# Zamolodchikov periodicity and integrability

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MIT

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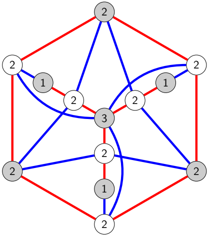
Infinite Analysis 17, Osaka City University, December 5, 2017

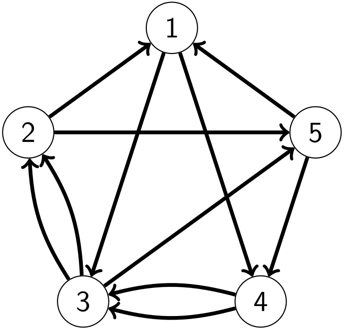
Joint work with Pavlo Pylyavskyy



# Part 1: $T$-systems

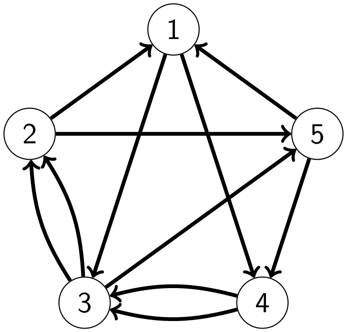
# General $T$-systems (Nakanishi, 2011)

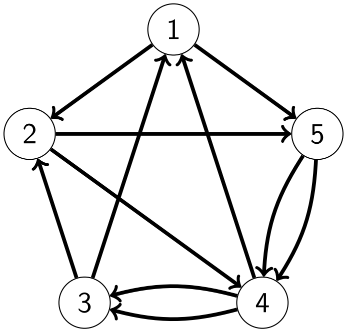
$Q$



# General $T$-systems (Nakanishi, 2011)

$Q$ $\mu_1(Q)$



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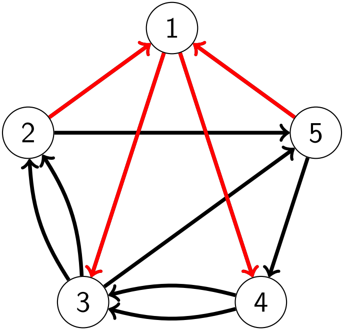


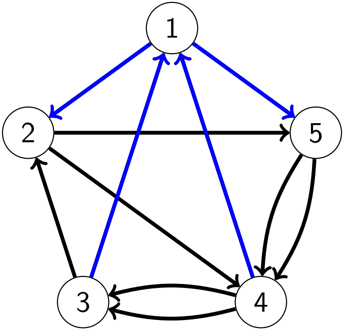

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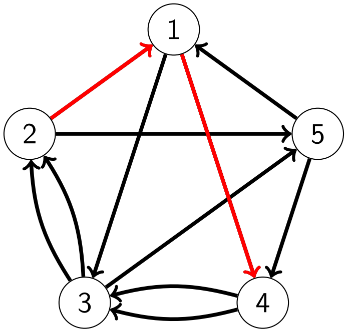
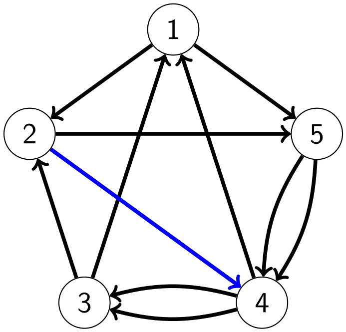



$\mu_1(Q)$



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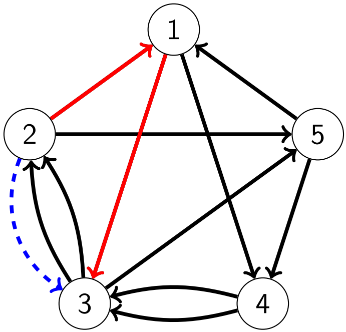


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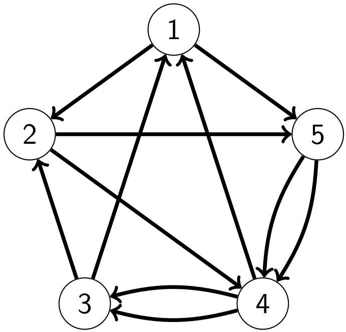

