Zamolodchikov periodicity and integrability

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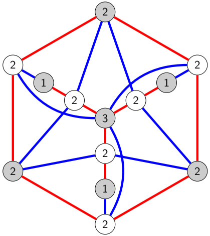
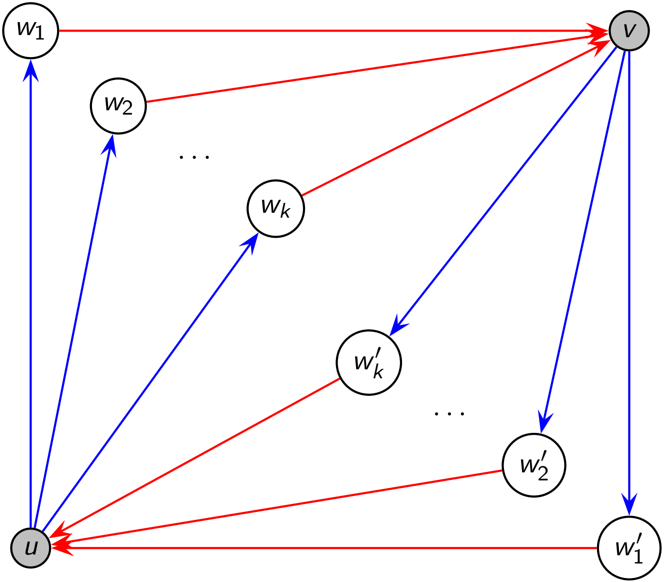
Infinite Analysis 17, Osaka City University, December 7, 2017

Joint work with Pavlo Pylyavskyy

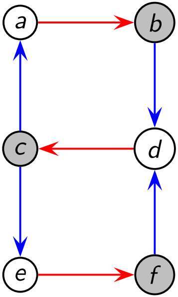


Part 0: Recall

Bipartite **recurrent** quivers



Bipartite T-system



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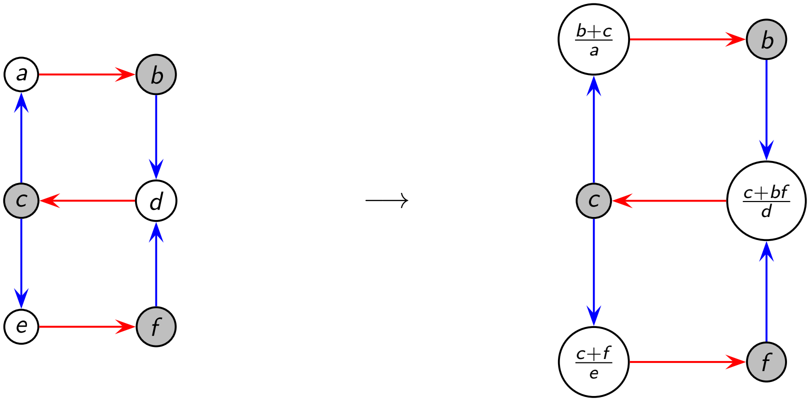

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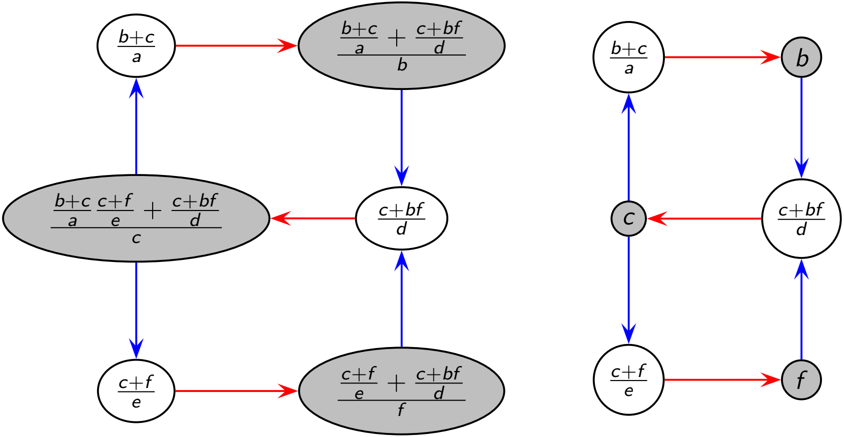
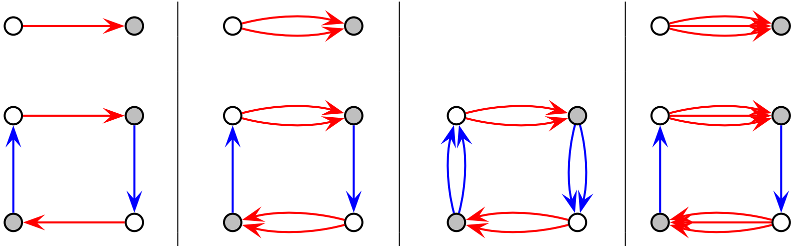
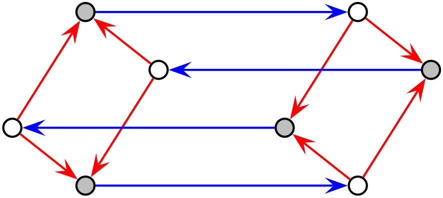


Four classes of quivers

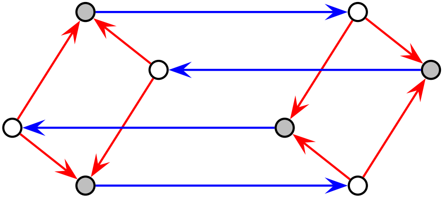


“finite ⊠ finite”	“affine ⊠ finite”	“affine ⊠ affine”	“wild”
periodic	linearizable	grows as $\exp(t^2)$	grows as $\exp(\exp(t))$

Affine ⊠ finite quivers

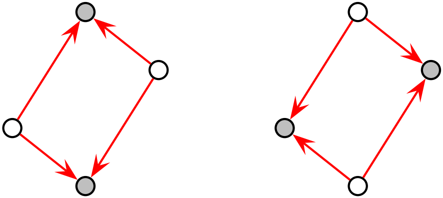


Affine ⊠ finite quivers



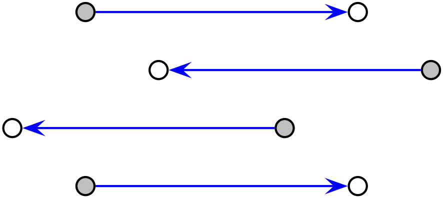
- Bipartite recurrent quiver

Affine ⊠ finite quivers



- Bipartite recurrent quiver
- All red components are **affine** Dynkin diagrams

Affine ⊠ finite quivers



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- All blue components are **finite** Dynkin diagrams

