Ising model and total positivity

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Joint work with Pavlo Pylyavskyy

arXiv:1807.03282

Part 1: Ising model

Ising model: definition

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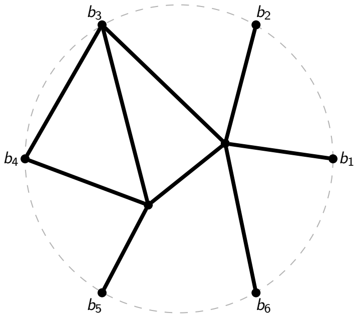

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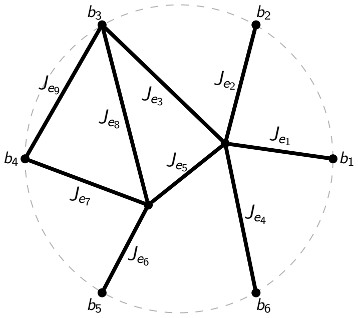

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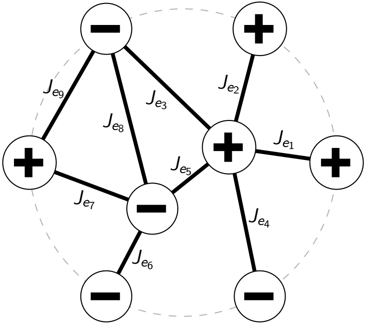

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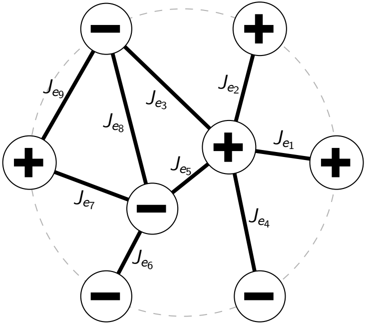

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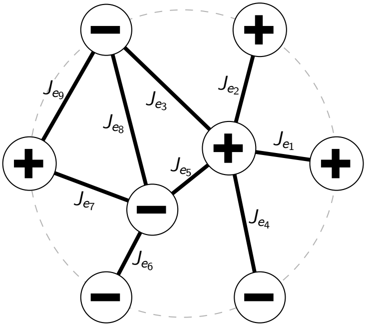

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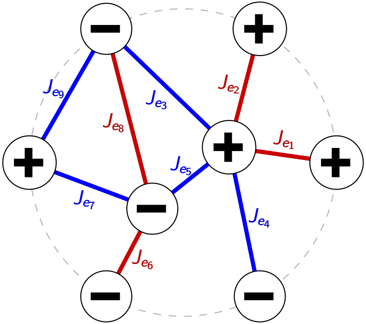

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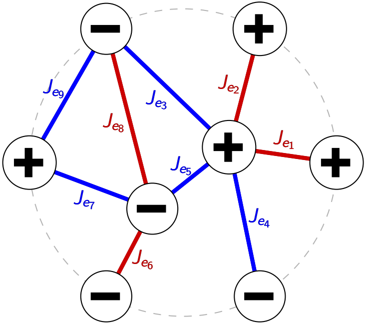

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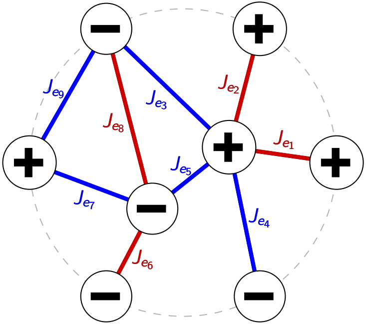

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$$\mathrm{Prob}(\sigma) := \frac{\mathrm{wt}(\sigma)}{Z}$$

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Theorem (G.–Pylyavskyy (2018))

Describe boundary correlations of the planar Ising model by inequalities.

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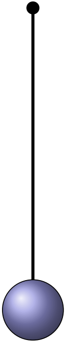

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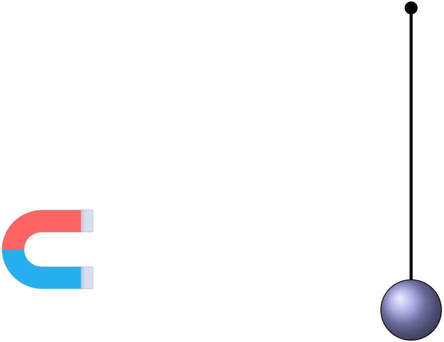

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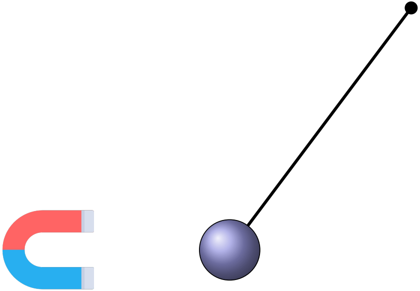

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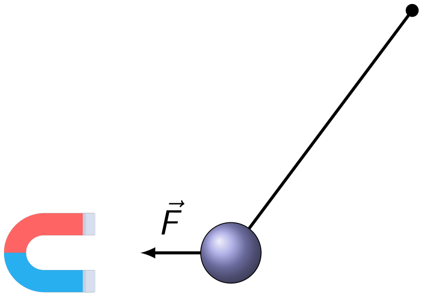

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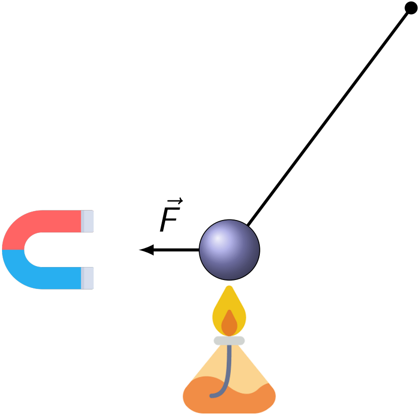