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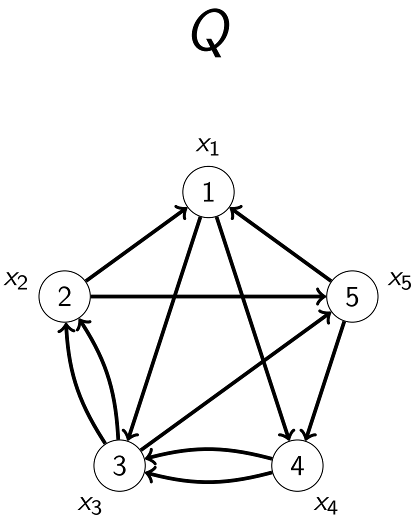
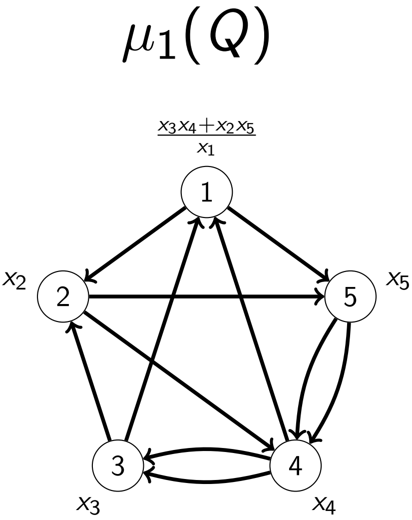
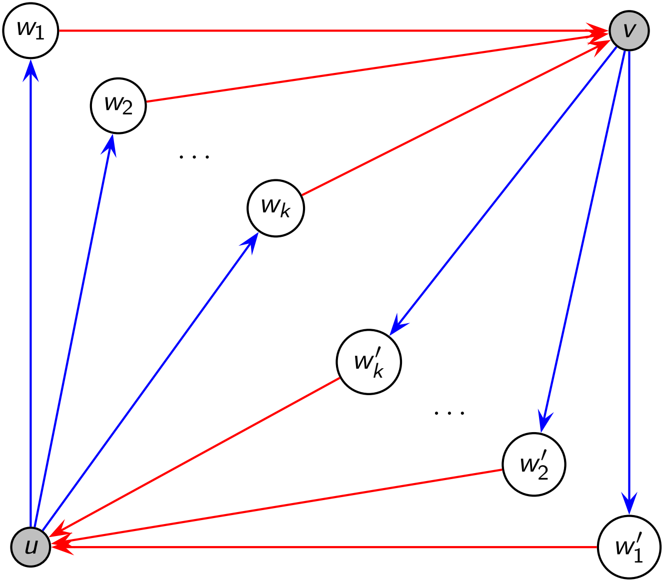
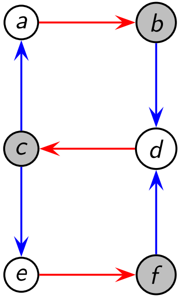


# Bipartite **recurrent** quivers



# Bipartite $T$-system



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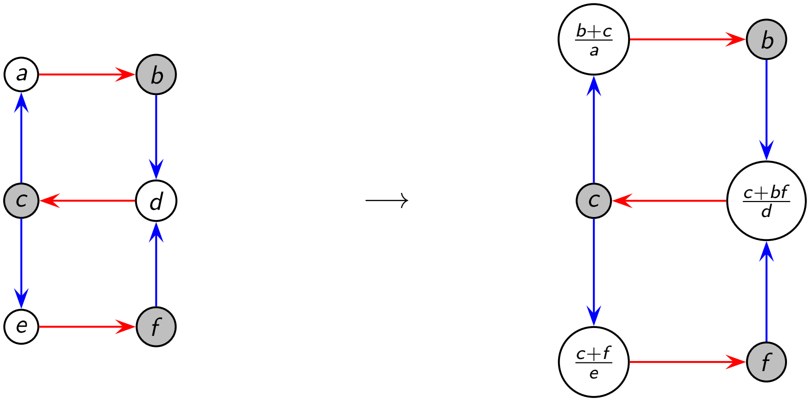
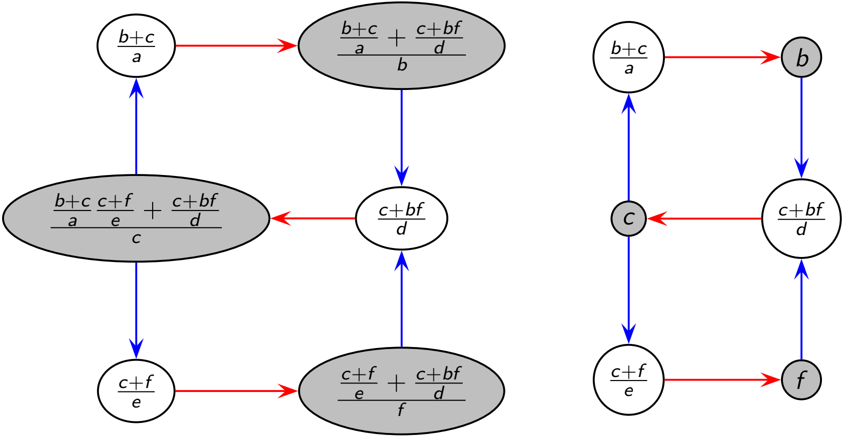
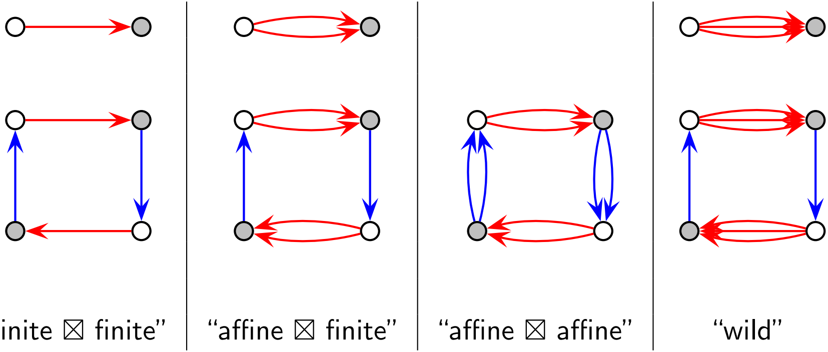


