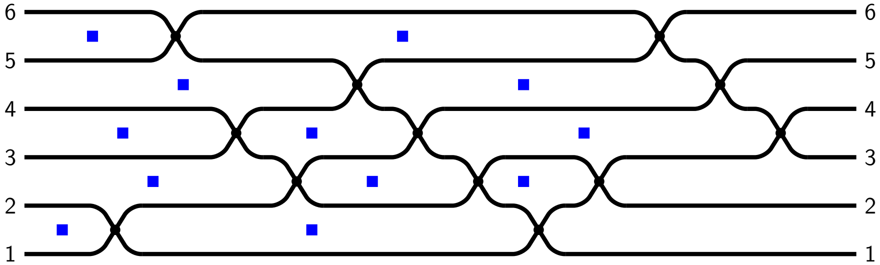

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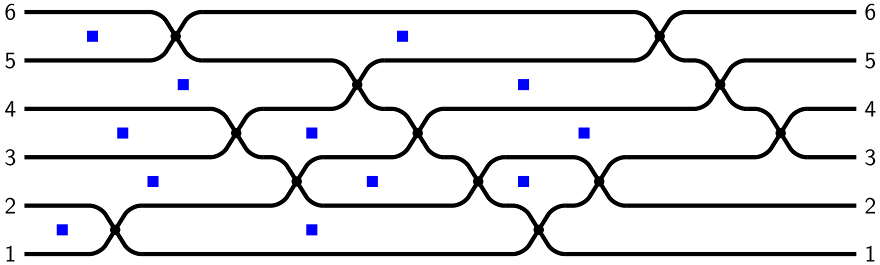

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Consider $R^\circ_{v,w}$ for arbitrary $v \leqslant w$. We have $\dim R^\circ_{v,w} = \ell(w) - \ell(v)$.

- Associate a module over the A_n preprojective algebra to each chamber.
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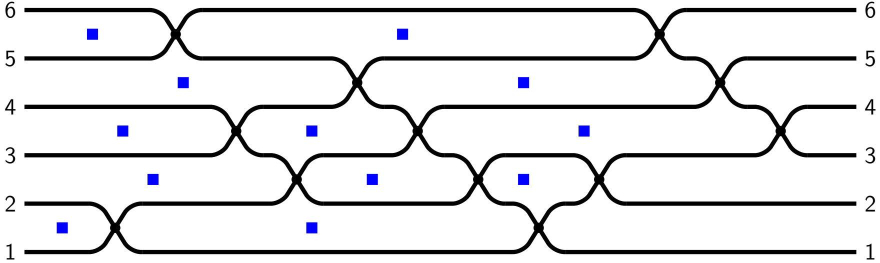

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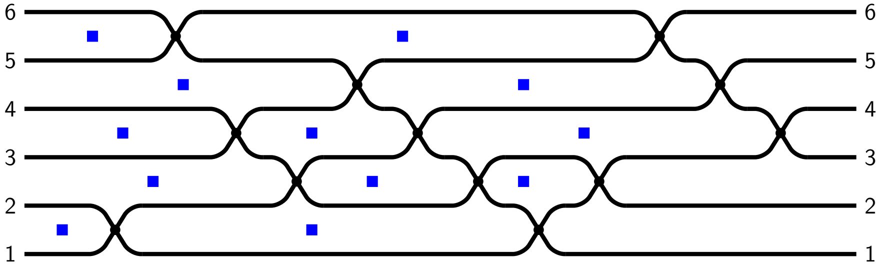

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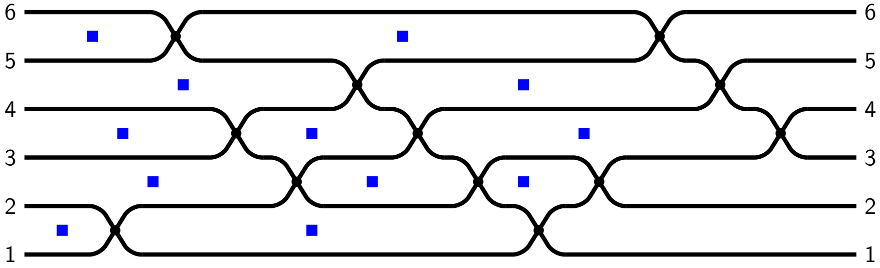

