Zamolodchikov periodicity and integrability

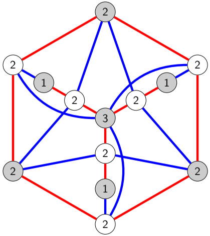
Pavel Galashin

MIT

galashin@mit.edu

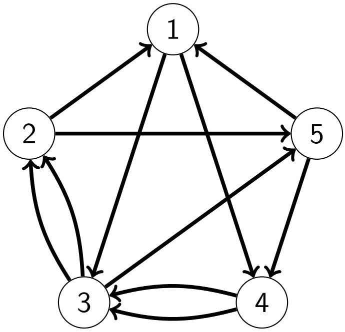
Hunter College, May 7, 2017

Joint work with Pavlo Pylyavskyy



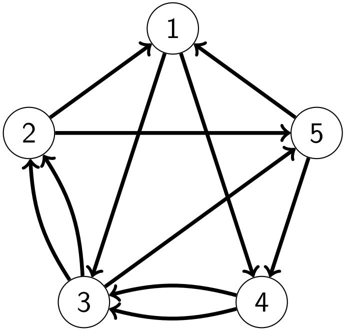
General T-systems (Nakanishi, 2011)

Q

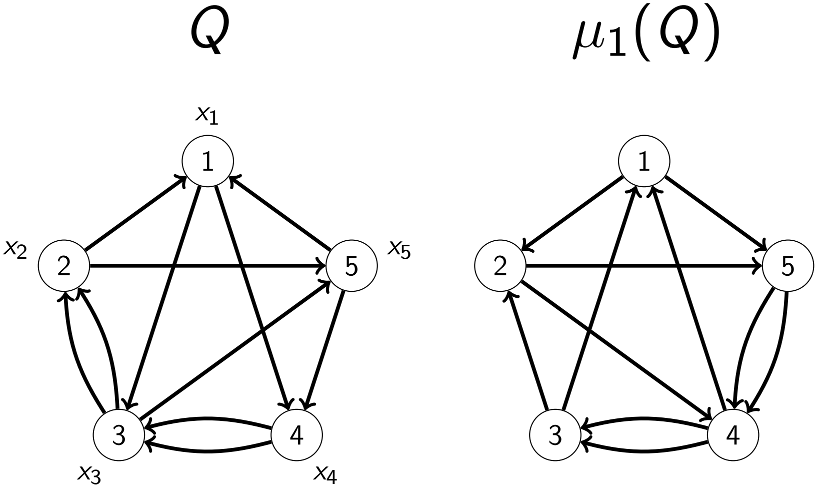


General T-systems (Nakanishi, 2011)

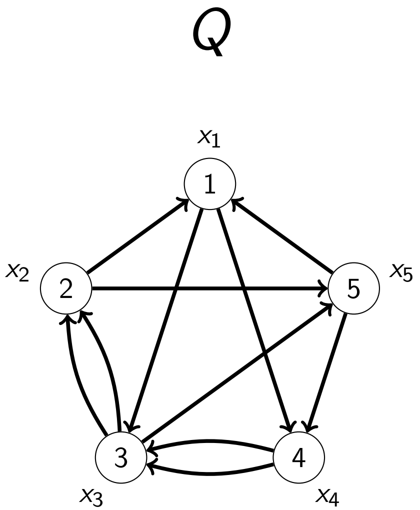
Q $\mu_1(Q)$

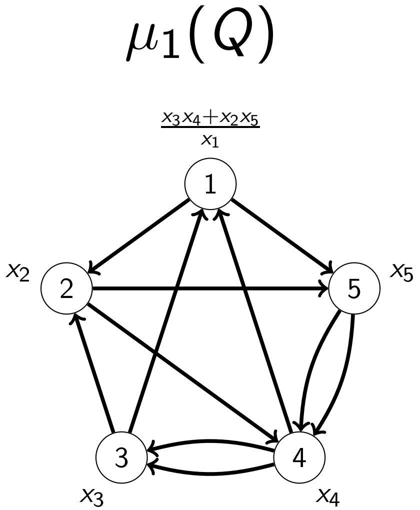


General T-systems (Nakanishi, 2011)

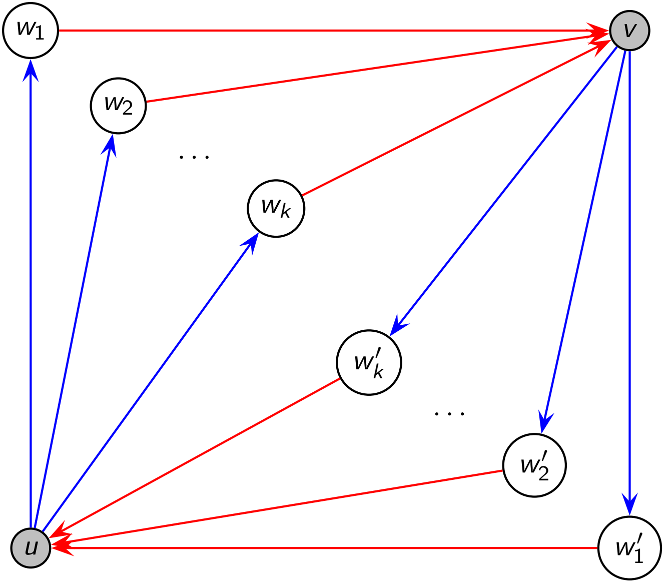


General T-systems (Nakanishi, 2011)