# Bipartite $T$-system

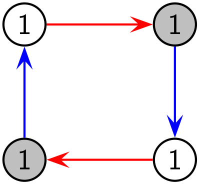


# Four classes of quivers

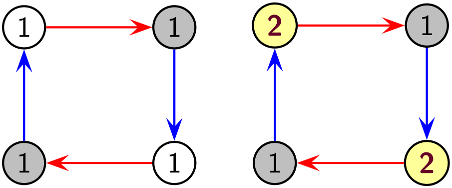


"finite ⊠ finite" | "affine ⊠ finite" | "affine ⊠ affine" | "wild"

# Example: finite ⊠ finite



# Example: finite $\boxtimes$ finite



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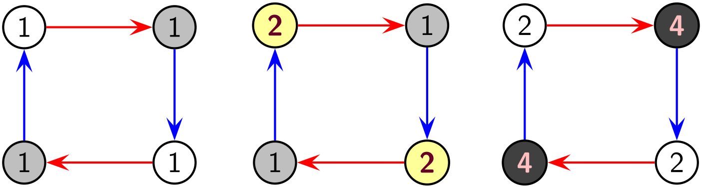

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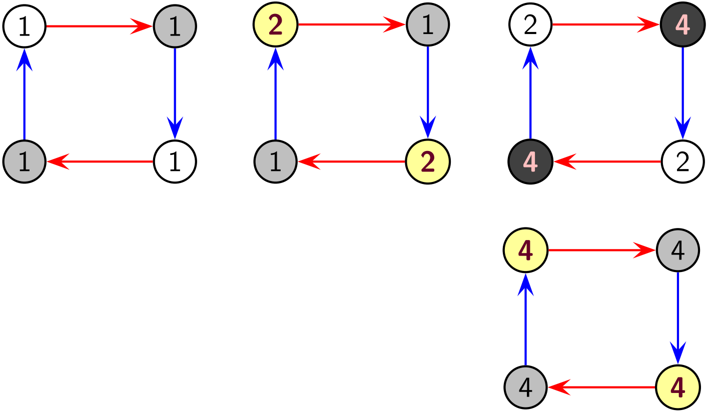

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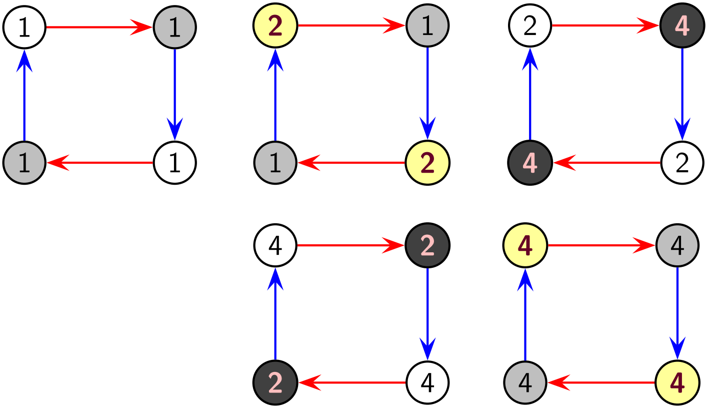

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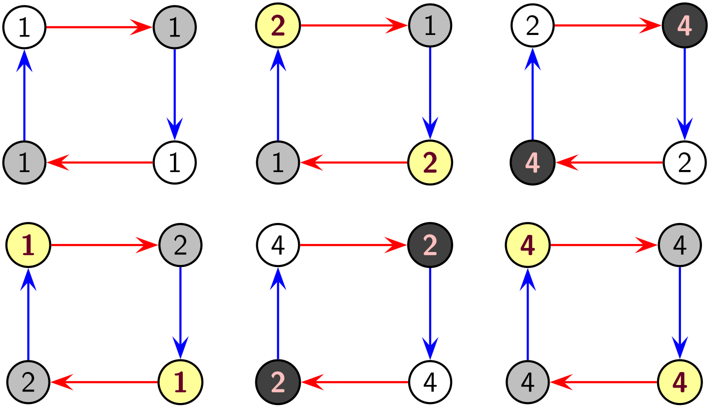

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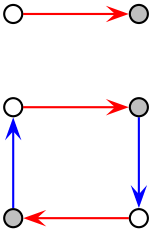
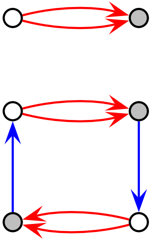
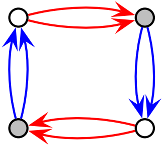


“finite $\boxtimes$ finite”

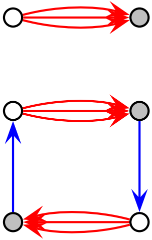
**periodic**



“affine $\boxtimes$ finite”



“affine $\boxtimes$ affine”



“wild”

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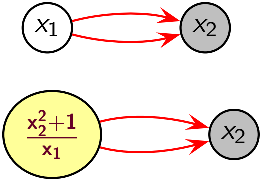

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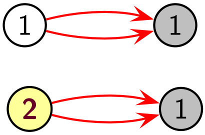

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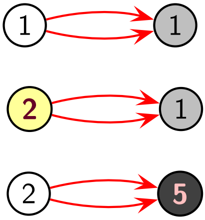

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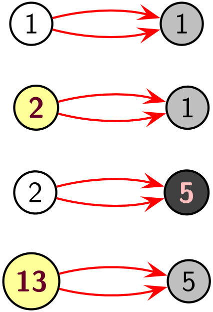