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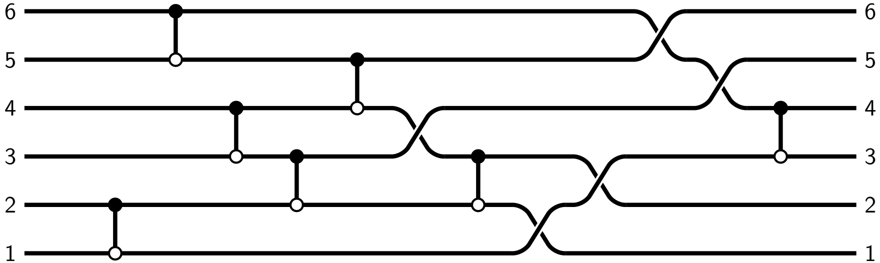

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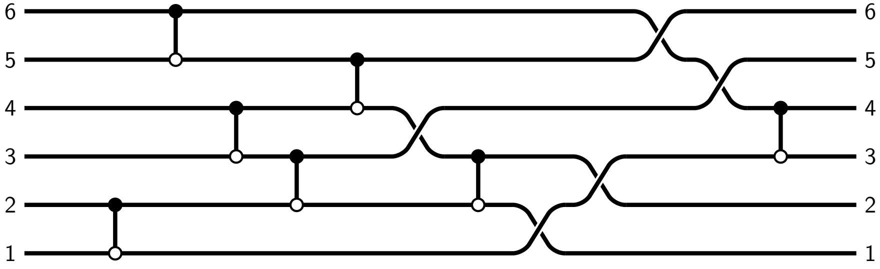

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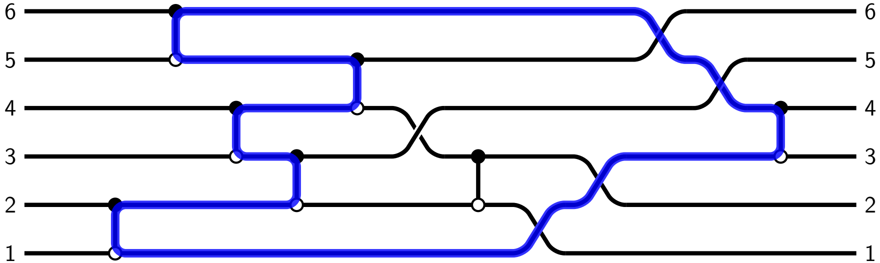

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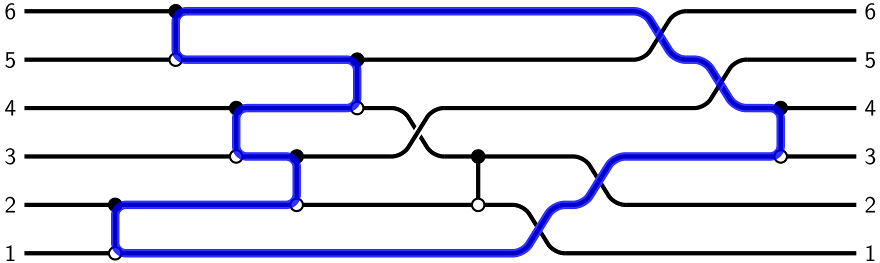

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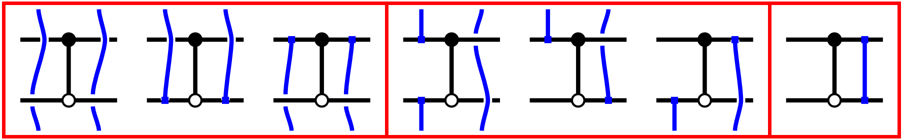