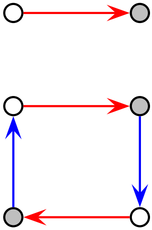




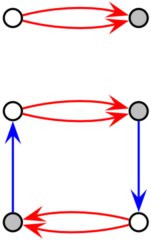
Bipartite **recurrent** quivers



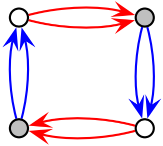
Four classes of quivers



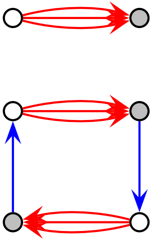
“finite ⊠ finite”



“affine ⊠ finite”

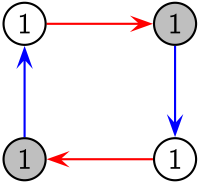


“affine ⊠ affine”

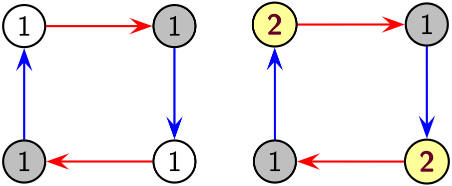


“wild”

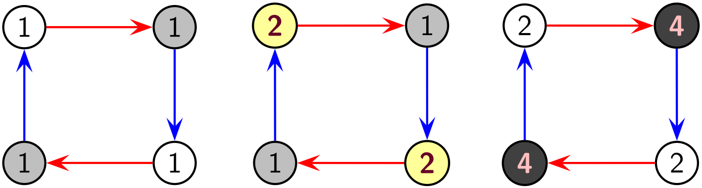
Example: finite ⊠ finite



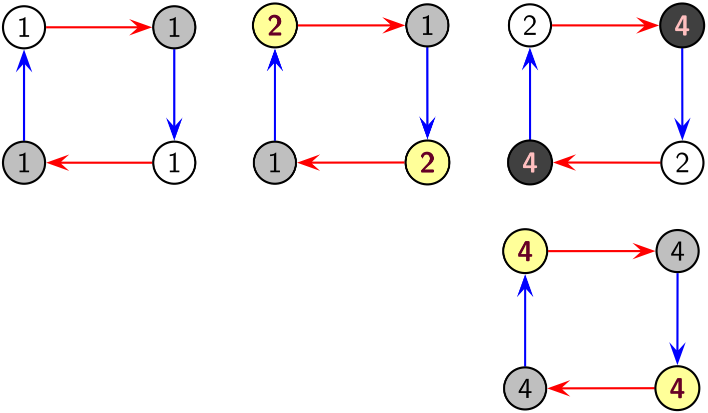
Example: finite $\boxtimes$ finite



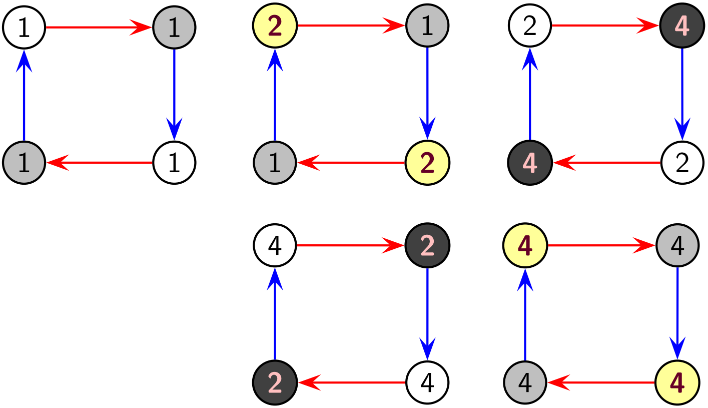
Example: finite ⊠ finite



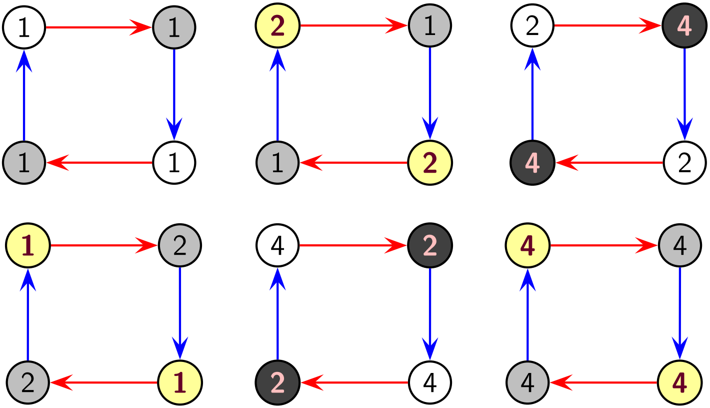
Example: finite ⊠ finite



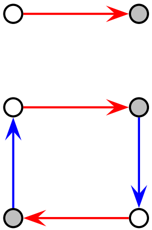
Example: finite ⊠ finite



Example: finite ⊠ finite

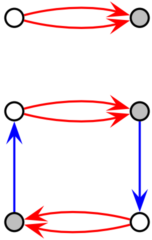


Four classes of quivers

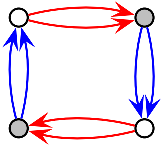


“finite $\boxtimes$ finite”

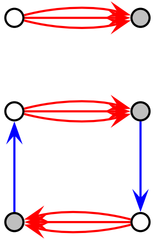
periodic



“affine $\boxtimes$ finite”



“affine $\boxtimes$ affine”



“wild”

Example: affine ⊠ finite

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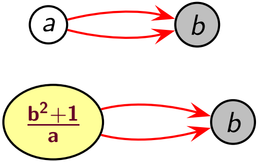

Example: affine ⊠ finite

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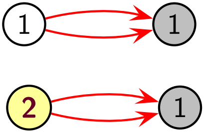