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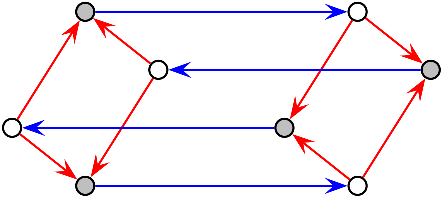


- Bipartite recurrent quiver
- All red components are **affine** Dynkin diagrams
- All blue components are **finite** Dynkin diagrams

↑

"**Affine ⊠ finite quiver**"

Master conjecture

Conjecture (G.-Pylyavskyy, 2016)

- *finite* $\boxtimes$ *finite* $\Longleftrightarrow$ *periodic*
- *affine* $\boxtimes$ *finite* $\Longleftrightarrow$ *linearizable, but not periodic*
- *affine* $\boxtimes$ *affine* $\Longleftrightarrow$ *grows as* $\exp(t^2)$
- *wild* $\Longleftrightarrow$ *grows as* $\exp(\exp(t))$

Results

Theorem (G.-Pylyavskyy, 2016)

Periodic $\iff$ *finite* $\boxtimes$ *finite*

Theorem (G.-Pylyavskyy, 2016)

Linearizable $\implies$ *affine* $\boxtimes$ *finite or finite* $\boxtimes$ *finite*

Theorem (G.-Pylyavskyy, 2017)

Grows slower than $\exp(\exp(t))$ $\implies$ *affine* $\boxtimes$ *affine, affine* $\boxtimes$ *finite, or finite* $\boxtimes$ *finite*

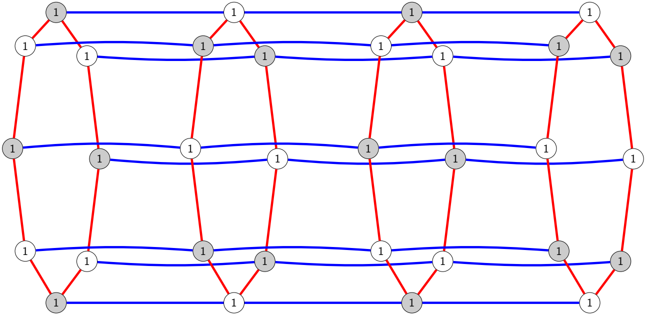
What is left:

Conjecture (G.-Pylyavskyy, 2017)

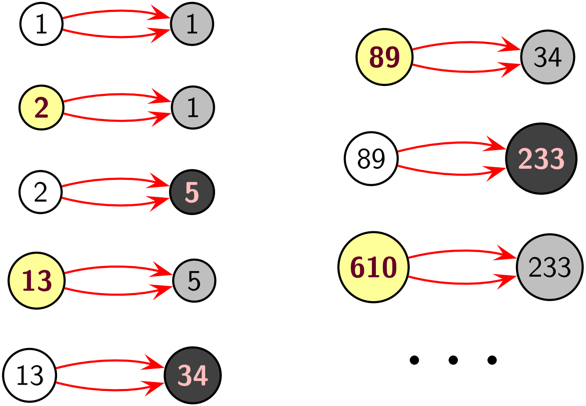
- *affine* $\boxtimes$ *finite* $\implies$ *linearizable*
- *affine* $\boxtimes$ *affine* $\implies$ *grows as* $\exp(t^2)$

Part 1: Type A

Type $A_m \otimes \hat{A}_{2n-1}$



Example: $A_1 \otimes \hat{A}_1$



$$x_{n+1} - 3x_n + x_{n-1} = 0$$

Type $A_m \otimes \hat{A}_{2n-1}$

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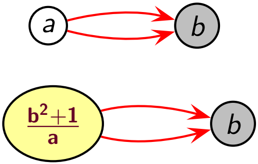

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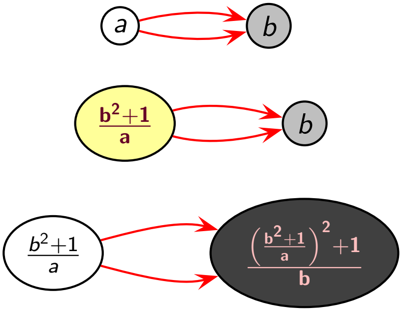

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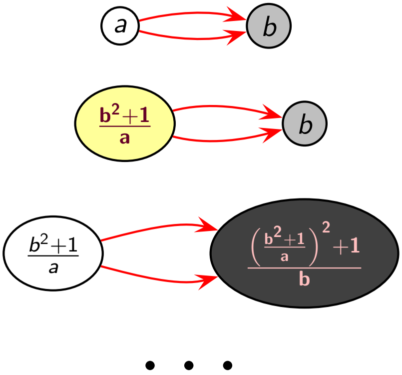

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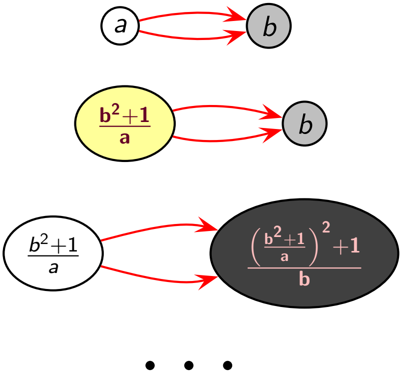

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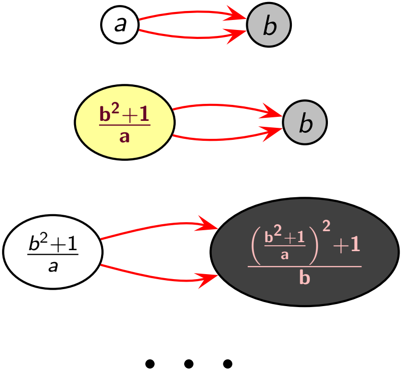
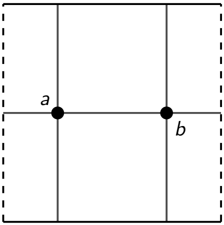


$$x_{n+1} - 3x_n + x_{n-1} = 0$$

$$\mathbf{1} \cdot x_{n+1} - \left(\frac{\mathbf{a}}{\mathbf{b}} + \frac{\mathbf{b}}{\mathbf{a}} + \frac{\mathbf{1}}{\mathbf{ab}}\right) \cdot x_n + \mathbf{1} \cdot x_{n-1} = 0$$

Domino tilings of the cylinder



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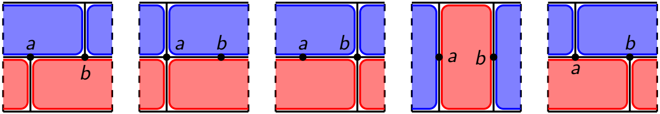