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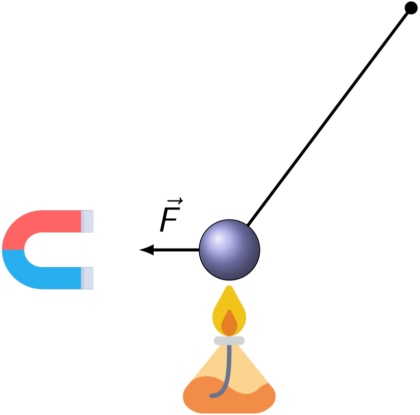

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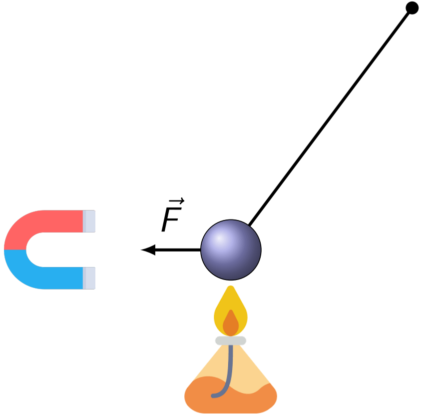

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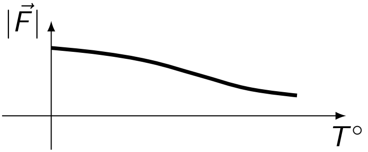

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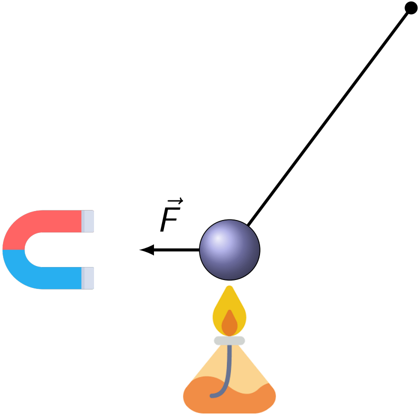

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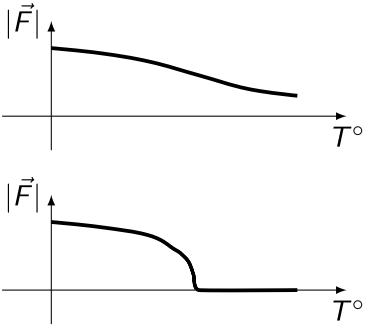

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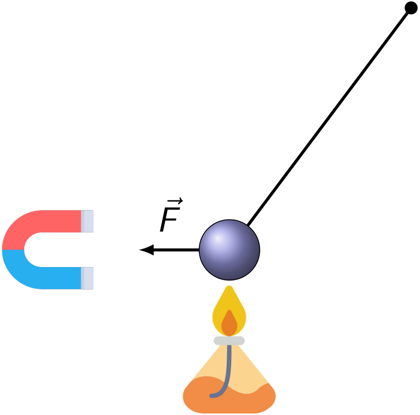

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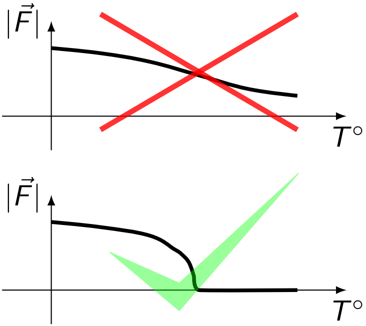

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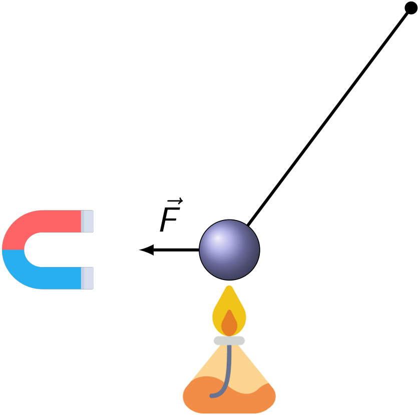

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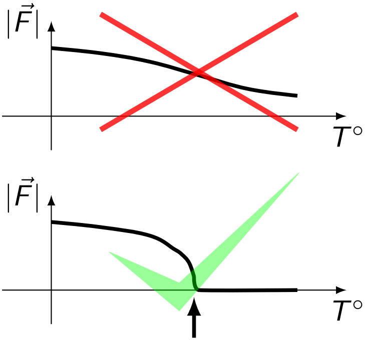

Curie point (P. Curie, 1895)

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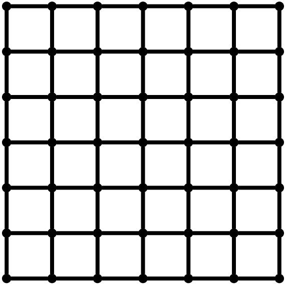

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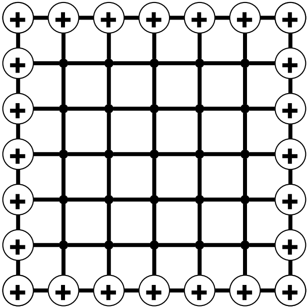

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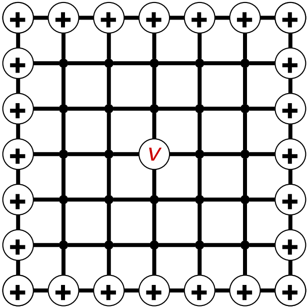

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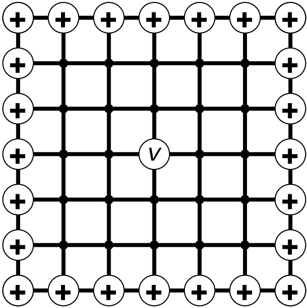
$$M(T) := \lim_{N\to\infty} \left(\mathrm{Prob}(\sigma_v = +1) - \mathrm{Prob}(\sigma_v = -1)\right)$$



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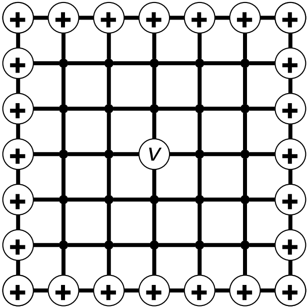