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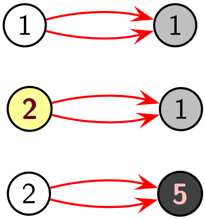

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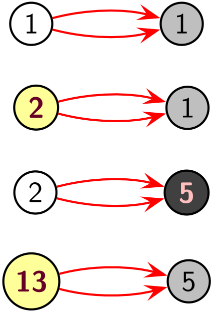
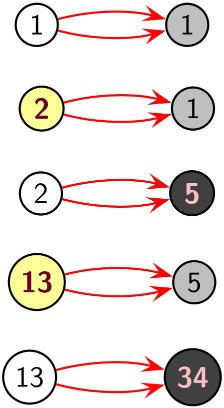
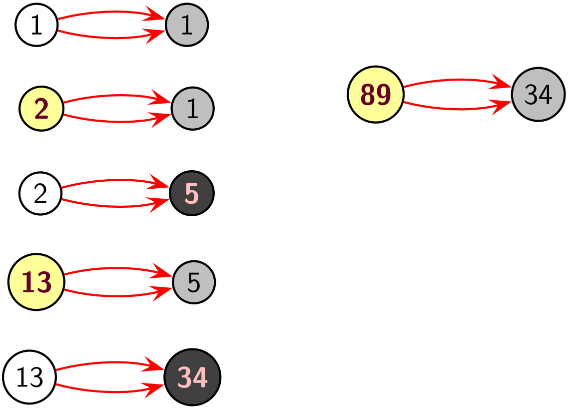


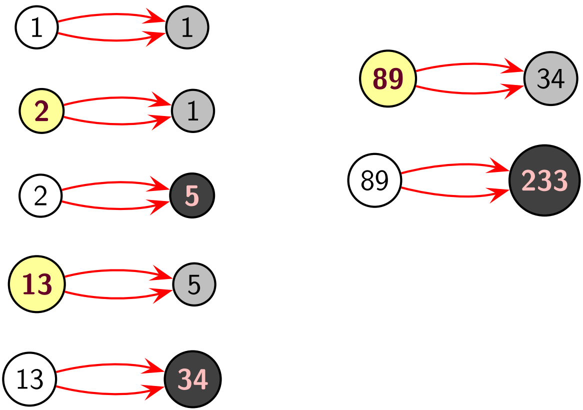
Example: affine ⊠ finite



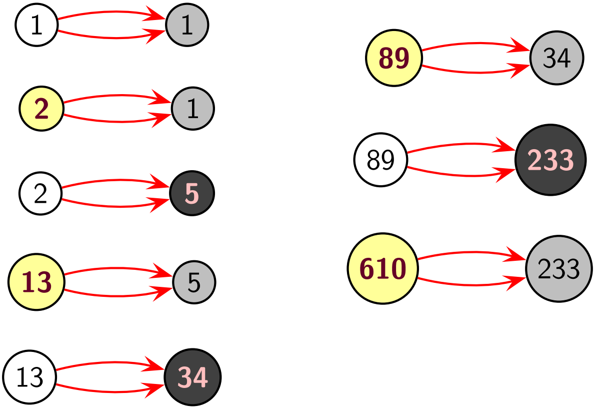
Example: affine ⊠ finite



Example: affine ⊠ finite



Example: affine ⊠ finite



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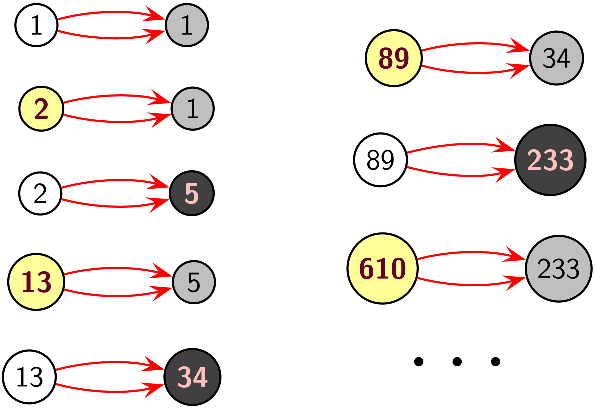

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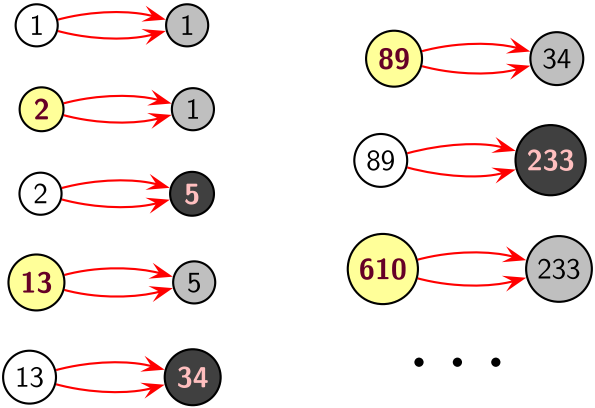
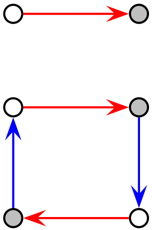


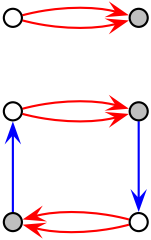
$$x_{n+1} = 3x_n - x_{n-1}$$

Four classes of quivers



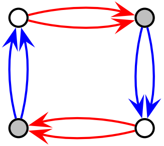
“finite $\boxtimes$ finite”

periodic

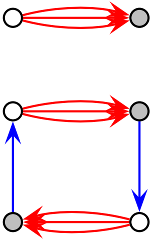


“affine $\boxtimes$ finite”

linearizable

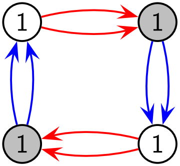


“affine $\boxtimes$ affine”