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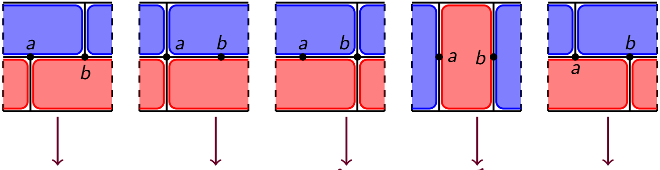

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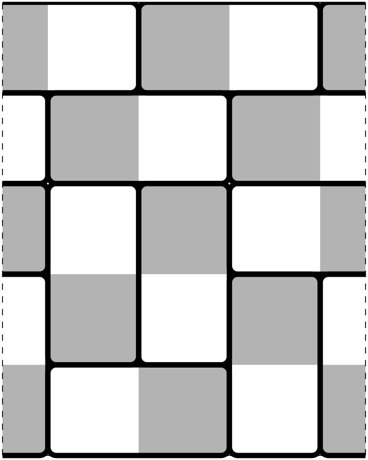
Thurston height



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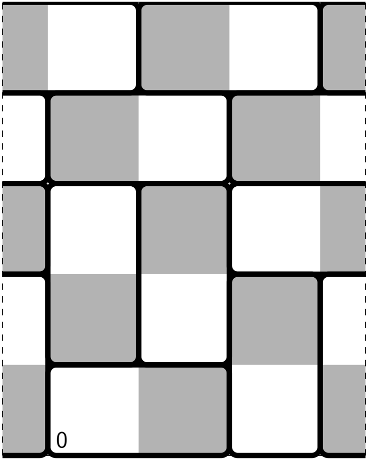

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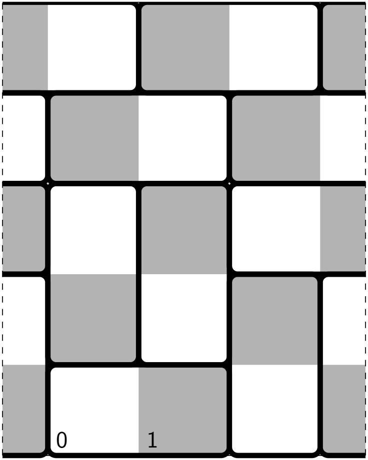

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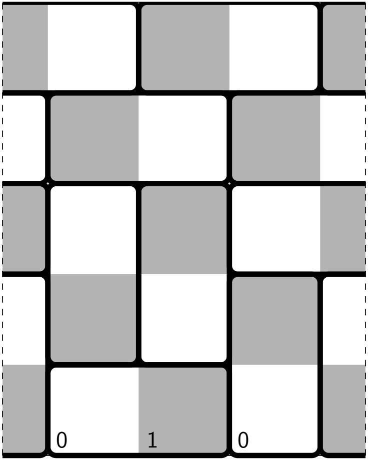

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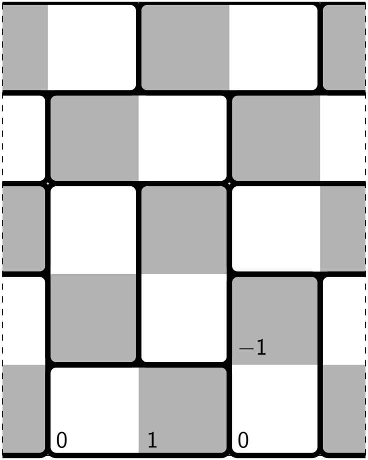

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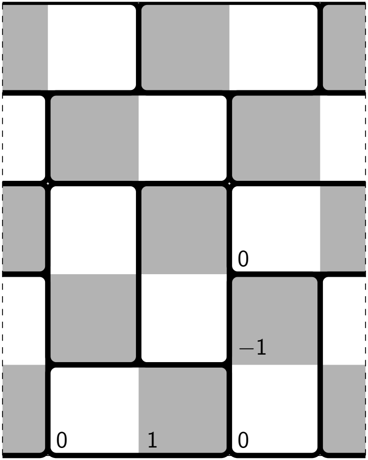

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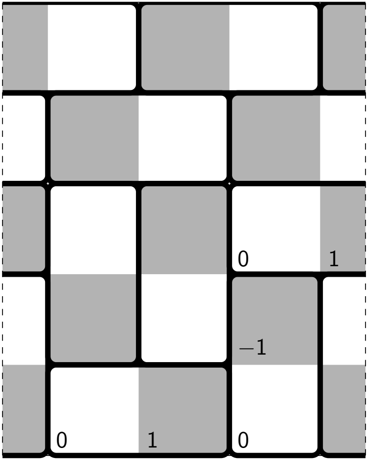

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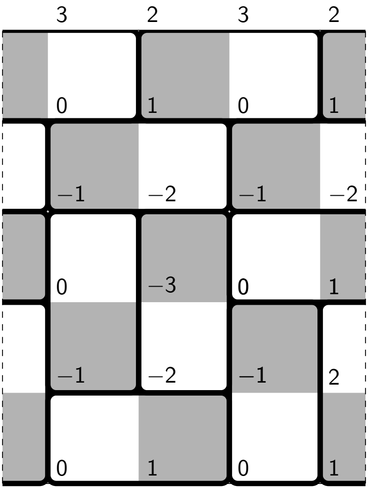

Type $A_m \otimes \hat{A}_{2n-1}$

Theorem (G.-Pylyavskyy, 2016)

- *Recurrence for* **boundary slice***:*

$$x_{t+(m+1)n} - H_1 x_{t+mn} + \ldots \pm H_m x_{t+n} \mp x_t = 0.$$

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$$H_i = \sum_{\substack{T \text{ – cylinder tiling} \\ \text{of Thurston height } i}} \mathrm{wt}(T).$$

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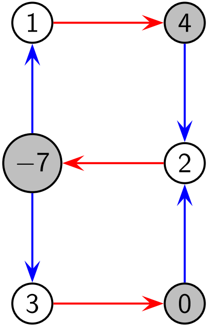
- *Recurrence for* r**-th slice**: *express* $e_j[e_r]$ *in* e_i*'s and send* $e_i \mapsto H_i$.