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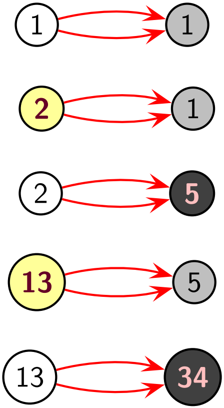

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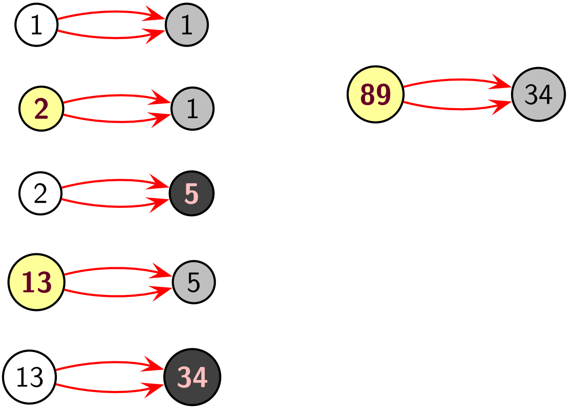

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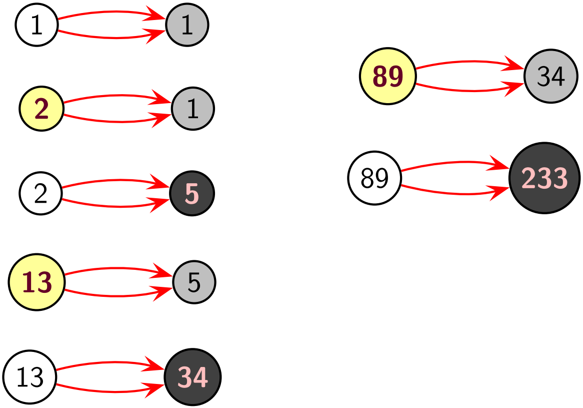

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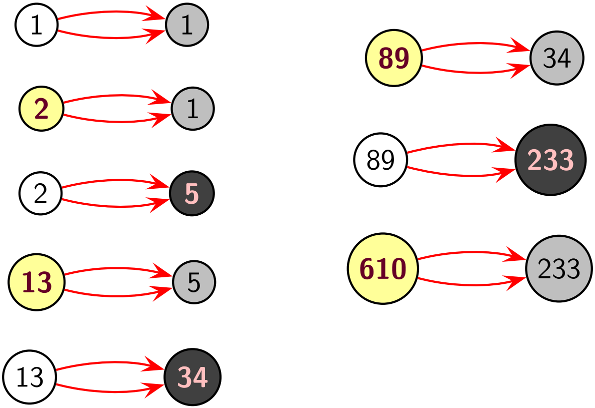

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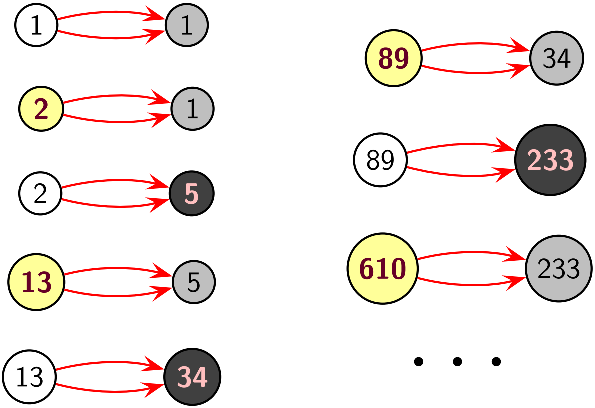

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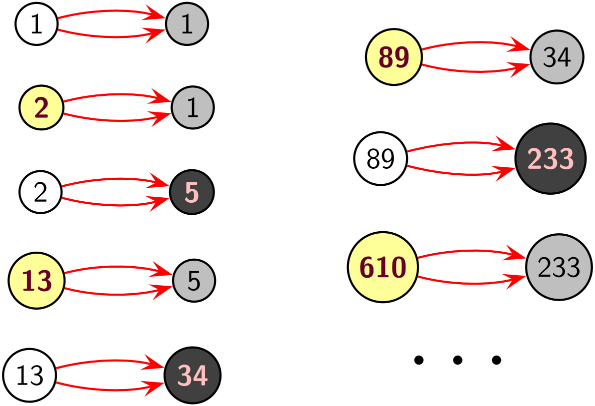
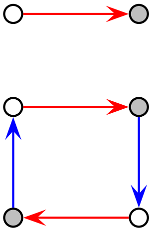


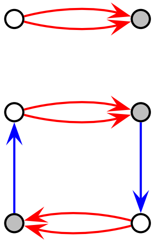

$$x_{n+1} = 3x_n - x_{n-1}$$

# Four classes of quivers



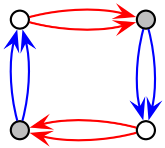
“finite ⊠ finite”

periodic

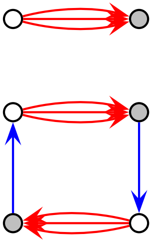


“affine ⊠ finite”

**linearizable**



“affine ⊠ affine”



“wild”

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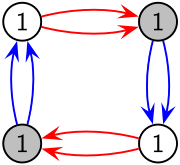

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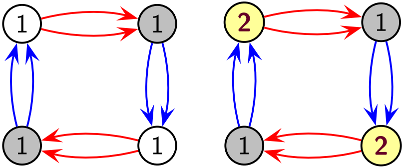