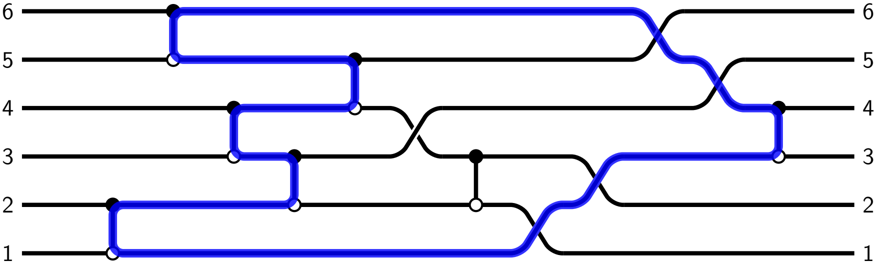

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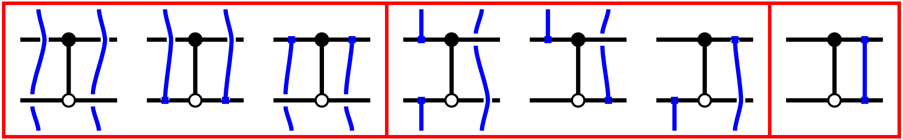

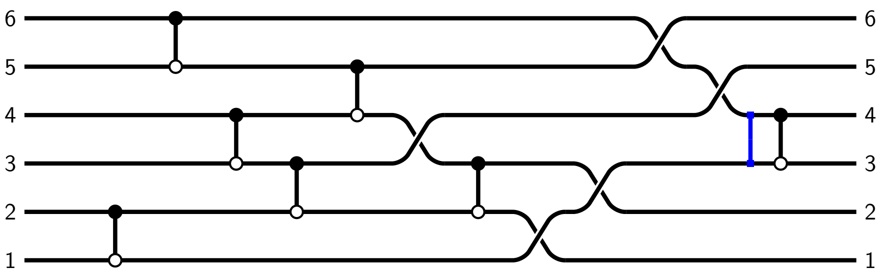

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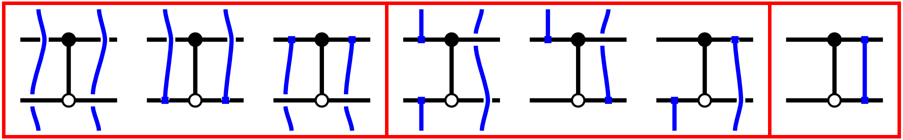

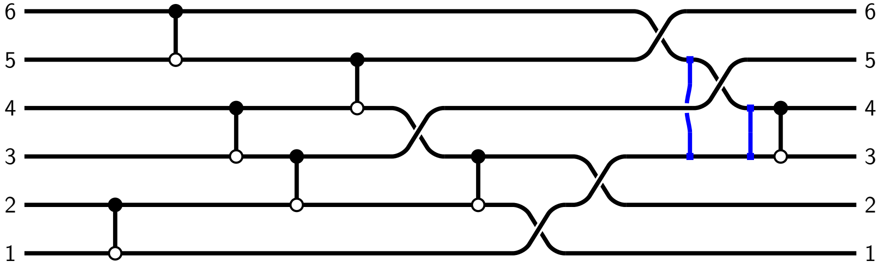

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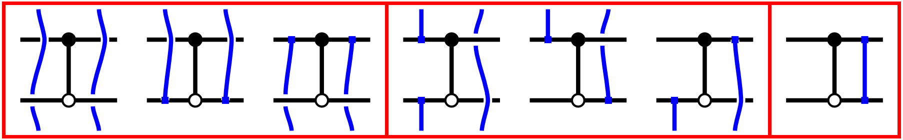