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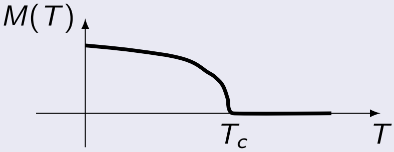

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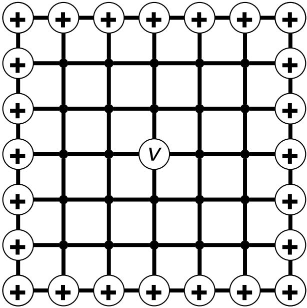

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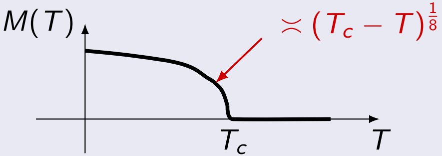

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- Belavin–Polyakov–Zamolodchikov (1984): conjectured conformal invariance of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$

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- Peierls (1937): phase transition in $\mathbb{Z}^d$ for $d \geq 2$
- Kramers–Wannier (1941): critical temperature $\frac{1}{T_c} = \frac{1}{2}\log\left(\sqrt{2}+1\right)$ for $\mathbb{Z}^2$
- Onsager, Kaufman, Yang (1944–1952): exact expressions for the free energy and spontaneous magnetization
- Belavin–Polyakov–Zamolodchikov (1984): conjectured conformal invariance of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$
- Smirnov, Chelkak, Hongler, Izyurov, ... (2010–2015): proved conformal invariance and universality of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$

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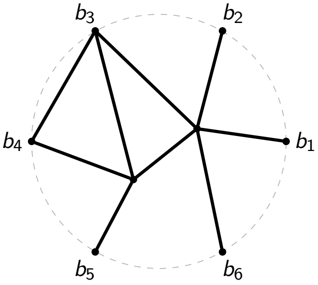

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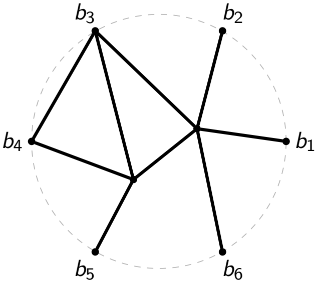

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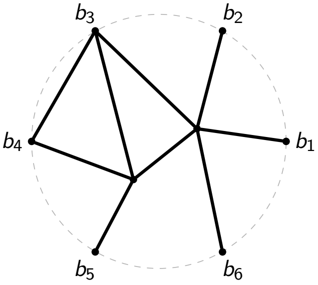

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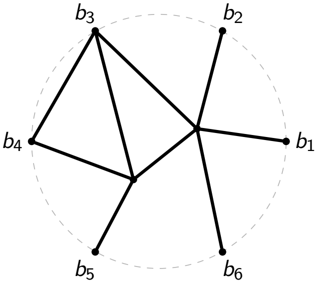

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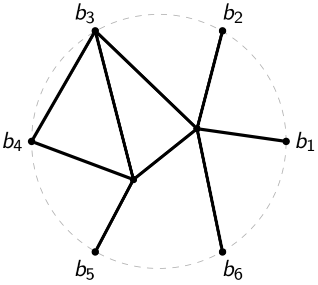

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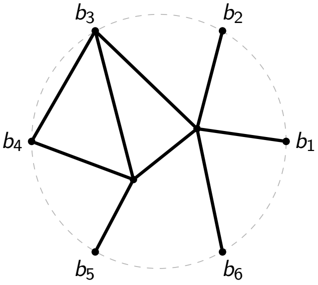

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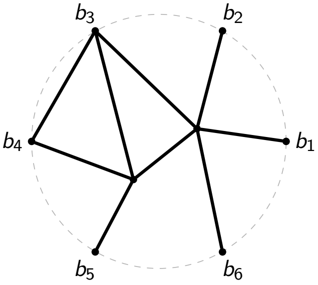

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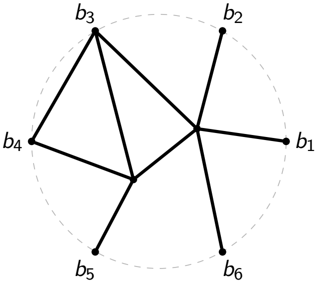

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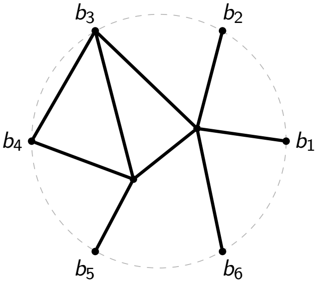

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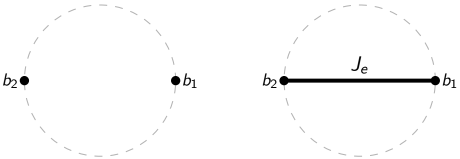

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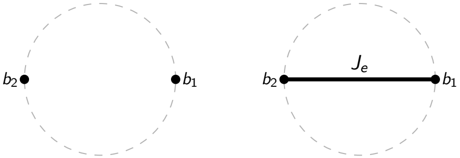

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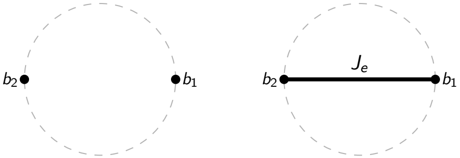

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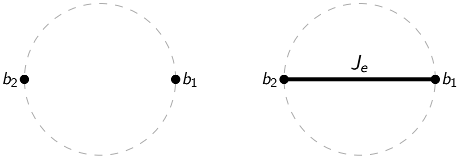

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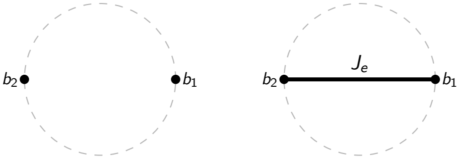