Part 2: Tropical T-system

Tropical T-system



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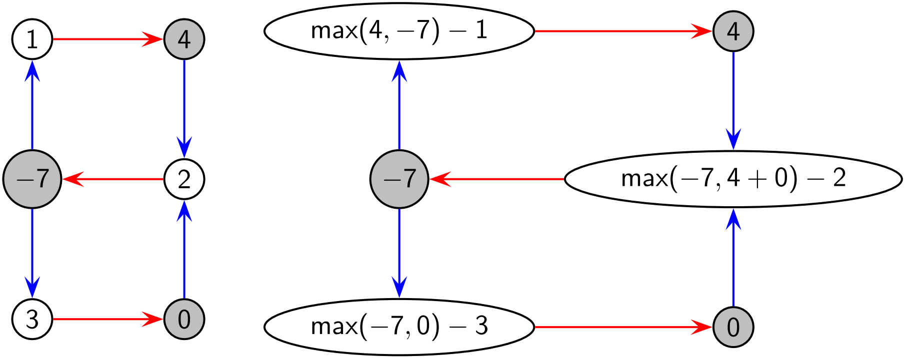

Tropical results

Theorem (G.-Pylyavskyy, 2016)

Tropical T-system is periodic $\iff$ finite $\boxtimes$ finite

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Tropical results

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Theorem (G.-Pylyavskyy, 2017)

Tropical T-system grows slower than $\exp(t) \Longrightarrow \begin{cases} \textit{finite} \boxtimes \textit{finite}, \\ \textit{affine} \boxtimes \textit{finite, or} \\ \textit{affine} \boxtimes \textit{affine.} \end{cases}$

Tropical results

Theorem (G.-Pylyavskyy, 2016)

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$$\textit{Tropical } T\textit{-system grows slower than } \exp(t) \Longrightarrow \begin{cases} \textit{finite} \boxtimes \textit{finite}, \\ \textit{affine} \boxtimes \textit{finite, or} \\ \textit{affine} \boxtimes \textit{affine}. \end{cases}$$

Conjecture (G.-Pylyavskyy, 2017)

- *affine* $\boxtimes$ *finite* $\Longrightarrow$ *tropical T-system grows linearly*
- *affine* $\boxtimes$ *affine* $\Longrightarrow$ *tropical T-system grows quadratically*

Tropical T-system: speed

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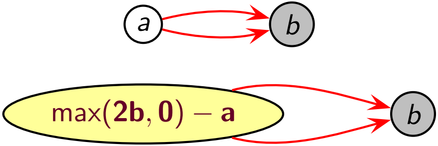
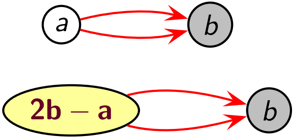


Tropical T-system: speed



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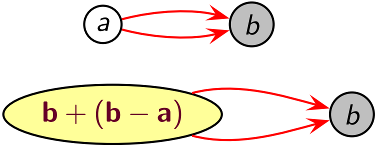

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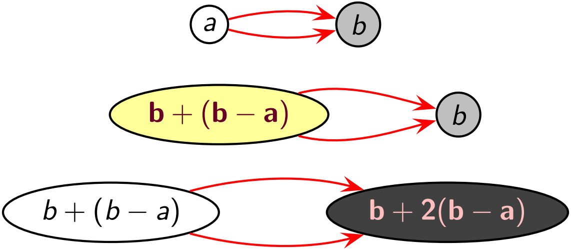

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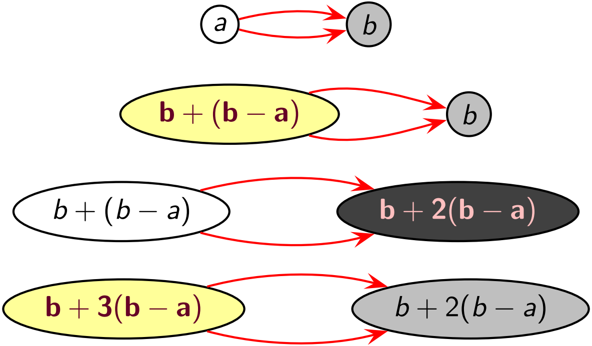

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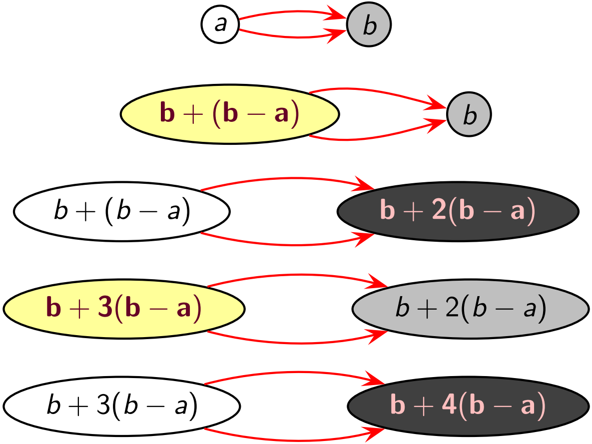

Affine ⊠ finite case

Definition

Blue mutation:

$$\sum_{(u,v)} \lambda_u > \sum_{(v,w)} \lambda_w.$$

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Proposition

The speed is non-decreasing and increases only during blue mutations.

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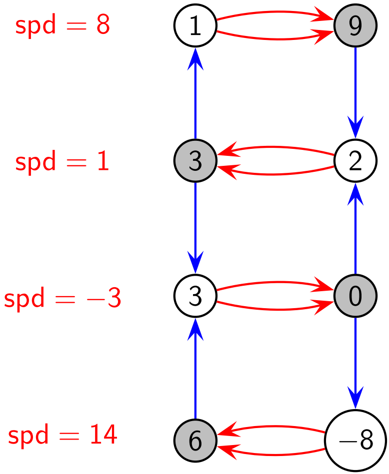
Proposition

The speed is non-decreasing and increases only during blue mutations.

Corollary

Tropical T-system grows linearly $\Leftrightarrow$ there are finitely many blue mutations.