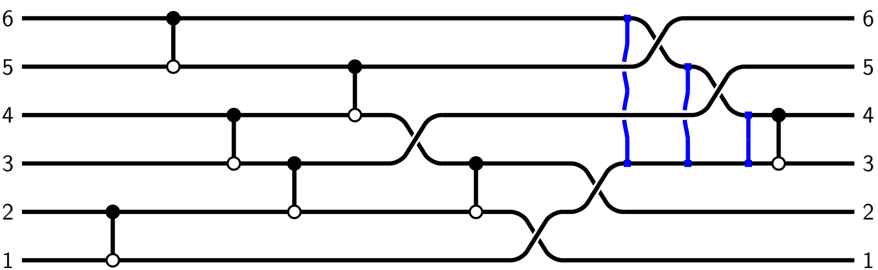

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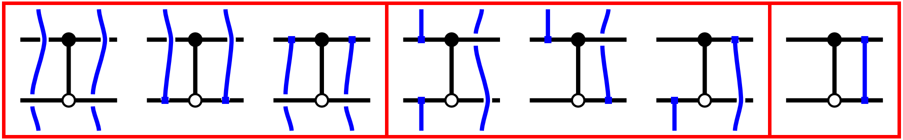

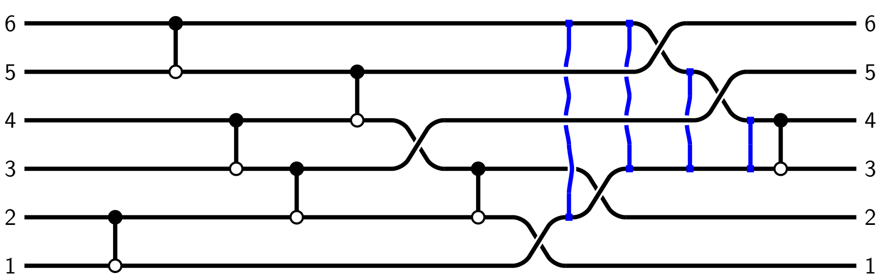

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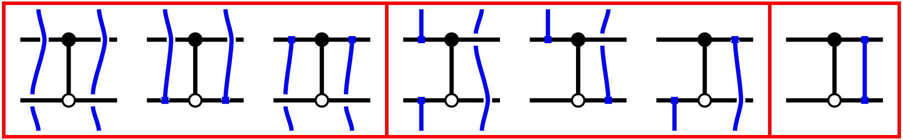

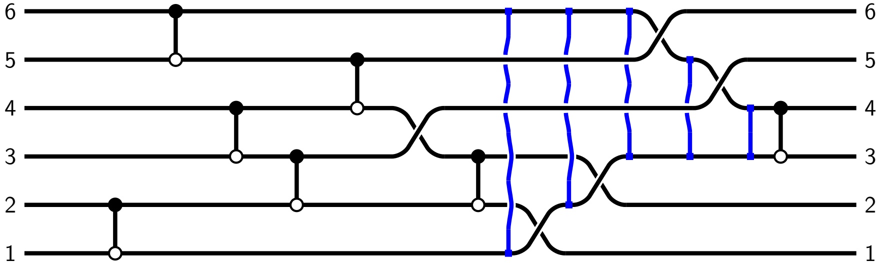

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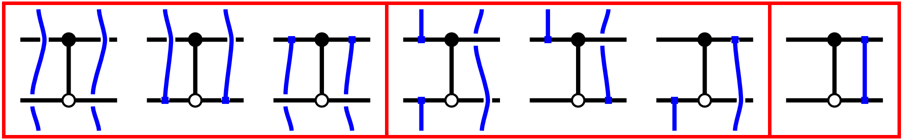

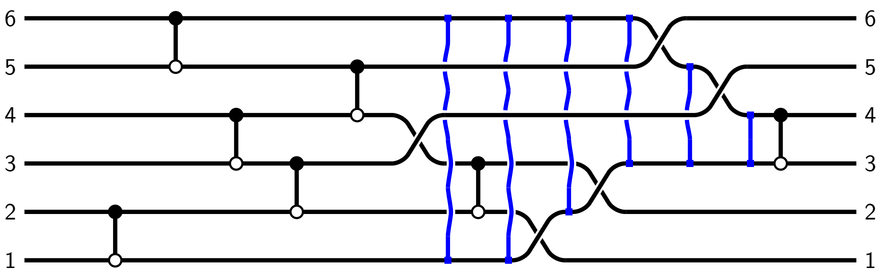

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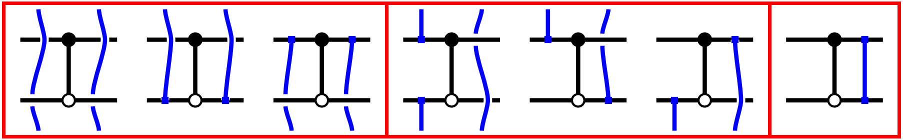