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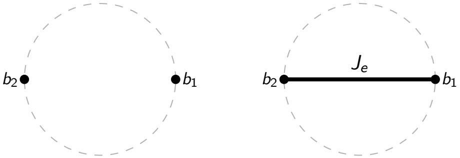

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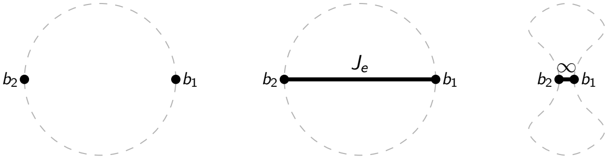

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Part 2: Total positivity

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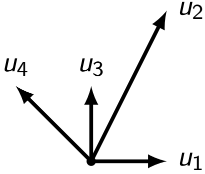

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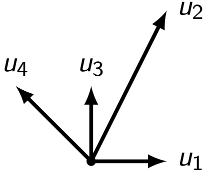

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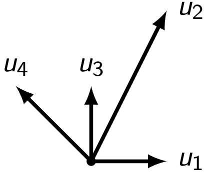

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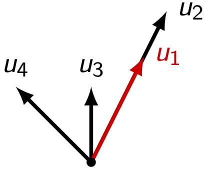

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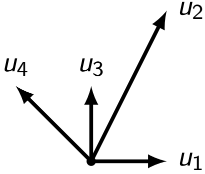

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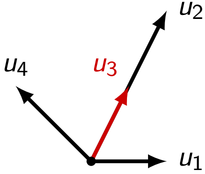

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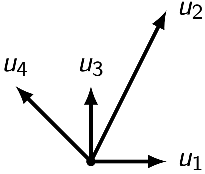

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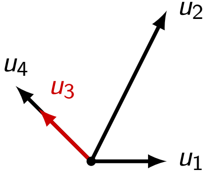

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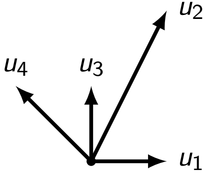

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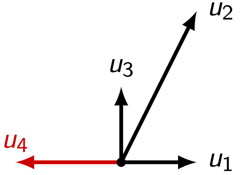

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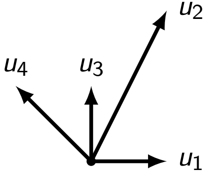

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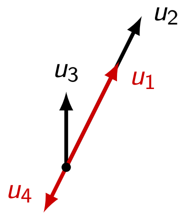

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Theorem (Postnikov (2006))

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Theorem (Smale (1960), Freedman (1982), Perelman (2003))

Let C be a compact contractible topological manifold whose boundary is homeomorphic to a sphere. Then C is homeomorphic to a closed ball.

The totally nonnegative orthogonal Grassmannian

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Theorem (G.–Pylyavskyy (2018))

$$\begin{array}{ccc} \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R},1) & \xhookrightarrow[\phi]{} & \mathrm{OG}(n,2n) \\ \uparrow & & \uparrow \\ \overline{\mathcal{X}}_n & & \mathrm{OG}_{\geq 0}(n,2n) \end{array}$$

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$$\begin{pmatrix} 1 & m_{12} & m_{13} & m_{14} \\ m_{12} & 1 & m_{23} & m_{24} \\ m_{13} & m_{23} & 1 & m_{34} \\ m_{14} & m_{24} & m_{34} & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & m_{12} & -m_{12} & -m_{13} & m_{13} & m_{14} & -m_{14} \\ -m_{12} & m_{12} & 1 & 1 & m_{23} & -m_{23} & -m_{24} & m_{24} \\ m_{13} & -m_{13} & -m_{23} & m_{23} & 1 & 1 & m_{34} & -m_{34} \\ -m_{14} & m_{14} & m_{24} & -m_{24} & -m_{34} & m_{34} & 1 & 1 \end{pmatrix}$$

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Main result

$\mathcal{X}_n := \{M(G,J) \mid (G,J) \text{ is a planar Ising network with } n \text{ boundary vertices}\}$
$\overline{\mathcal{X}}_n :=$ closure of $\mathcal{X}_n$ inside the space of $n \times n$ matrices.

We have $\mathcal{X}_n, \overline{\mathcal{X}}_n \subset \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R},1) := \left\{ \begin{array}{c} \text{symmetric } n \times n \text{ matrices} \\ \text{with 1's on the diagonal} \end{array} \right\}$.

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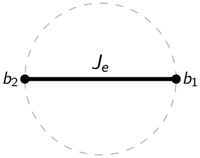

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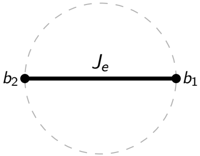

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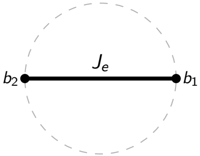

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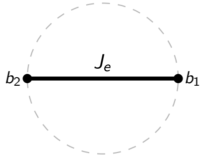

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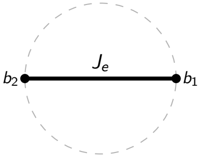

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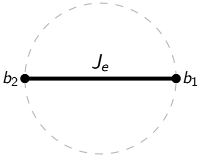

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Kramers–Wannier's duality

Ising model: history

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- Ising (1925): no phase transition in 1D $\Longrightarrow$ not a good model for ferromagnetism

Historically, we let $G := \mathbb{Z}^d \cap \Omega$ for some $\Omega \subset \mathbb{R}^d$
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- Onsager, Kaufman, Yang (1944–1952): exact expressions for the free energy and spontaneous magnetization
- Belavin–Polyakov–Zamolodchikov (1984): conjectured conformal invariance of the scaling limit at $T = T_c$ for $\mathbb{Z}^2$
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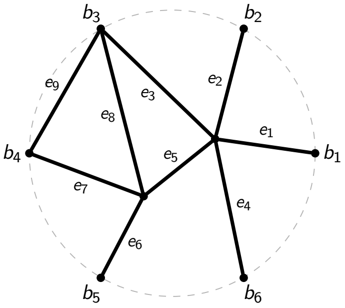