“wild”

Example: affine $\boxtimes$ affine



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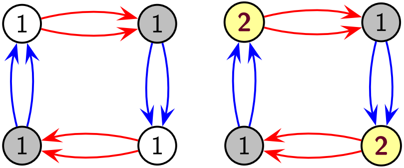
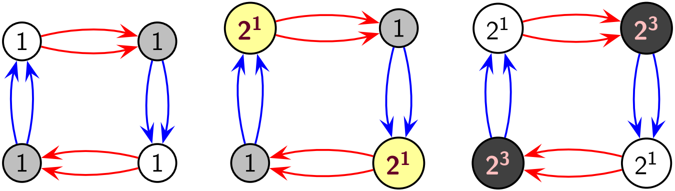


Example: affine ⊠ affine



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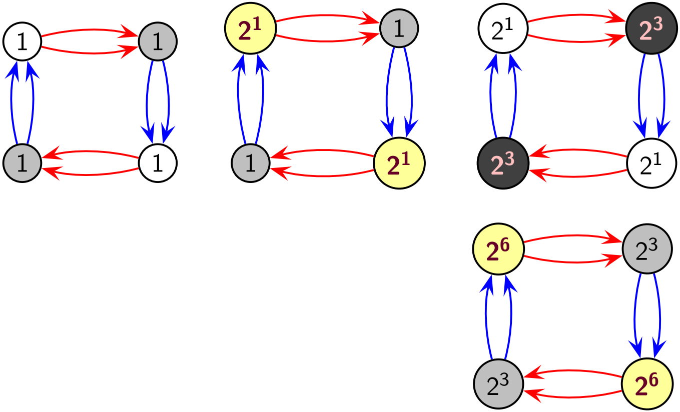

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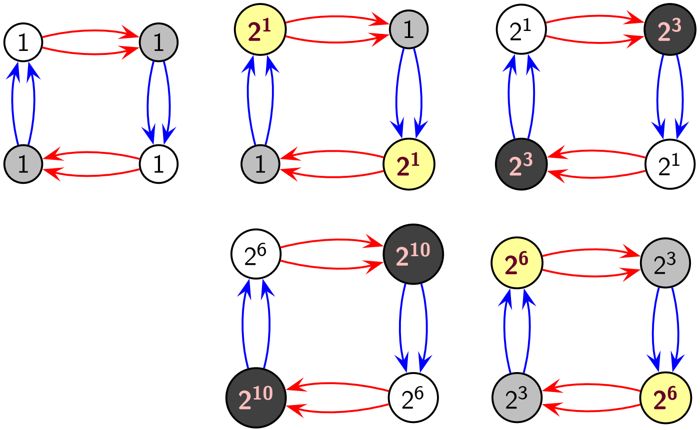

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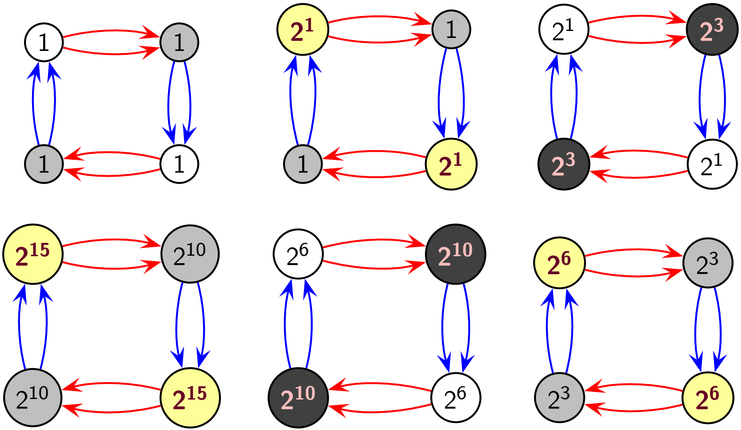

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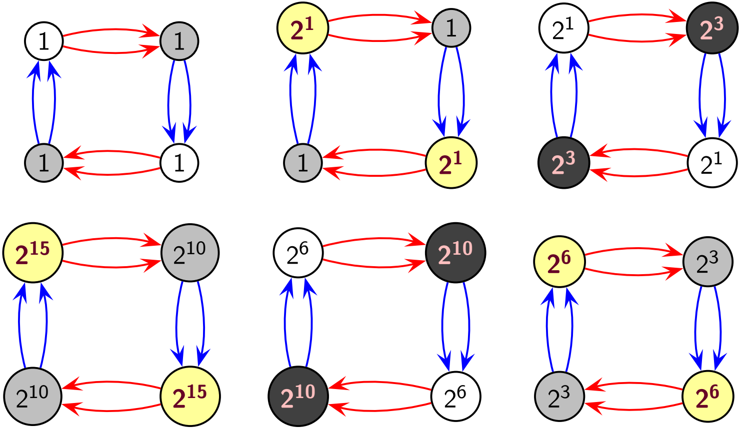
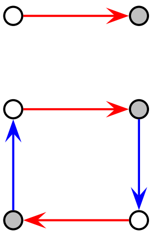


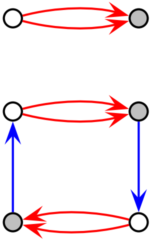
$$2^{\binom{n}{2}}$$

Four classes of quivers



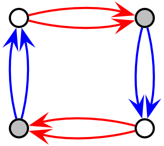
"finite ⊠ finite"

periodic



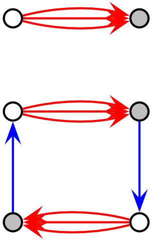
"affine ⊠ finite"

linearizable



"affine ⊠ affine"

grows as $\exp(t^2)$



"wild"

