Positroid Catalan numbers

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Joint work with Thomas Lam (arXiv:2012.09745, arXiv:2104.05701)

Motivation: Geometry

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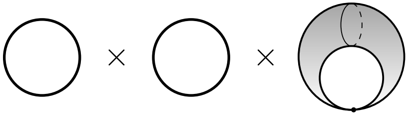
$\Pi^\circ_{2,4}(\mathbb{C}) \cong$ 

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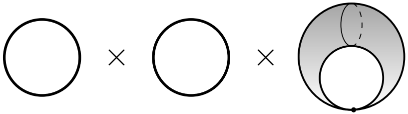
$\Pi^\circ_{2,4}(\mathbb{C}) \cong$ 

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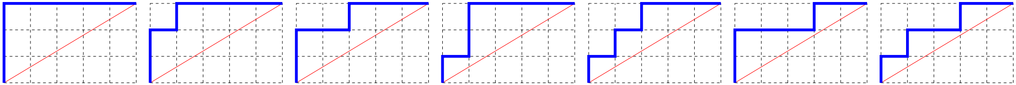
- Counts the number of Dyck paths inside an $a \times b$ rectangle. E.g. $C_{3,5} = 7$:



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- $f\in S_n$ is a single cycle $\Longrightarrow \mathcal{P}(\Pi^\circ_f;q,t)$ is divisible by $\left(q^{\frac{1}{2}}+t^{\frac{1}{2}}\right)^{n-1}$.

Theorem (G.–Lam (2020))

Let $\gcd(k,n) = 1$*. Then the bigraded Poincaré polynomial of* $\Pi^\circ_{k,n}$ *is given by*

$$\mathcal{P}(\Pi^\circ_{k,n}; q, t) = \left(q^{\frac{1}{2}} + t^{\frac{1}{2}}\right)^{n-1} C_{k,n-k}(q,t).$$

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- *Both* $\mathcal{P}(\Pi^\circ_f;q,t)$ *and* $C_f(q,t)$ *are* q,t*-symmetric and unimodal.*

The combinatorics

Assume that $f \in S_n$ is a single cycle.

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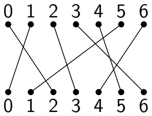

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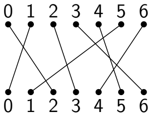

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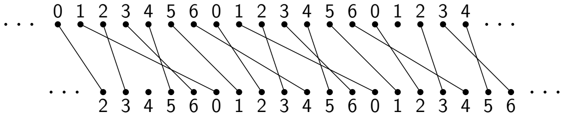

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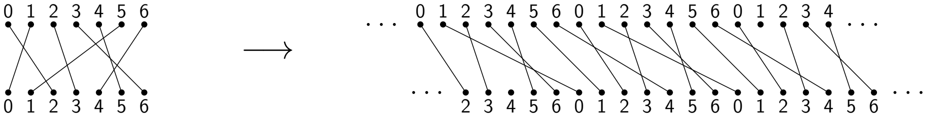

permutation $f \in S_n$ $\longrightarrow$ bounded affine permutation $\tilde{f} : \mathbb{Z} \to \mathbb{Z}$

$$\tilde{f}(i) \equiv f(i) \pmod{n}, \quad \tilde{f}(i+n) = \tilde{f}(i) + n, \quad i < \tilde{f}(i) < i + n \quad \text{for all } i.$$

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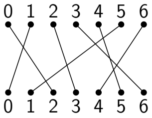

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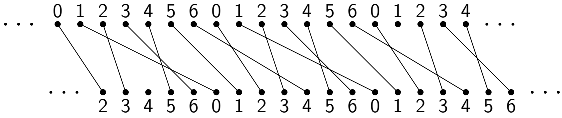

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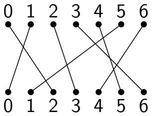

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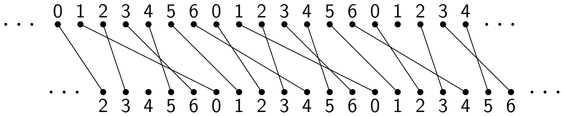

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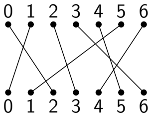

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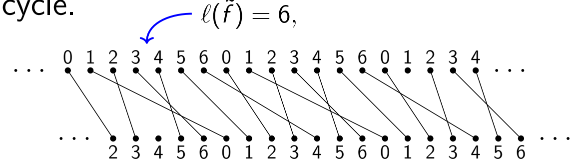

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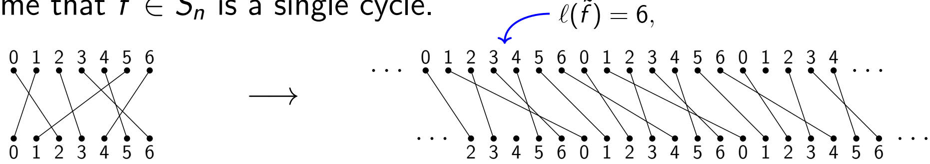