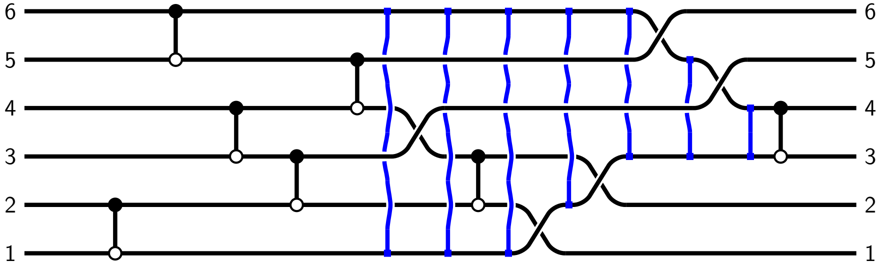

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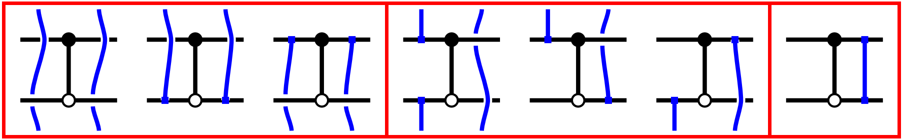

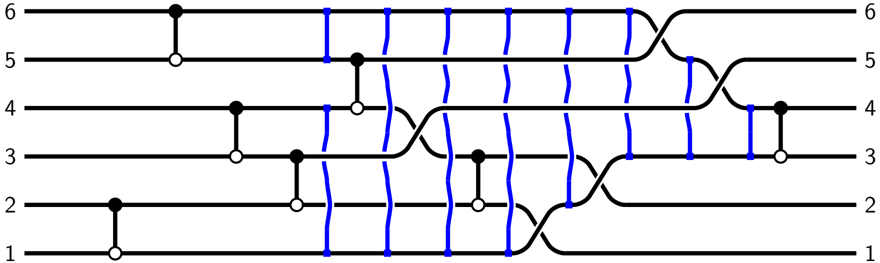

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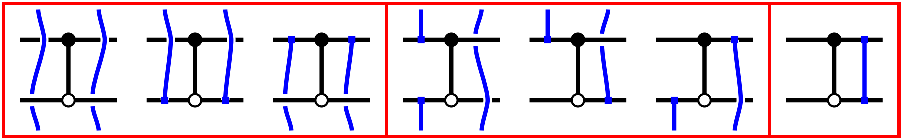

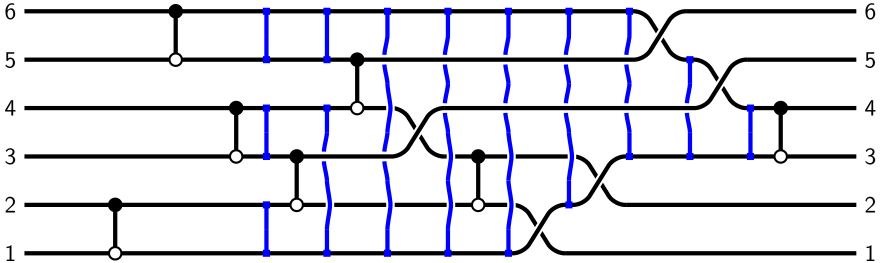

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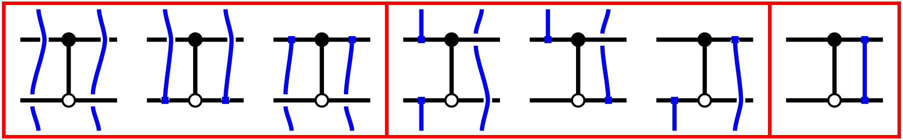