Example



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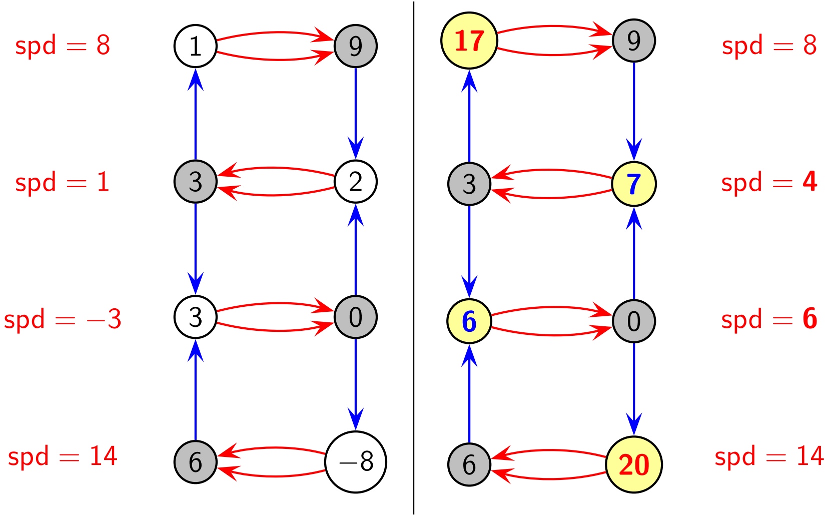
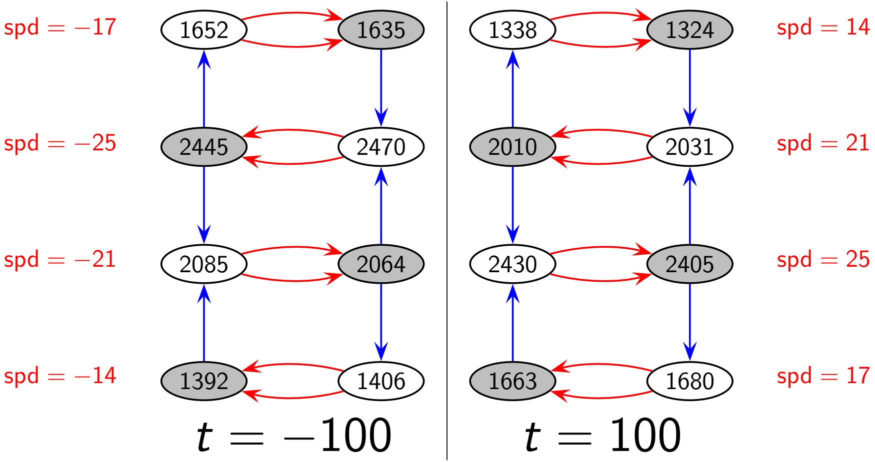

spd = 8
spd = 1
spd = −3
spd = 14
1
9
3
2
3
0
6
−8
17
9
3
7
6
0
6
20
spd = 8
spd = 4
spd = 6
spd = 14

Solitonic behavior



Type $A_m \otimes \hat{A}_{2n-1}$: solitonic behavior

Theorem (G.-Pylyavskyy, 2016)

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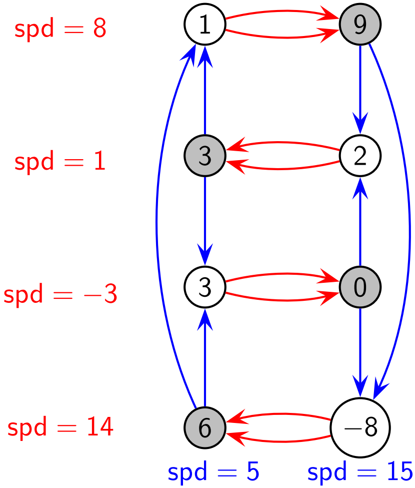
Type $A_m \otimes \hat{A}_{2n-1}$: solitonic behavior

Theorem (G.-Pylyavskyy, 2016)

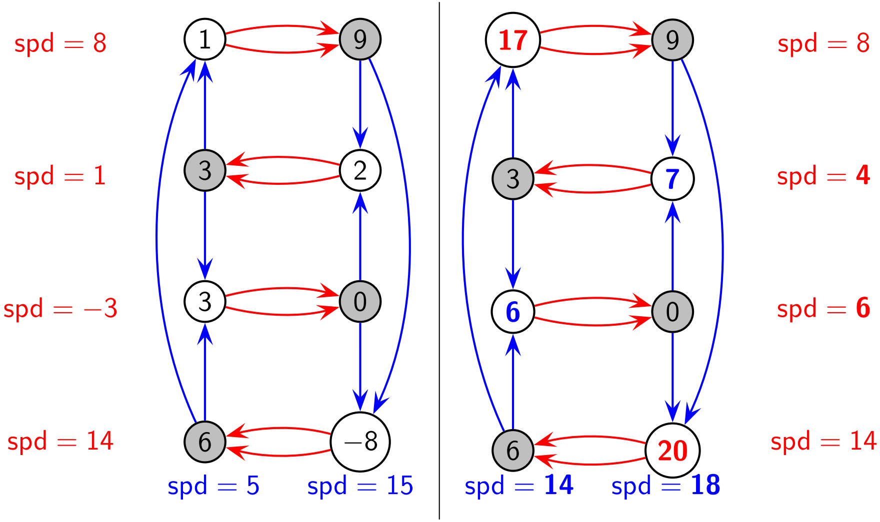
In Type $A_m \otimes \hat{A}_{2n-1}$ we have:

- **("soliton resolution")** *t sufficiently large $\Longrightarrow$ each affine slice moves independently with constant speed*
- **("speed conservation")** *the speed of each slice at $t \to +\infty$ equals the speed of its mirror image at $t \to -\infty$*

Affine ⊠ affine case



Affine ⊠ affine case



Acceleration

Definition

$$\text{Acceleration} = \left| \sum_{(u,v)} \lambda_u - \sum_{(v,w)} \lambda_w \right|.$$

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$$\text{Acceleration} = \left| \sum_{(u,v)} \lambda_u - \sum_{(v,w)} \lambda_w \right|.$$

Proposition

Tropical T-system grows at most quadratically $\Leftarrow$ bounded acceleration.

Tropical problems

Problem

Affine $\boxtimes$ *finite quivers: show that the number of blue mutations is finite.*

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Case $A \otimes \hat{A}$: follows from Pylyavskyy (2016);

Tropical problems

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Case $A_1 \otimes \hat{\Lambda}$: follows from Keller-Scherotzke (2011)

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Problem

Affine $\boxtimes$ finite quivers: prove solitonic behavior.

Tropical problems

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Case $A \otimes \hat{A}$: G.-Pylyavskyy (2016)

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Problem

Affine $\boxtimes$ finite quivers: prove solitonic behavior.

Case $A \otimes \hat{A}$: G.-Pylyavskyy (2016)

Problem

Affine $\boxtimes$ affine quivers: show that the acceleration is bounded.

Tropical problems

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Affine $\boxtimes$ finite quivers: show that the number of blue mutations is finite.

Case $A \otimes \hat{A}$: follows from Pylyavskyy (2016);
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Problem

Affine $\boxtimes$ finite quivers: prove solitonic behavior.