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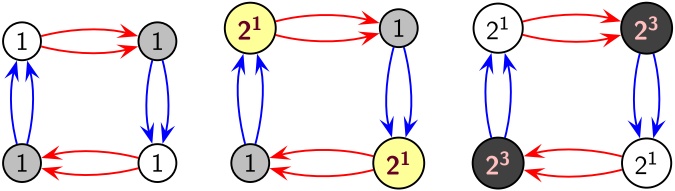

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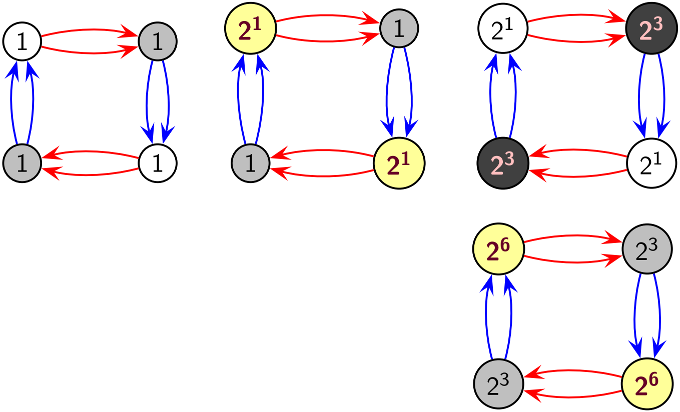
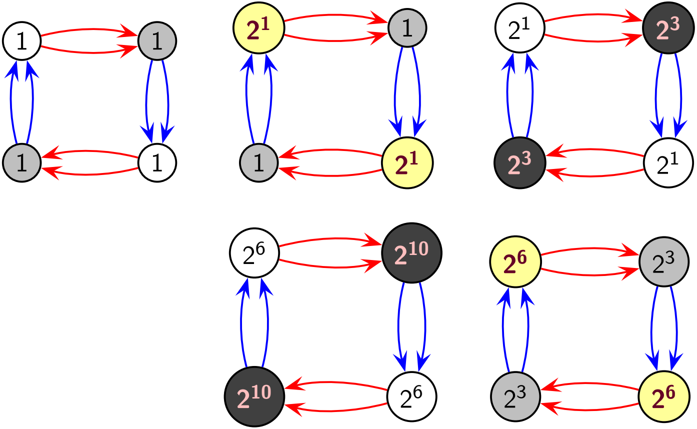


# Example: affine $\boxtimes$ affine



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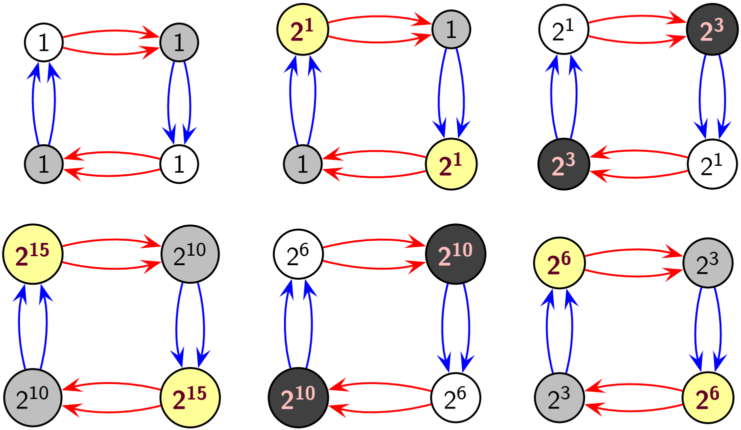

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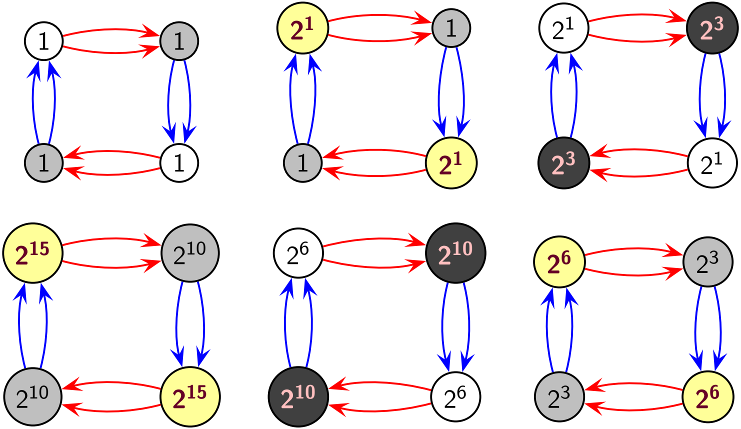
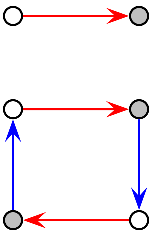


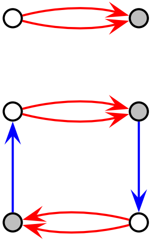

$$2^{\binom{n}{2}}$$

# Four classes of quivers



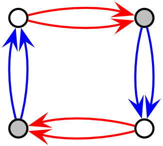
“finite ⊠ finite”

periodic



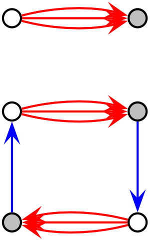
“affine ⊠ finite”

linearizable



“affine ⊠ affine”