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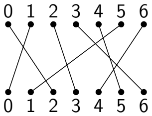
$\ell(\tilde{f}) = 6$, $n(f) = 7$, $k(f) = 3$, $\gamma(f) = (3,4)$



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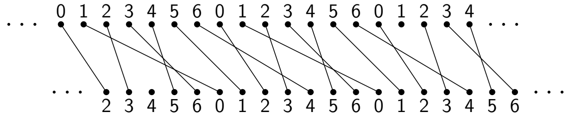

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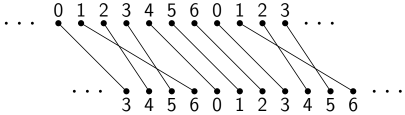

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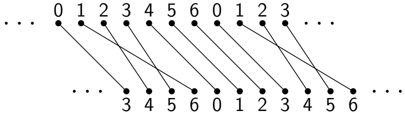

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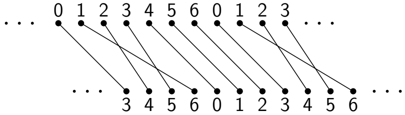

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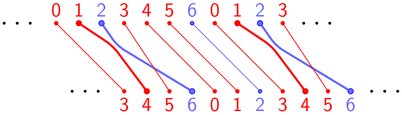

$\gamma(f^{(1,2)}) = (2,3)$

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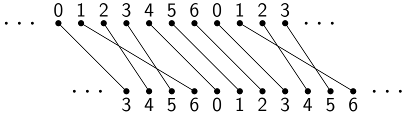

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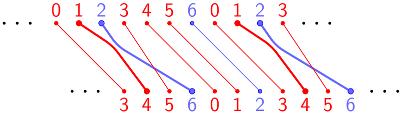

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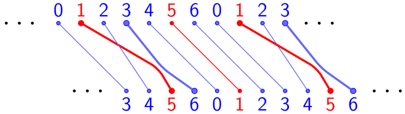

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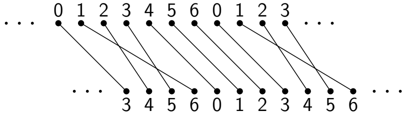

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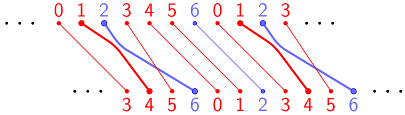

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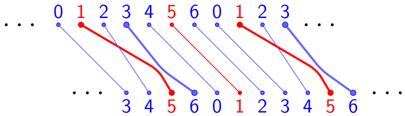

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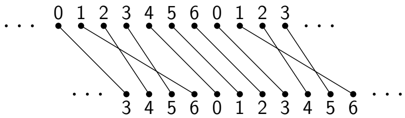

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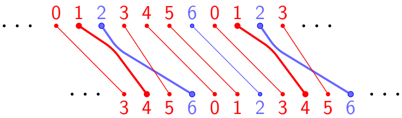

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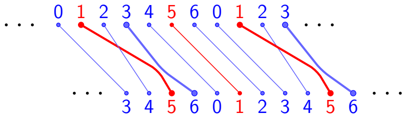

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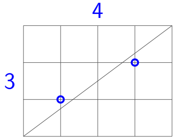

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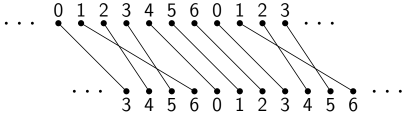
Definition

f is repetition-free if the inversion multiset $\Gamma(f)$ has no repeated elements.



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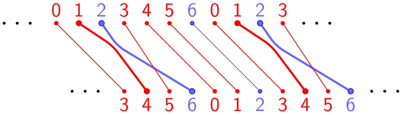

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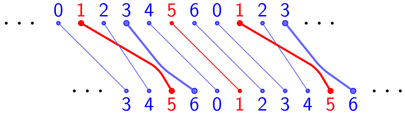

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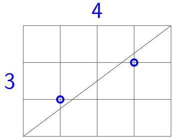

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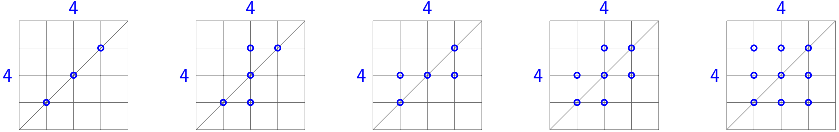
- Here are all possible repetition-free $\Gamma(f)$'s for $k = 4$, $n = 8$:



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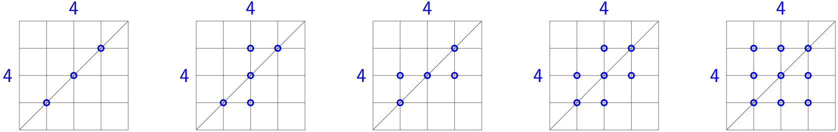

- For a subset Γ of the $k \times (n-k)$ rectangle, let $\widehat{\Gamma} := \Gamma \sqcup \{(0,0),(k,n-k)\}$.