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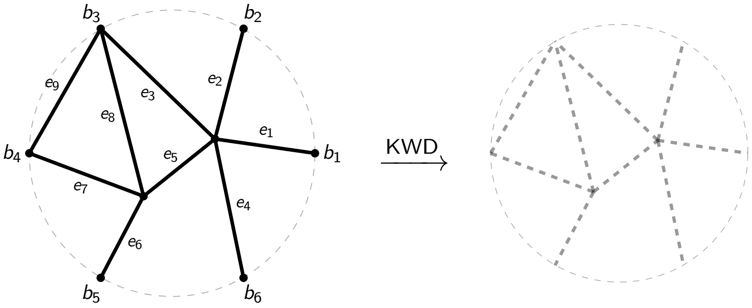

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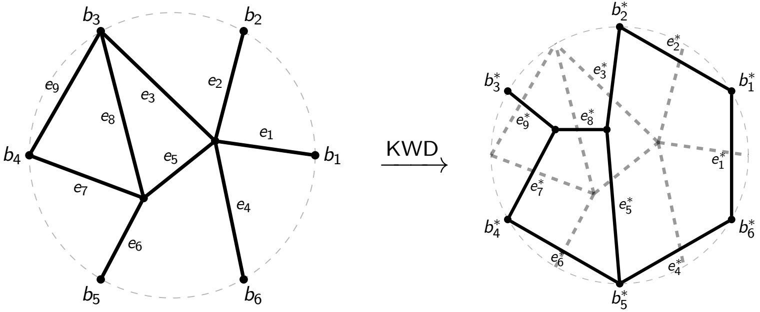

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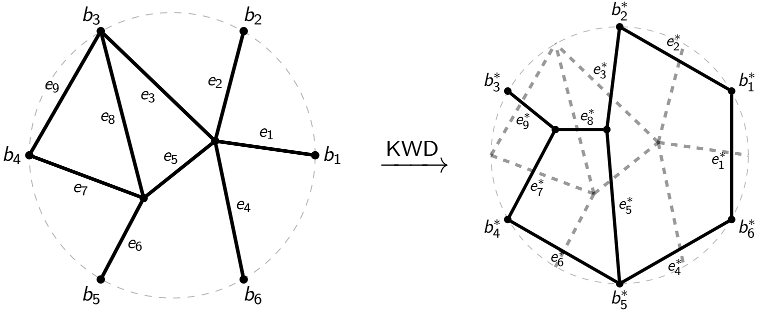

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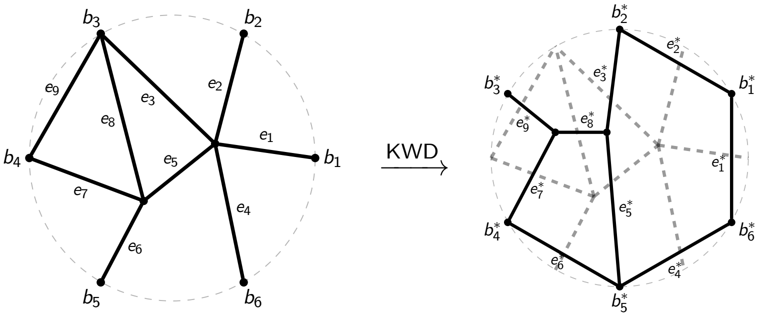

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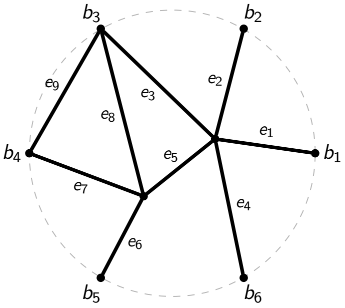

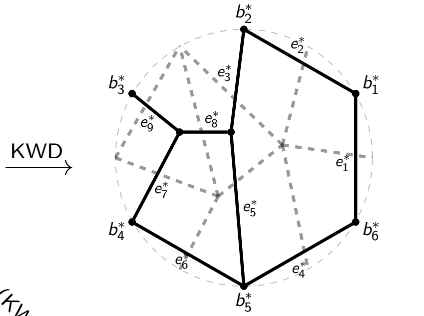

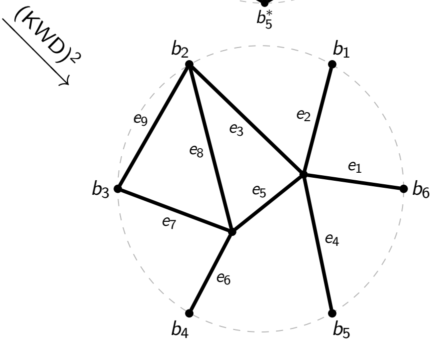

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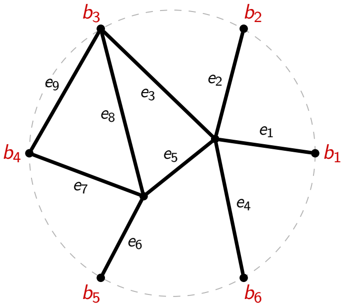

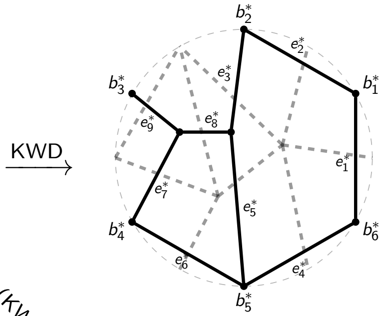

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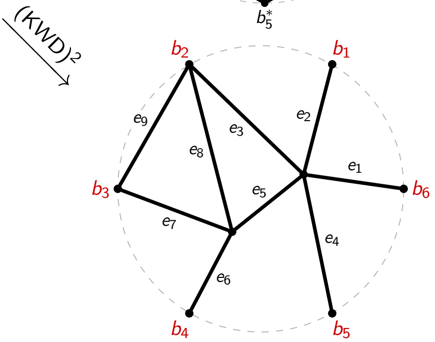