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Problem

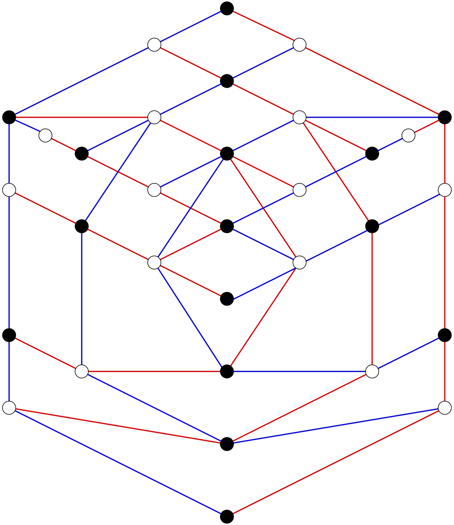
Affine $\boxtimes$ affine quivers: show that the acceleration is bounded.

Case $\hat{A} \otimes \hat{A}$: G.-Pylyavskyy (2017).

Part 3: The classification

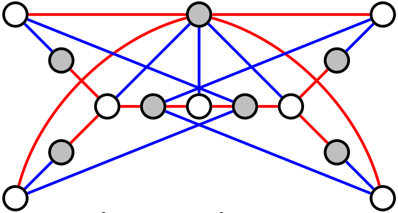
Finite ⊠ finite classification (Stembridge, 2010)

5 infinite families and 11 exceptional quivers

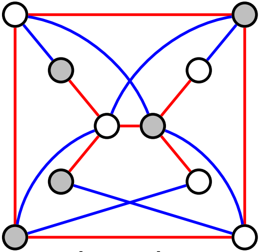


Affine ⊠ finite classification

15 infinite families and 4 exceptional cases



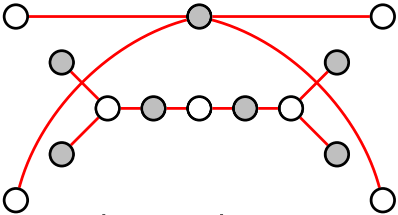
$\hat{D}_{n+1} * \hat{D}_{3n-1}$
for $n = 3$



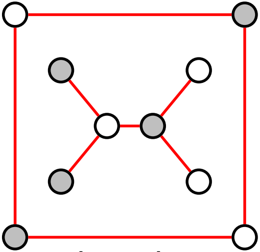
$\hat{A}_3 * \hat{D}_5$

Affine ⊠ finite classification

15 infinite families and 4 exceptional cases



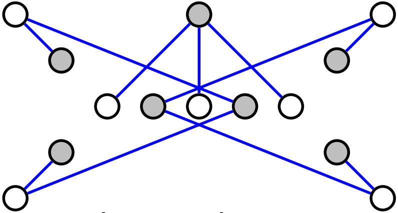
$\hat{D}_{n+1} * \hat{D}_{3n-1}$
for $n = 3$



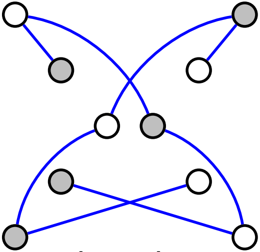
$\hat{A}_3 * \hat{D}_5$

Affine ⊠ finite classification

15 infinite families and 4 exceptional cases



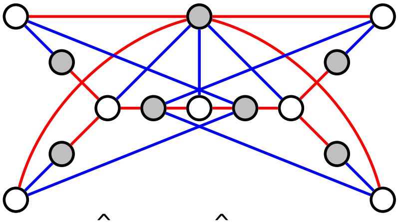
$\hat{D}_{n+1} * \hat{D}_{3n-1}$
for $n = 3$