Example: wild

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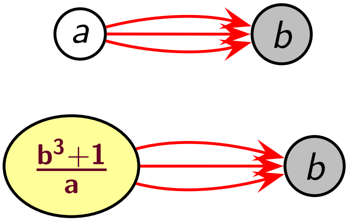

Example: wild

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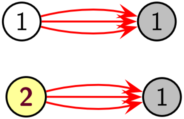

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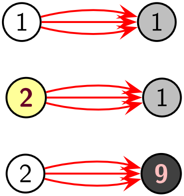

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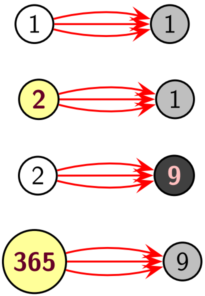

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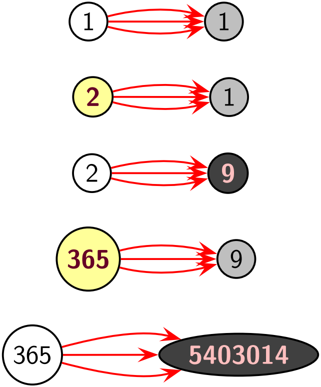
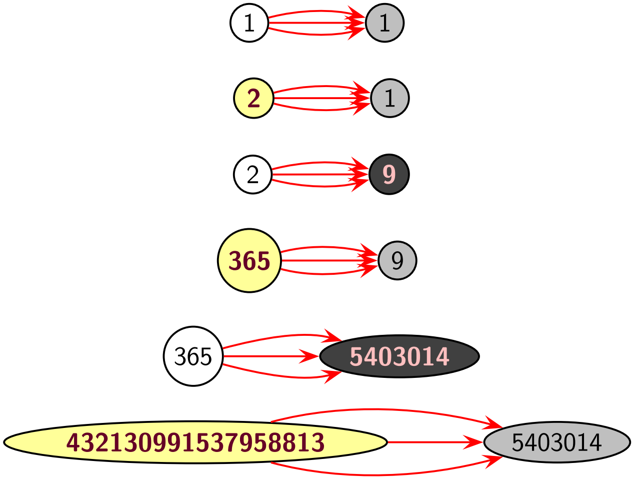


Example: wild



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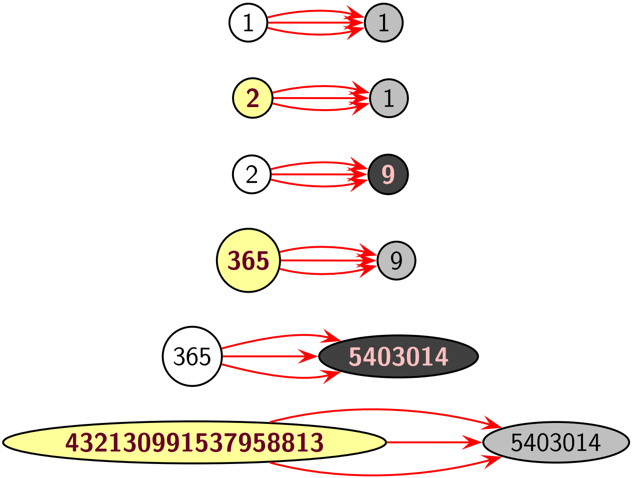
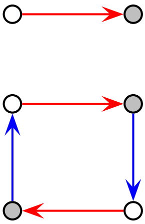


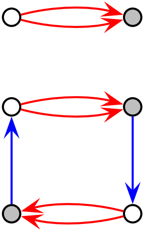
the next number is 14935169284101525874491673463268414536523593057

Four classes of quivers



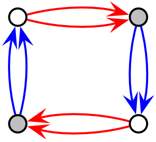
“finite ⊠ finite”

periodic



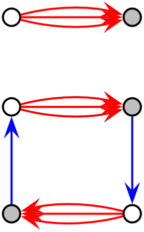
“affine ⊠ finite”

linearizable



“affine ⊠ affine”

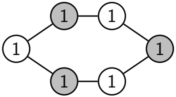
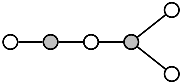
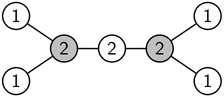
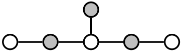
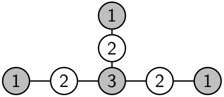
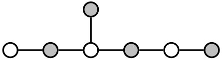
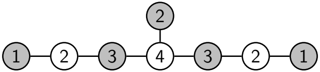
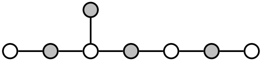
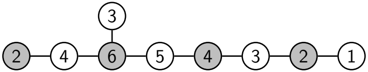
grows as $\exp(t^2)$



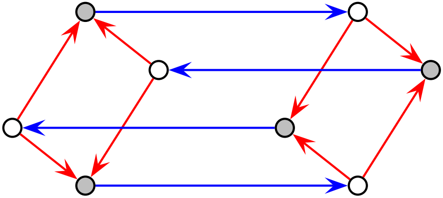
“wild”

grows as $\exp(\exp(t))$

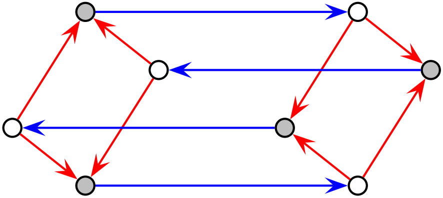
ADE Dynkin diagrams

Name	Finite diagram	Affine diagram	Name
A_n			$\hat{A}_{n-1}$
D_n			$\hat{D}_{n-1}$
E_6			$\hat{E}_6$
E_7			$\hat{E}_7$
E_8			$\hat{E}_8$