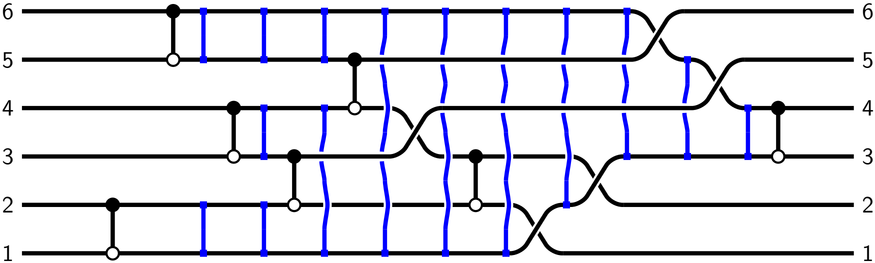

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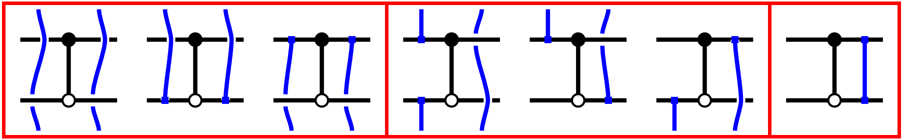

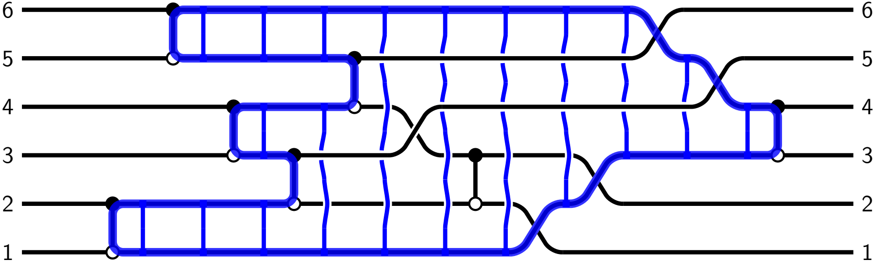

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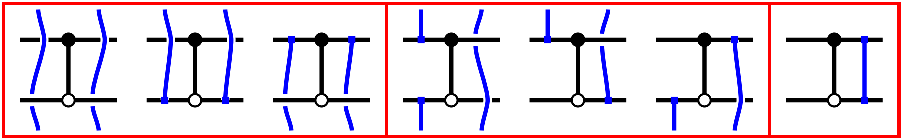

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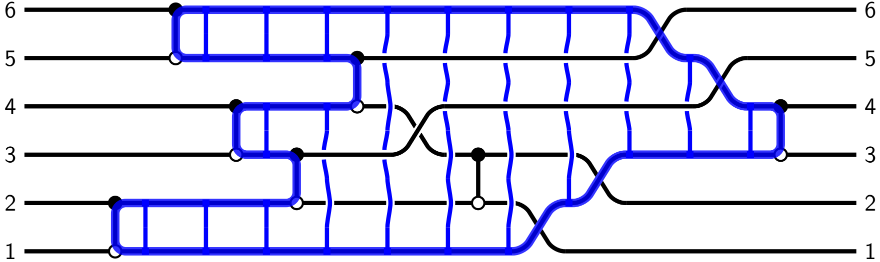

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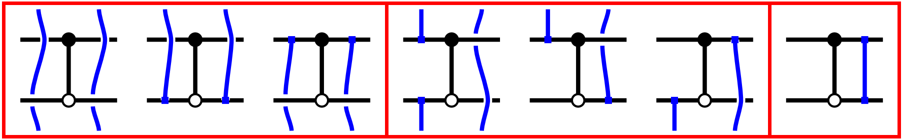

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