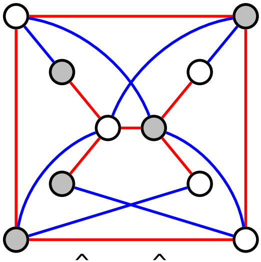
$\hat{A}_3 * \hat{D}_5$

Affine ⊠ finite classification

15 infinite families and 4 exceptional cases

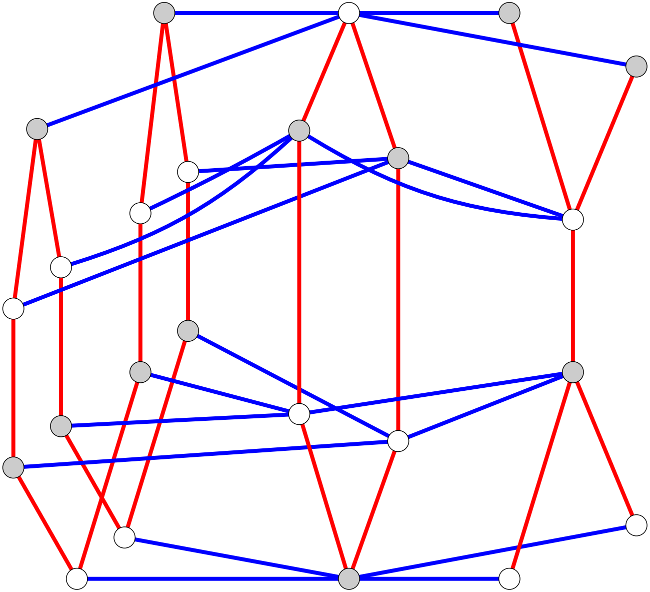


$\hat{D}_{n+1} * \hat{D}_{3n-1}$
for $n = 3$

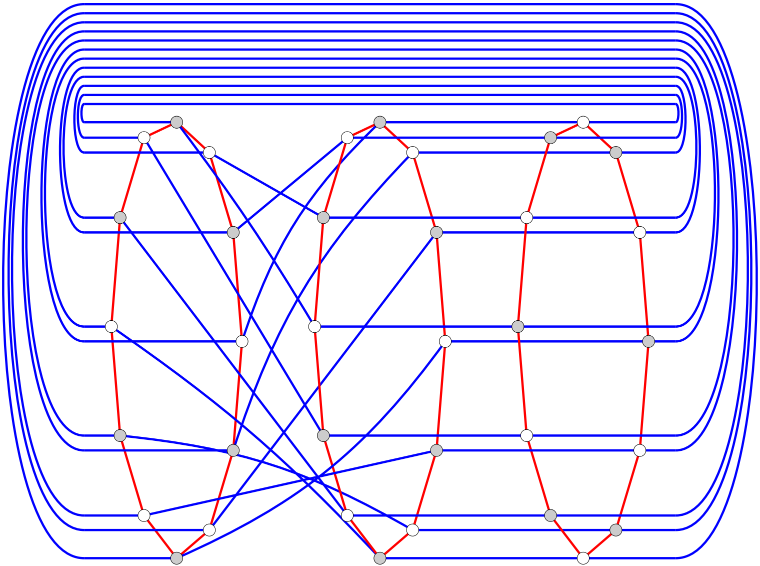


$\hat{A}_3 * \hat{D}_5$

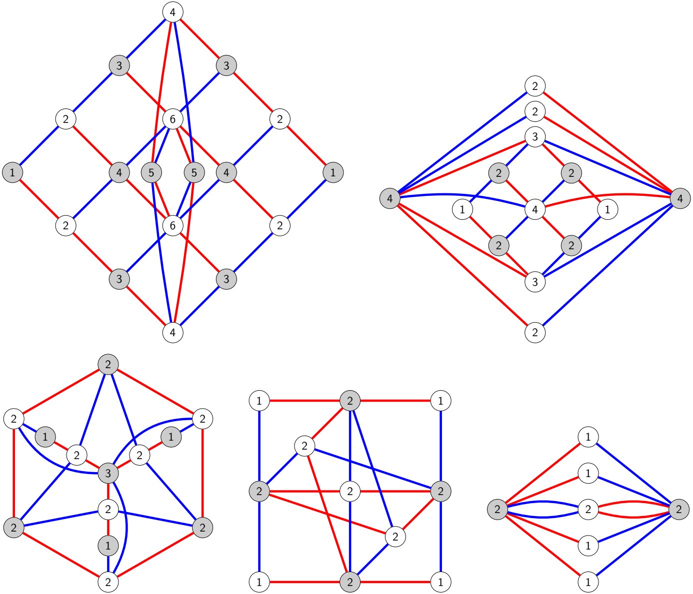
Affine ⊠ affine quivers



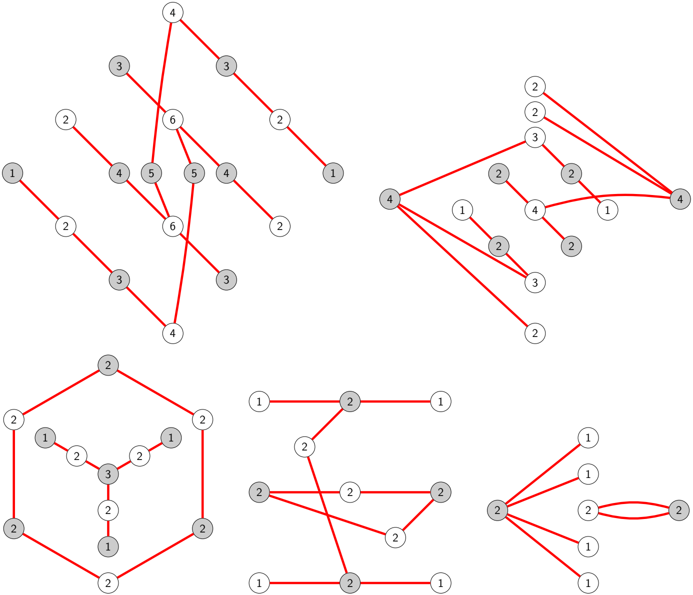
Toric quivers



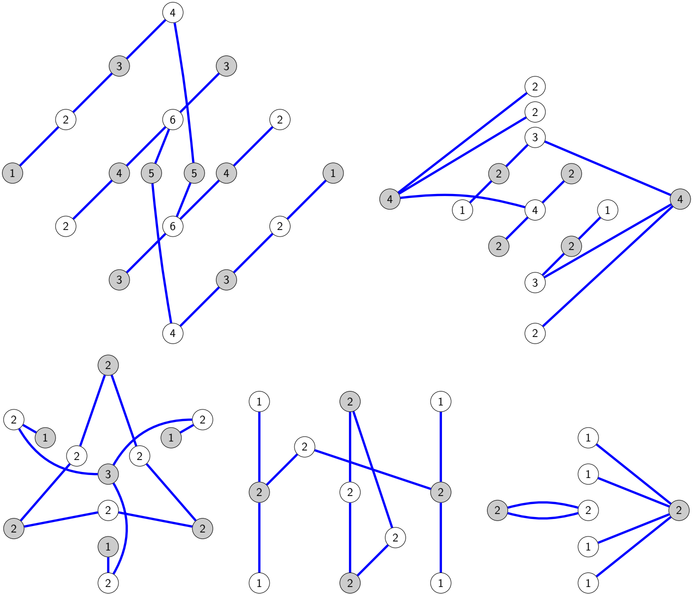
Affine ⊠ affine classification: 40 infinite, 13 exceptional



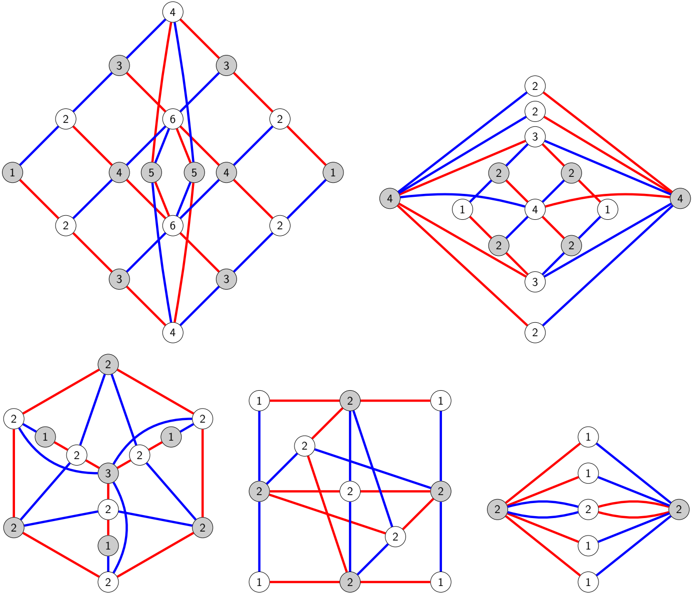
Affine ⊠ affine classification: 40 infinite, 13 exceptional



Affine ⊠ affine classification: 40 infinite, 13 exceptional



Affine ⊠ affine classification: 40 infinite, 13 exceptional



Results

Theorem (G.-Pylyavskyy, 2016)

Periodic $\iff$ *finite* $\boxtimes$ *finite*

Theorem (G.-Pylyavskyy, 2016)

Linearizable $\Longrightarrow$ *affine* $\boxtimes$ *finite or finite* $\boxtimes$ *finite*

Theorem (G.-Pylyavskyy, 2017)

Grows slower than $\exp(\exp(t)) \Longrightarrow$ **affine** $\boxtimes$ **affine, affine** $\boxtimes$ **finite, or finite** $\boxtimes$ **finite**

What is left:

Conjecture (G.-Pylyavskyy, 2017)

- *affine* $\boxtimes$ *finite* $\Longrightarrow$ *linearizable*
- *affine* $\boxtimes$ *affine* $\Longrightarrow$ *grows as* $\exp(t^2)$

Results

Theorem (G.-Pylyavskyy, 2016)