Affine ⊠ finite quivers

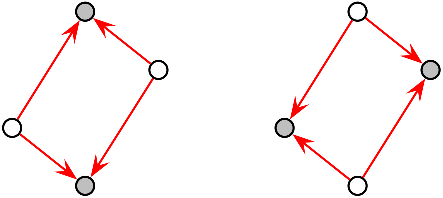


Affine $\boxtimes$ finite quivers



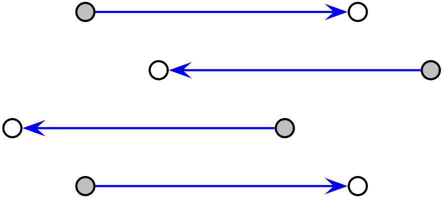
- Bipartite recurrent quiver

Affine ⊠ finite quivers



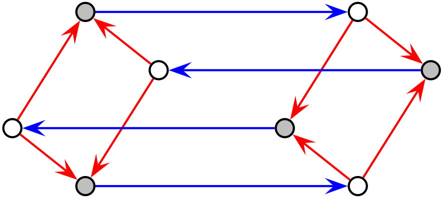
- Bipartite recurrent quiver
- All red components are **affine** Dynkin diagrams

Affine ⊠ finite quivers



- Bipartite recurrent quiver
- All red components are **affine** Dynkin diagrams
- All blue components are **finite** Dynkin diagrams

Affine ⊠ finite quivers



- Bipartite recurrent quiver
- All red components are **affine** Dynkin diagrams
- All blue components are **finite** Dynkin diagrams

$\uparrow$

"Affine ⊠ finite quiver"

Master conjecture

Conjecture (G.-Pylyavskyy, 2016)

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- *finite* $\boxtimes$ *finite* $\Longleftrightarrow$ *periodic*

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Master conjecture

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- *wild* $\Longleftrightarrow$ *grows as* $\exp(\exp(t))$

Results

Results

Theorem (G.-Pylyavskyy, 2016)

Periodic $\Longleftrightarrow$ *finite* $\boxtimes$ *finite*

Results

Theorem (G.-Pylyavskyy, 2016)

Periodic $\iff$ *finite* $\boxtimes$ *finite*

Theorem (G.-Pylyavskyy, 2016)

Linearizable $\implies$ *affine* $\boxtimes$ *finite or finite* $\boxtimes$ *finite*

Results

Theorem (G.-Pylyavskyy, 2016)

Periodic $\Longleftrightarrow$ *finite* $\boxtimes$ *finite*

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Theorem (G.-Pylyavskyy, 2017)

Grows slower than $\exp(\exp(t))$ $\Longrightarrow$ *affine* $\boxtimes$ *affine, affine* $\boxtimes$ *finite, or finite* $\boxtimes$ *finite*

Results

Theorem (G.-Pylyavskyy, 2016)

Periodic $\Longleftrightarrow$ *finite* $\boxtimes$ *finite*

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What is left:

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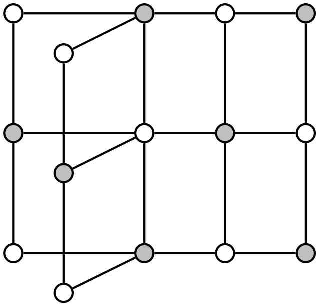
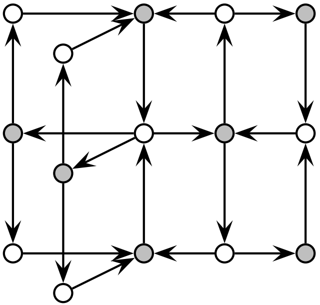
Tensor product



$D_5 \otimes A_3$

Tensor product



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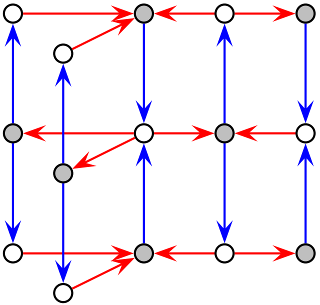

$D_5 \otimes A_3$

Zamolodchikov periodicity

Theorem (B. Keller, 2013)

Tensor product of **finite** *Dynkin diagrams* $\Longrightarrow$ *the* T*-system is periodic.*

Results

Theorem (G.-Pylyavskyy, 2016)

Periodic $\Longleftrightarrow$ finite $\boxtimes$ finite

Theorem (G.-Pylyavskyy, 2016)

Linearizable $\Longrightarrow$ *affine* $\boxtimes$ *finite or finite* $\boxtimes$ *finite*

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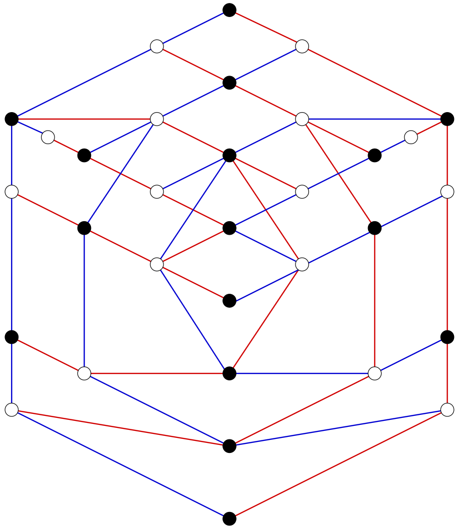
finite ⊠ finite classification (Stembridge, 2010)