**grows as** $\exp(t^2)$



“wild”

# Example: wild

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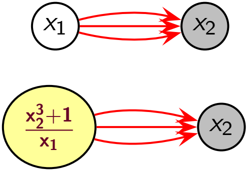

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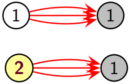

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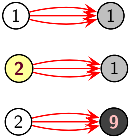

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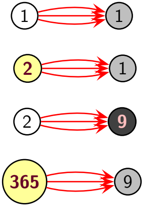

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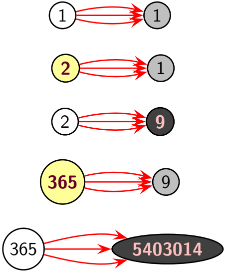

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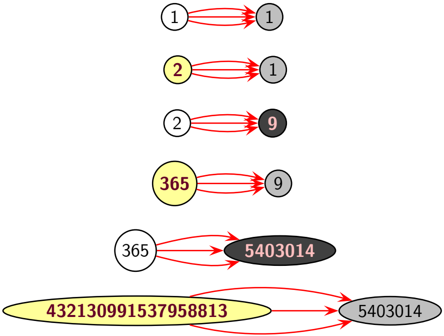

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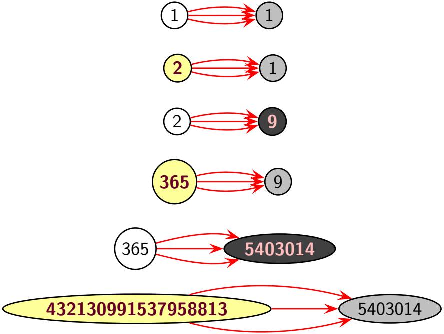


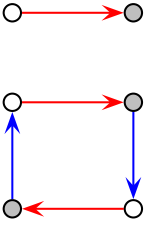

A003818 a(1)=a(2)=1, a(n+1) = (a(n)^3 +1)/a(n-1).

1, 1, 2, 9, 365, 5403014, 432130991537958813, 14935169284101525874491673463268414536523593057 (list; graph; refs; lis

OFFSET 1,3

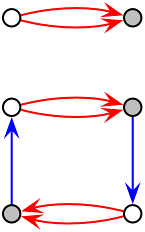
COMMENTS The term a(9) has 121 digits. - Harvey P. Dale,

# Four classes of quivers



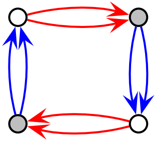
"finite ⊠ finite"

periodic



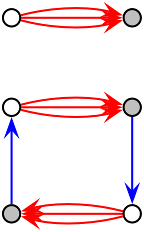
"affine ⊠ finite"

linearizable



"affine ⊠ affine"

grows as $\exp(t^2)$

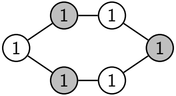
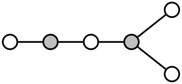
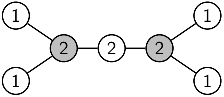
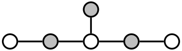
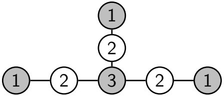
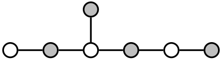
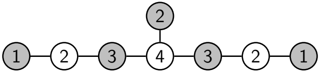
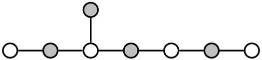
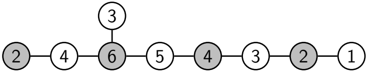


"wild"

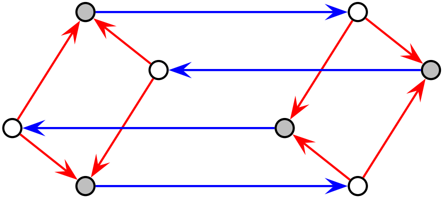
**grows as** $\exp(\exp(t))$

# Part 2: The master conjecture

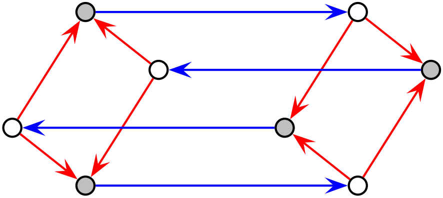
# $ADE$ Dynkin diagrams

| Name | Finite diagram | Affine diagram | Name |
|---|---|---|---|
| $A_n$ |  |  | $\hat{A}_{n-1}$ |
| $D_n$ |  |  | $\hat{D}_{n-1}$ |
| $E_6$ |  |  | $\hat{E}_6$ |
| $E_7$ |  |  | $\hat{E}_7$ |
| $E_8$ |  |  | $\hat{E}_8$ |

# Affine ⊠ finite quivers

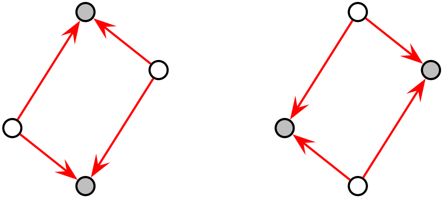


# Affine ⊠ finite quivers



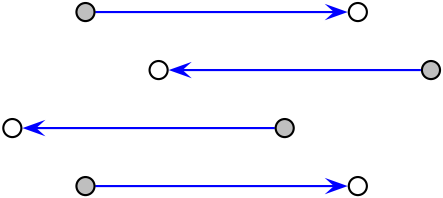
- Bipartite recurrent quiver

# Affine ⊠ finite quivers



- Bipartite recurrent quiver
- All red components are **affine** Dynkin diagrams

# Affine ⊠ finite quivers



- Bipartite recurrent quiver
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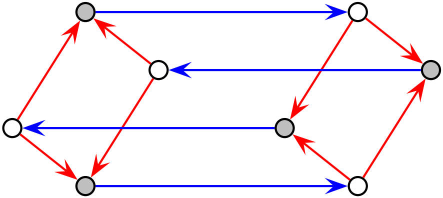
- All blue components are **finite** Dynkin diagrams