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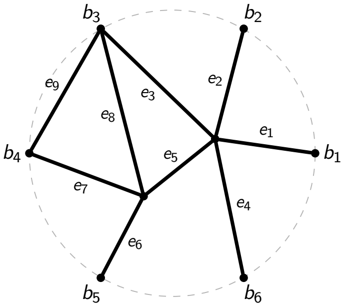

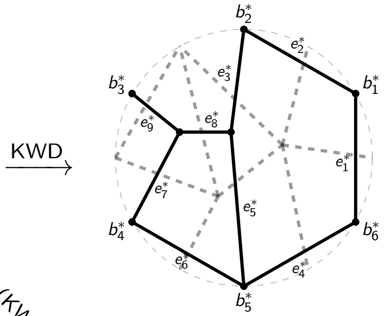

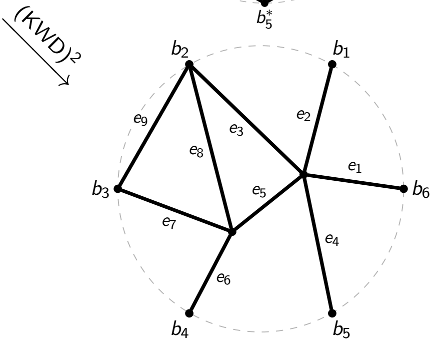

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$(\mathrm{KWD})^2 =$ cyclic shift!

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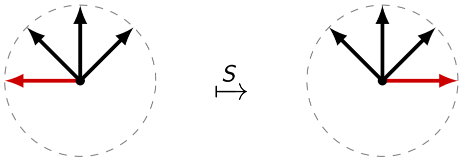

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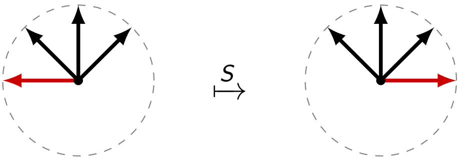

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This map preserves $\mathrm{Gr}_{\geq 0}(k, n)$.

Example: For $\mathrm{Gr}_{\geq 0}(2, 4)$, we have

$$X_0 = \begin{pmatrix} 1 & 0 & -1 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 & 0 \end{pmatrix} \overset{S}{\mapsto} \begin{pmatrix} \sqrt{2} & 1 & 0 & -1 \\ 0 & 1 & \sqrt{2} & 1 \end{pmatrix} = X_0 \in \mathrm{Gr}_{\geq 0}(2, 4)$$

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Cyclic shift on $\mathrm{Gr}_{\geq 0}(k, n)$

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Kramers–Wannier's duality vs. cyclic shift

Theorem (G.–Pylyavskyy (2018))

- *The map ϕ restricts to a homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n, 2n)$.*
- *Each of the spaces is homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*

$$\begin{array}{ccc} \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) & \xrightarrow[\phi]{} & \mathrm{OG}(n, 2n) \\ \uparrow & & \uparrow \\ \overline{\mathcal{X}}_n & \xrightarrow[\phi]{\sim} & \mathrm{OG}_{\geq 0}(n, 2n) \end{array}$$

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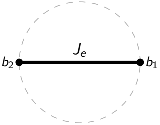
Example:



$$M_0 \leftrightarrow J_e = \frac{1}{2}\log(\sqrt{2}+1)$$

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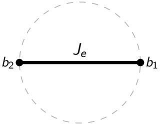

$M_0 \leftrightarrow J_e = \frac{1}{2}\log(\sqrt{2} + 1)$

Fixed point M_0 of KWD $\leftrightarrow$ Ising model at critical temperature $\leftrightarrow$ X_0?

Kramers–Wannier's duality vs. cyclic shift

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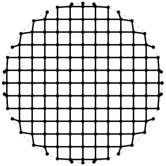
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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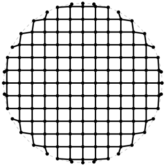
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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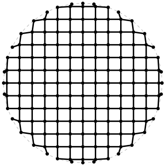
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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The entries of $M_0 = (m_{ij})_{i,j=1}^{n}$ *are given by* $m_{ij} = \dfrac{\sum_I \Delta_I(X_0)}{\sum_{I'} \Delta_{I'}(X_0)}$.

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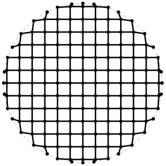
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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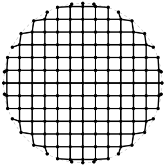
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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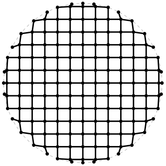
Let $G =$  with n boundary vertices and $J_e := \frac{1}{2}\log\left(\sqrt{2}+1\right)$.

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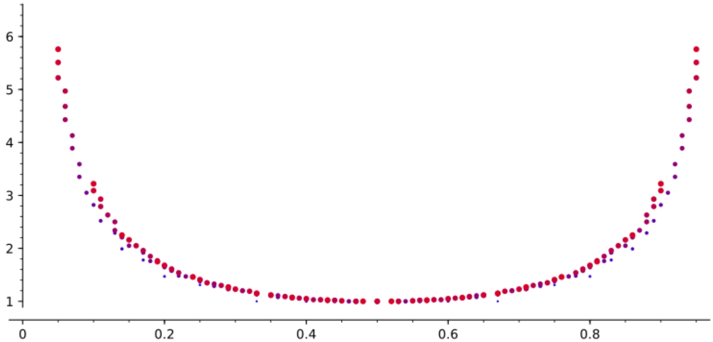

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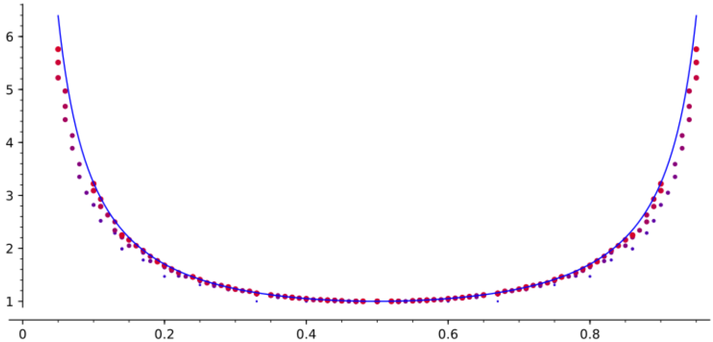