5 infinite families and 11 exceptional quivers



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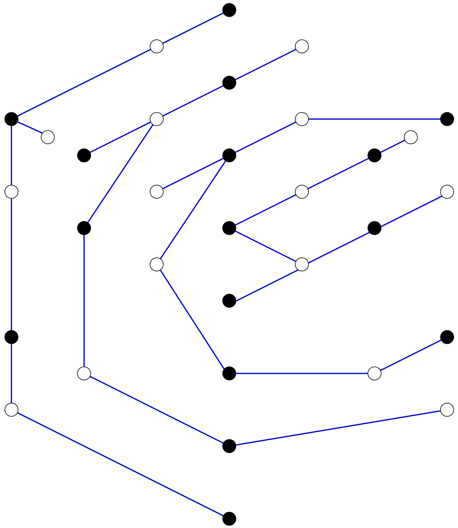

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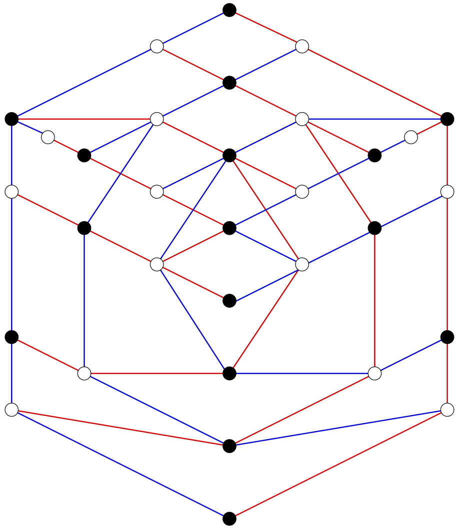
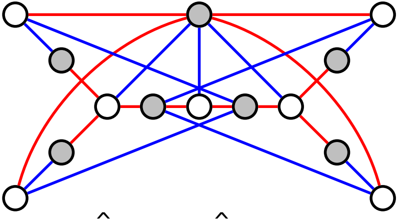
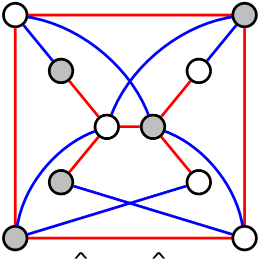


affine ⊠ finite classification

15 infinite families and 4 exceptional cases



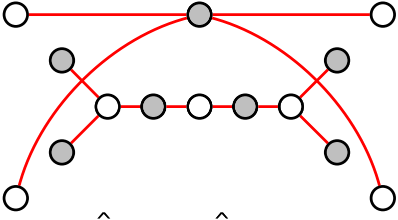
$\hat{D}_{n+1} * \hat{D}_{3n-1}$
for $n = 3$



$\hat{A}_3 * \hat{D}_5$

affine ⊠ finite classification

15 infinite families and 4 exceptional cases



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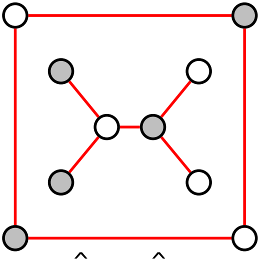

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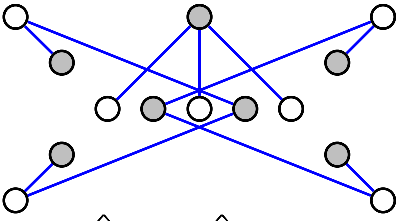

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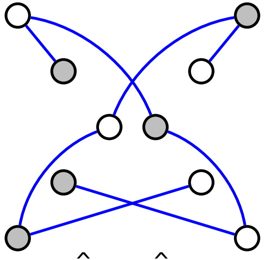

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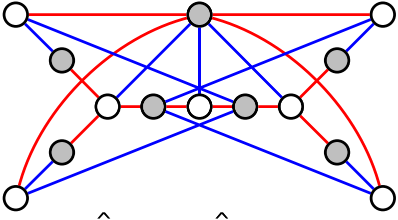

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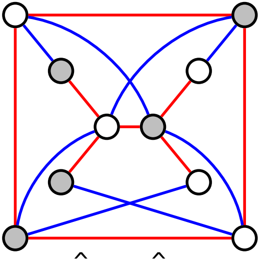

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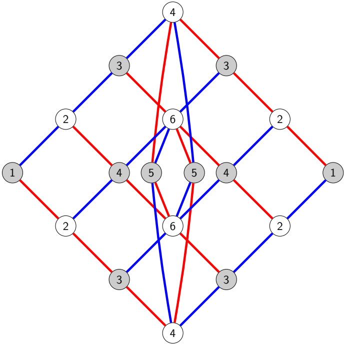
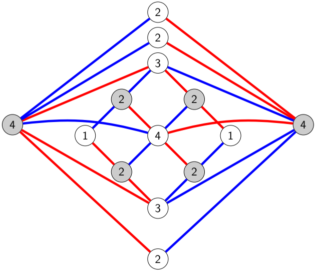
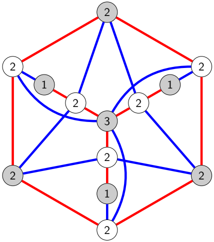
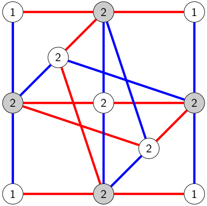
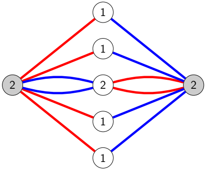
affine ⊠ affine classification: 41 infinite, 13 exceptional