# Affine ⊠ finite quivers

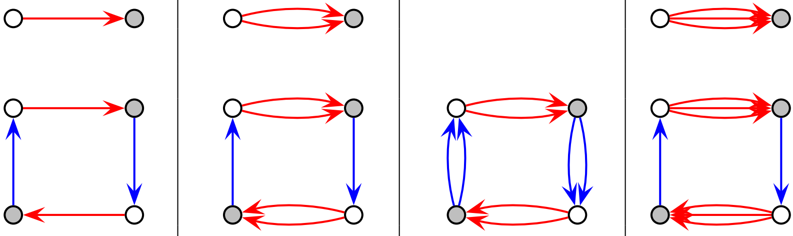


- Bipartite recurrent quiver
- All red components are **affine** Dynkin diagrams
- All blue components are **finite** Dynkin diagrams

↑

"**Affine ⊠ finite quiver**"

# Four classes of quivers



| "finite ⊠ finite" | "affine ⊠ finite" | "affine ⊠ affine" | "wild" |
|---|---|---|---|
| periodic | linearizable | grows as $\exp(t^2)$ | grows as $\exp(\exp(t))$ |

# Master conjecture

**Conjecture (G.-Pylyavskyy, 2016)**

# Master conjecture

## Conjecture (G.-Pylyavskyy, 2016)

- *finite* $\boxtimes$ *finite* $\iff$ *periodic*

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# Master conjecture

**Conjecture (G.-Pylyavskyy, 2016)**

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- *affine* $\boxtimes$ *finite* $\Longleftrightarrow$ *linearizable, but not periodic*
- *affine* $\boxtimes$ *affine* $\Longleftrightarrow$ *grows as* $\exp(t^2)$
- *wild* $\Longleftrightarrow$ *grows as* $\exp(\exp(t))$

# Results

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**Theorem (G.-Pylyavskyy, 2016)**

*Periodic* $\iff$ *finite* $\boxtimes$ *finite*

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# Results

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*Periodic* $\iff$ *finite* $\boxtimes$ *finite*

**Theorem (G.-Pylyavskyy, 2016)**

*Linearizable* $\implies$ *affine* $\boxtimes$ *finite or finite* $\boxtimes$ *finite*

**Theorem (G.-Pylyavskyy, 2017)**

*Grows slower than* $\exp(\exp(t)) \implies$ *affine* $\boxtimes$ *affine, affine* $\boxtimes$ *finite, or finite* $\boxtimes$ *finite*

# Results

**Theorem (G.-Pylyavskyy, 2016)**

*Periodic* $\iff$ *finite* $\boxtimes$ *finite*

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*Linearizable* $\Longrightarrow$ *affine* $\boxtimes$ *finite or finite* $\boxtimes$ *finite*

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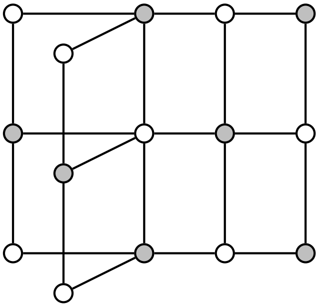
*Grows slower than* $\exp(\exp(t))$ $\Longrightarrow$ *affine* $\boxtimes$ *affine, affine* $\boxtimes$ *finite, or finite* $\boxtimes$ *finite*

What is left:

**Conjecture (G.-Pylyavskyy, 2017)**

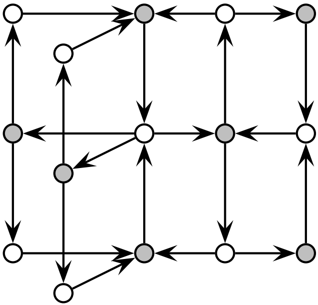
- *affine* $\boxtimes$ *finite* $\Longrightarrow$ *linearizable*
- *affine* $\boxtimes$ *affine* $\Longrightarrow$ *grows as* $\exp(t^2)$

# Tensor product



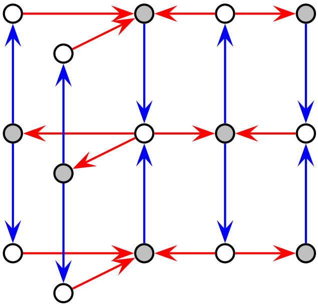
$D_5 \otimes A_3$

# Tensor product



$D_5 \otimes A_3$

# Tensor product



$D_5 \otimes A_3$

# Zamolodchikov periodicity

**Theorem (B. Keller, 2013)**

*Tensor product of* **finite** *Dynkin diagrams* $\Longrightarrow$ *the $T$-system is periodic.*