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Matchings

Boundary measurement map

Theorem (Postnikov (2006))

Each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

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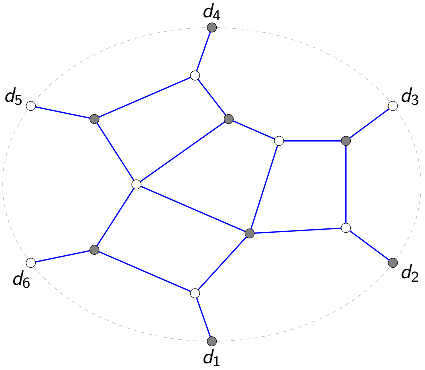

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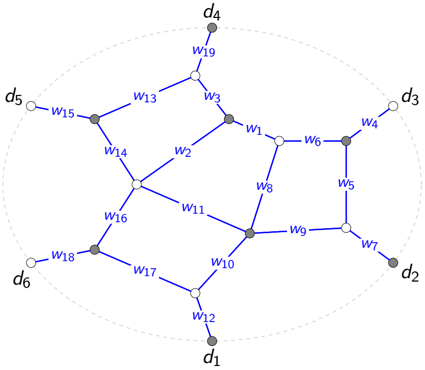

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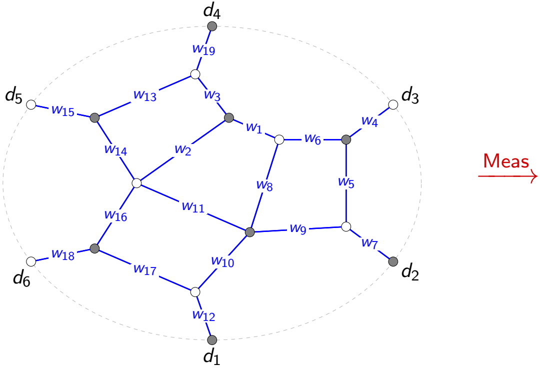

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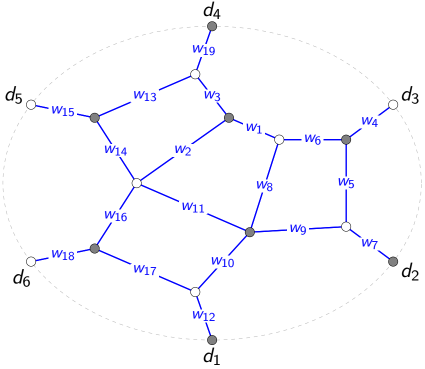

$$\xrightarrow{\text{Meas}} \qquad \Delta_I := \sum_{\mathcal{A}:\ \partial(\mathcal{A})=I} \mathrm{wt}(\mathcal{A})$$

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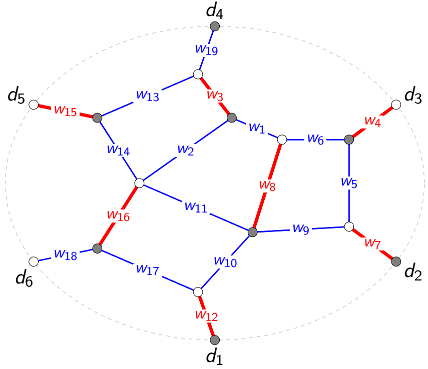

$\mathcal{A}$: almost perfect matching

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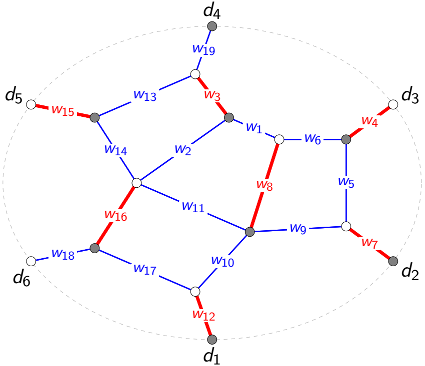

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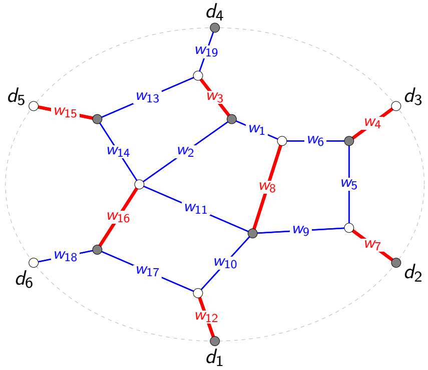

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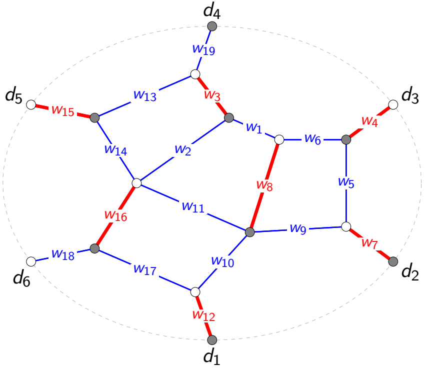

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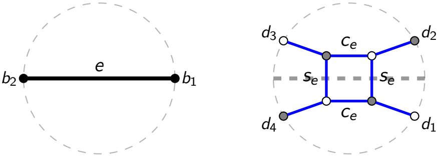
Postnikov (2006), Talaska (2007), Postnikov–Speyer–Williams (2009), Lam (2016)

Ising network $\to$ planar bipartite graph



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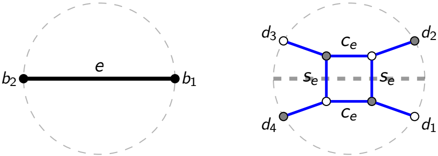