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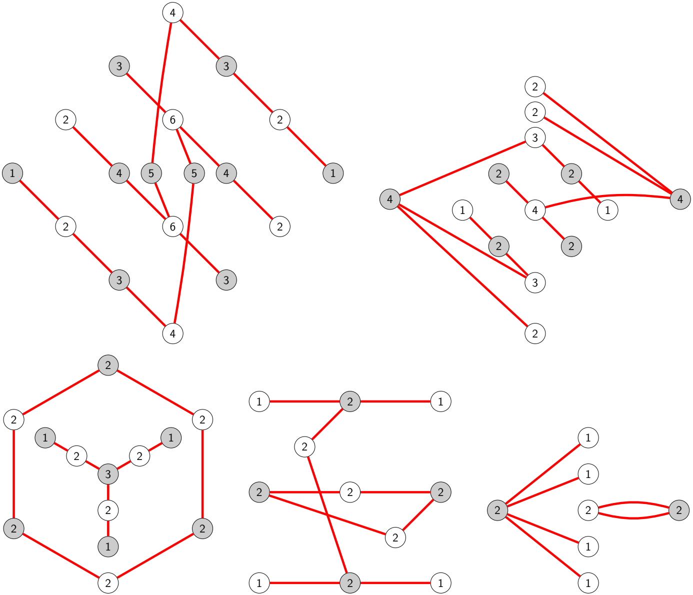

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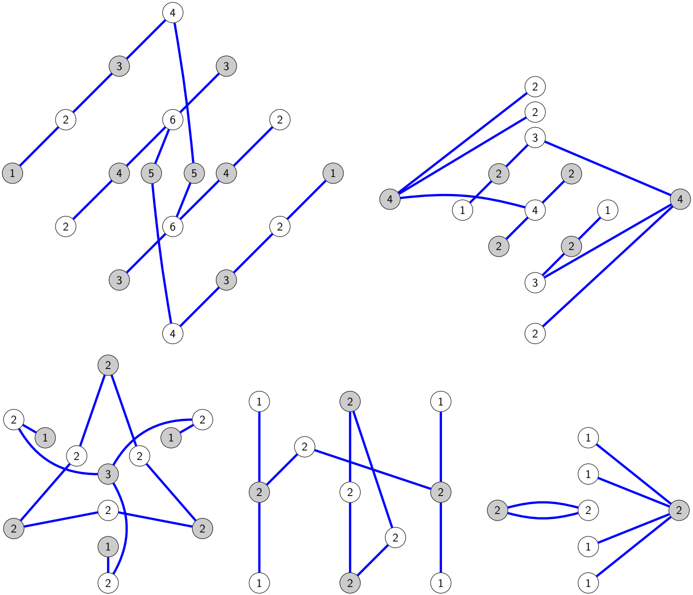

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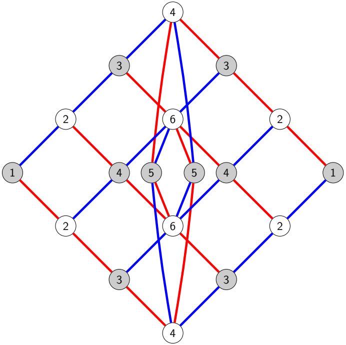

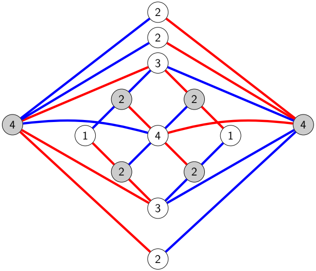

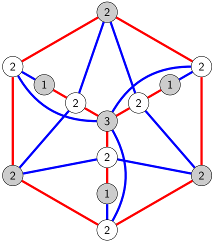

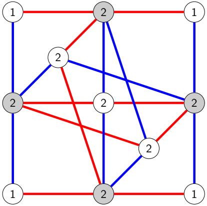

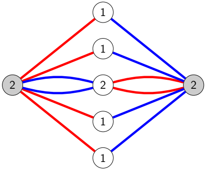

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Bibliography

Slides: `http://math.mit.edu/~galashin/slides/nyc.pdf`

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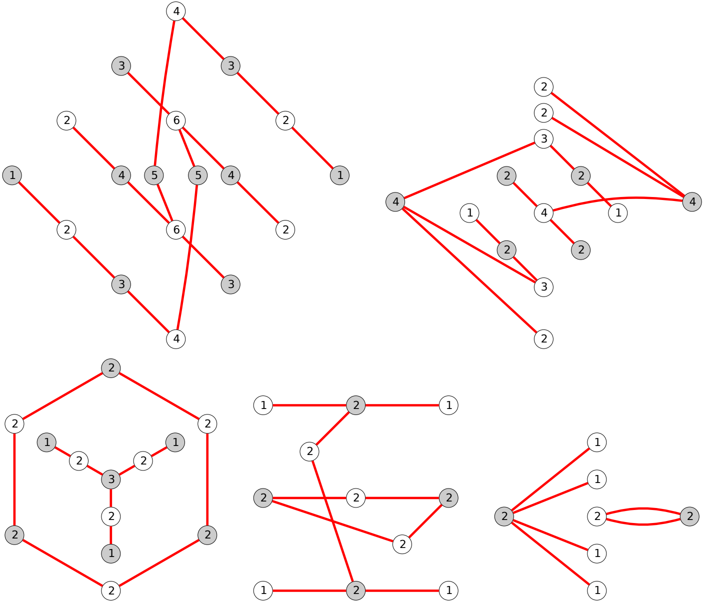
Thank you!



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