Periodic $\iff$ *finite* $\boxtimes$ *finite*

Theorem (G.-Pylyavskyy, 2016)

Linearizable $\implies$ *affine* $\boxtimes$ *finite or finite* $\boxtimes$ *finite*

Theorem (G.-Pylyavskyy, 2017)

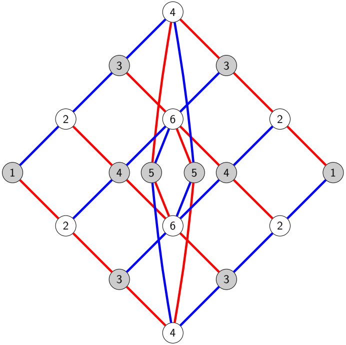
Grows slower than $\exp(\exp(t))$ $\implies$ *affine* $\boxtimes$ *affine, affine* $\boxtimes$ *finite, or finite* $\boxtimes$ *finite*

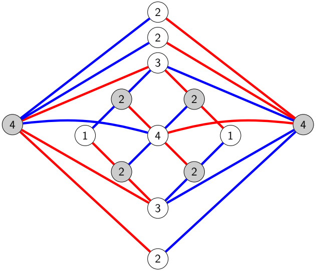
What is left:

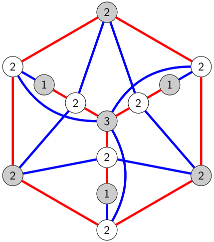
Conjecture (G.-Pylyavskyy, 2017)

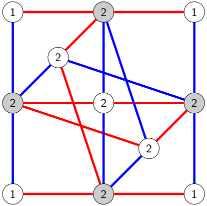
- **affine** $\boxtimes$ **finite** $\implies$ **linearizable**
- **affine** $\boxtimes$ **affine** $\implies$ **grows as** $\exp(t^2)$

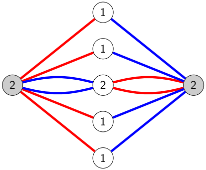
Affine ⊠ affine classification: 40 infinite, 13 exceptional



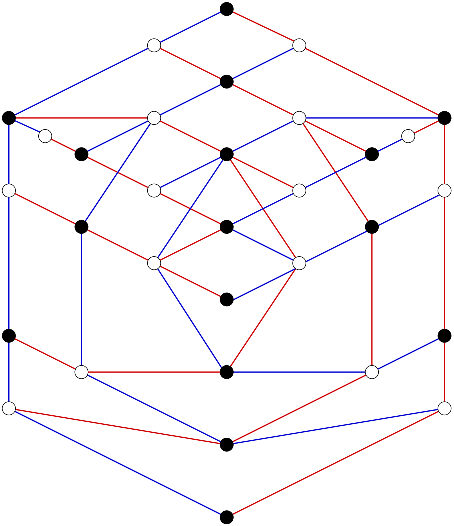








Stembridge's classification (2010)



“Ocneanu graphs” (2001)

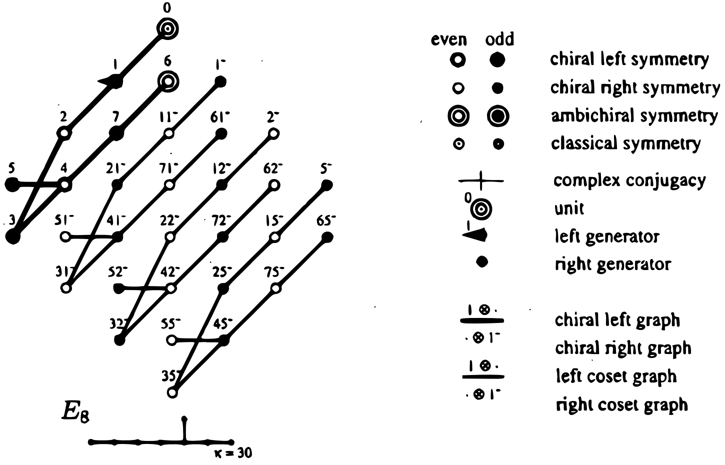


Fig. 23. Quantum symmetries for Coxeter graphs E_8.

Bibliography

Slides: `http://math.mit.edu/~galashin/slides/japan2.pdf`

Pavel Galashin and Pavlo Pylyavskyy.
The classification of Zamolodchikov periodic quivers.
Amer. J. Math., to appear.
arXiv:1603.03942 (2016).

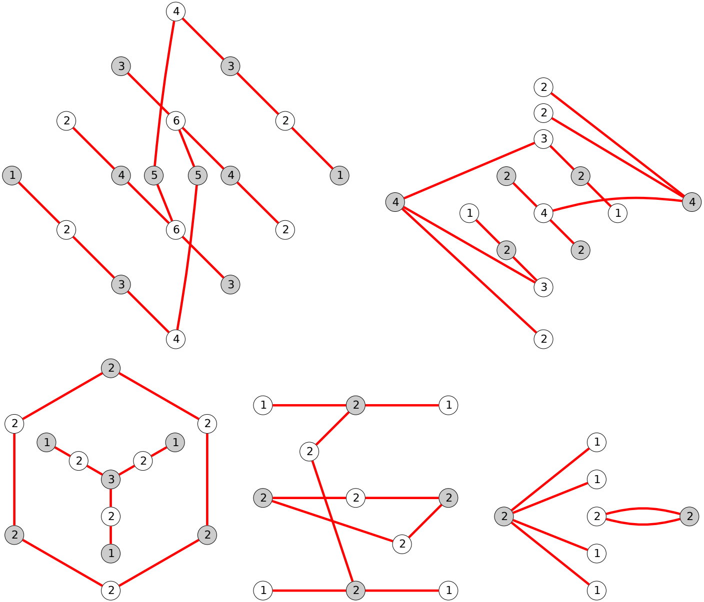
Pavel Galashin and Pavlo Pylyavskyy
Quivers with subadditive labelings: classification and integrability
arXiv:1606.04878 (2016).

Pavel Galashin and Pavlo Pylyavskyy
Quivers with additive labelings: classification and algebraic entropy.
arXiv:1704.05024 (2017).

Pavel Galashin and Pavlo Pylyavskyy
Linear recurrences for cylindrical networks.
Int. Math. Res. Not. IMRN, to appear.
arXiv:1704.05160 (2017).

Pavel Galashin.
Periodicity and integrability for the cube recurrence.
arXiv:1704.05570 (2017).

Thank you!



Bibliography

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Pavel Galashin and Pavlo Pylyavskyy.
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arXiv:1704.05160 (2017).

Pavel Galashin.
Periodicity and integrability for the cube recurrence.
arXiv:1704.05570 (2017).