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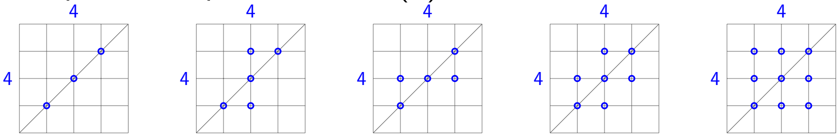

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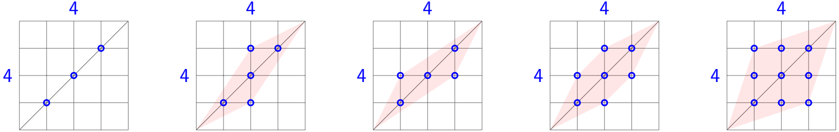

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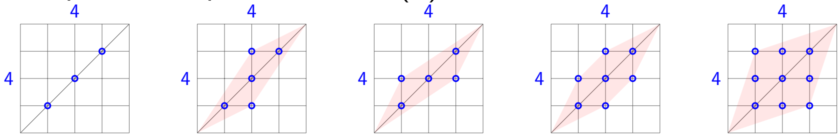

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Theorem (G.–Lam (2021))

- *f is repetition-free $\Longrightarrow$ $\Gamma(f)$ is convex and centrally symmetric.*

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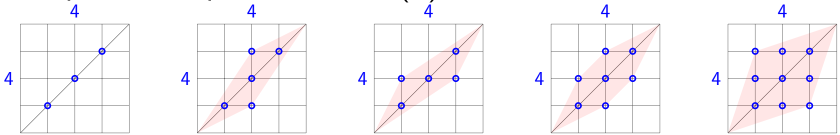

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- *f is repetition-free $\Longrightarrow$ $\Gamma(f)$ is convex and centrally symmetric.*
- *Conversely, any convex centrally symmetric Γ arises from some repetition-free f.*

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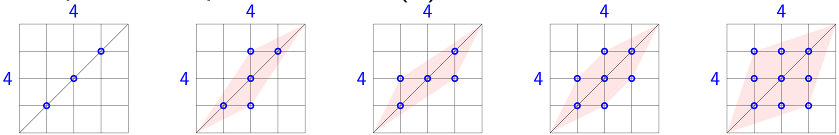

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- We say that Γ is convex if $\widehat{\Gamma} = \mathbb{Z}^2 \cap \mathrm{Conv}(\widehat{\Gamma})$.

Theorem (G.–Lam (2021))

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Recall: $\mathcal{P}(\Pi_f^\circ; q, t) = \left(q^{\frac{1}{2}} + t^{\frac{1}{2}}\right)^{n-1} C_f(q,t)$.

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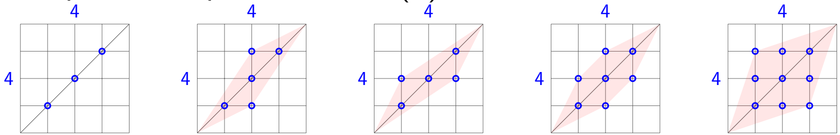

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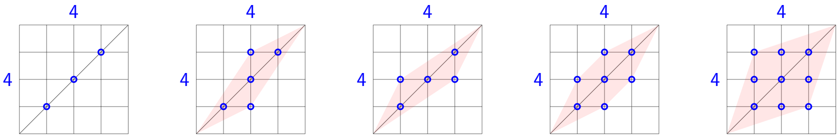

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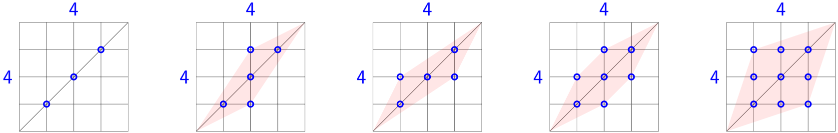

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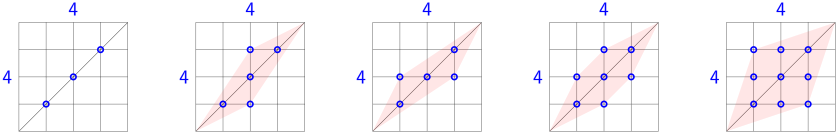

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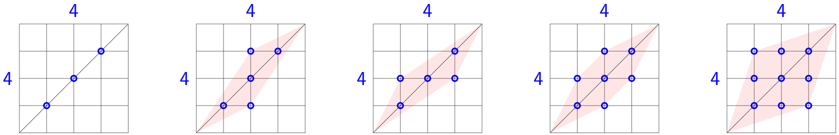

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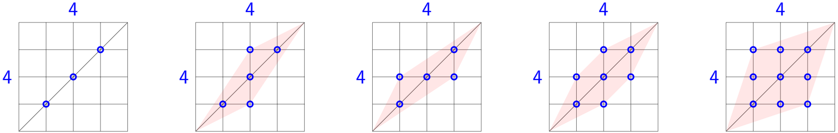

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