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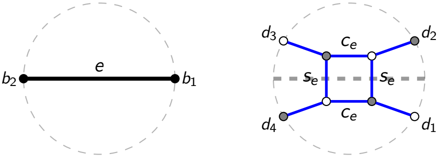

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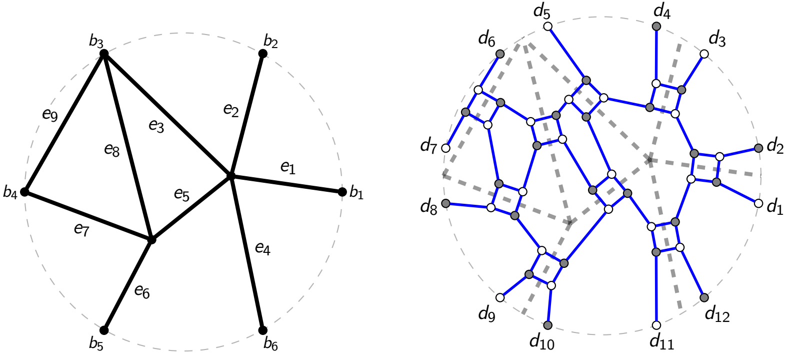
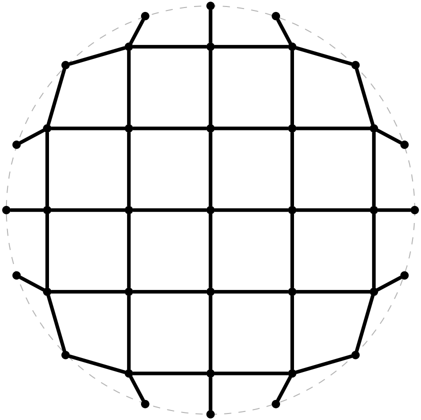


Random almost perfect matchings



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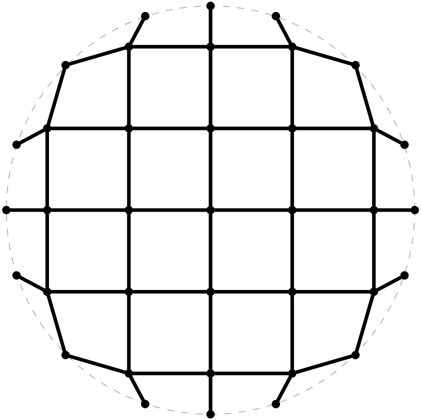

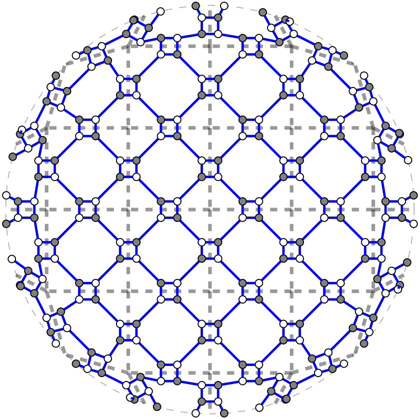

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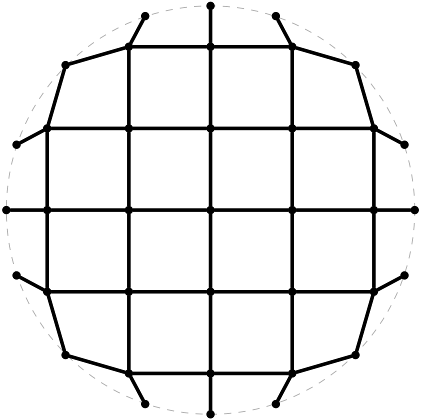

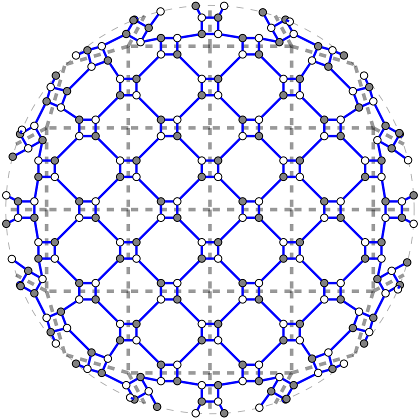

Question

- *What is the* *shape* *of a random almost perfect matching?*

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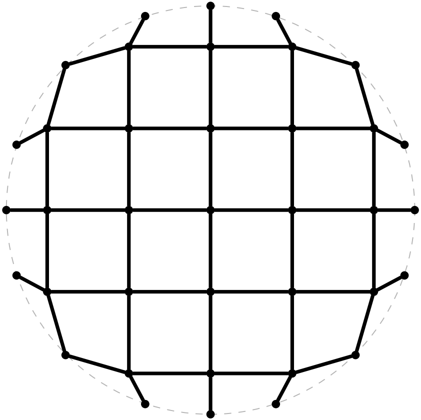 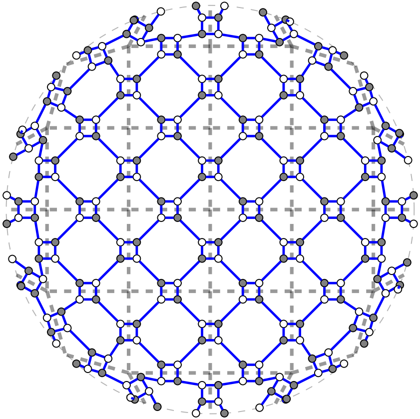

Question

- *What is the shape of a random almost perfect matching?*
- *Is there a* *phase transition* *as* (s_e, c_e) *changes from* $(1, 0)$ *to* $(0, 1)$*?*

Thank you!

Slides: `http://math.mit.edu/~galashin/slides/mit_ising.pdf`

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Ising model and the positive orthogonal Grassmannian
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The totally nonnegative Grassmannian is a ball.
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J. Stat. Phys., 166(1):72–89, 2017.

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Total positivity, Grassmannians, and networks.
arXiv:math/0609764.

Thomas Lam.
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Current developments in mathematics 2014, pages 51–152. Int. Press, 2016