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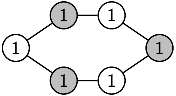
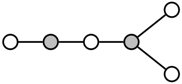
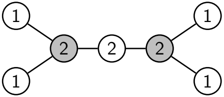
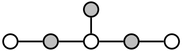
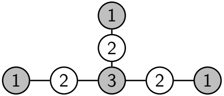
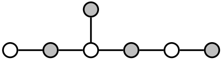
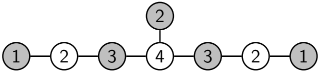
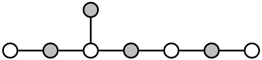
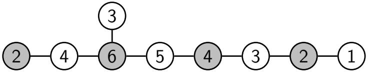
- Fomin-Zelevinsky (2003): $\Lambda \otimes A_1$;
- Volkov (2005): $A_n \otimes A_m$;

# *ADE* Dynkin diagrams

| Name | Finite diagram | h | Affine diagram | Name |
| --- | --- | --- | --- | --- |
| $A_n$ |  | $\mathbf{n+1}$ |   | $\hat{A}_{n-1}$ |
| $D_n$ |  | $\mathbf{2n-2}$ |   | $\hat{D}_{n-1}$ |
| $E_6$ |  | **12** |   | $\hat{E}_6$ |
| $E_7$ |  | **18** |   | $\hat{E}_7$ |
| $E_8$ |  | **30** |   | $\hat{E}_8$ |

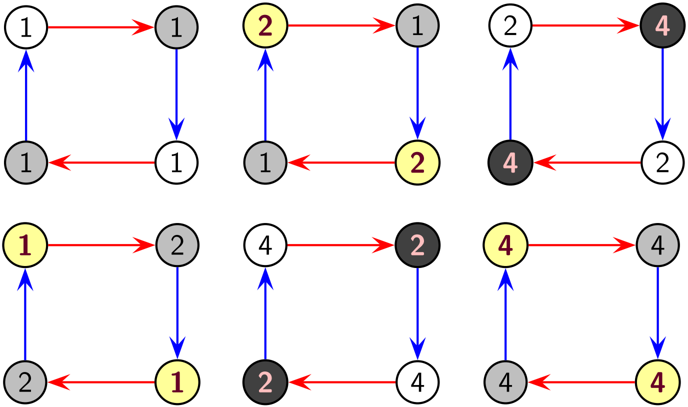
# Zamolodchikov periodicity

**Theorem (B. Keller, 2013)**

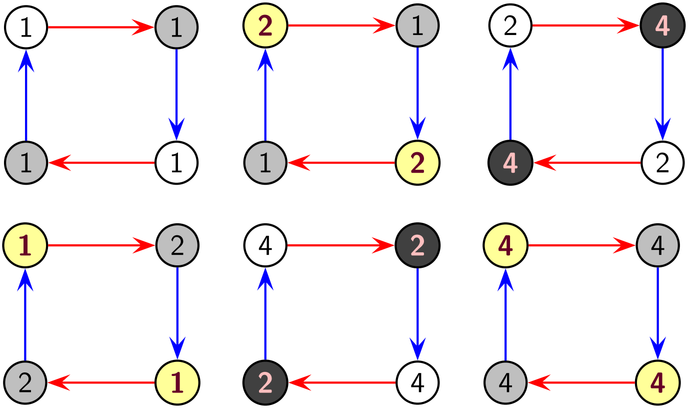
*Tensor product of* **finite** *Dynkin diagrams* $\Longrightarrow$ *the* $T$*-system is periodic* **with period dividing**

$$2(h + h').$$

# Example: $A_2 \otimes A_2$

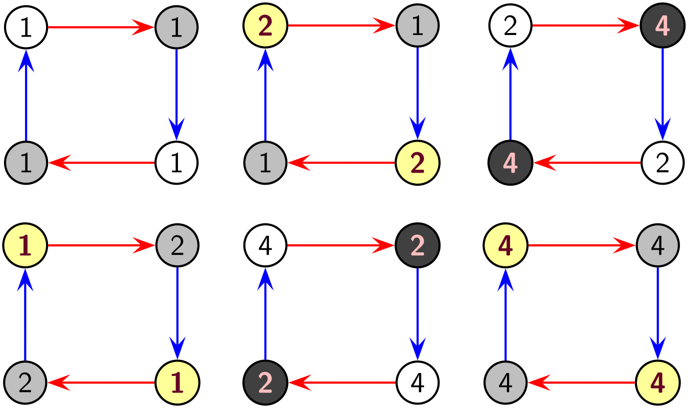


# Example: $A_2 \otimes A_2$




**6 steps!**

# Example: $A_2 \otimes A_2$




~~6~~ 12 **steps!**

# Results

**Theorem (G.-Pylyavskyy, 2016)**

**Periodic $\iff$ finite $\boxtimes$ finite**

**Theorem (G.-Pylyavskyy, 2016)**

*Linearizable $\implies$ affine $\boxtimes$ finite or finite $\boxtimes$ finite*

**Theorem (G.-Pylyavskyy, 2017)**

*Grows slower than* $\exp(\exp(t)) \implies$ *affine $\boxtimes$ affine, affine $\boxtimes$ finite, or finite $\boxtimes$ finite*

What is left:

**Conjecture (G.-Pylyavskyy, 2017)**

- *affine $\boxtimes$ finite $\implies$ linearizable*
- *affine $\boxtimes$ affine $\implies$ grows as* $\exp(t^2)$

# Part 3: Periodicity

# The result

**Theorem**

*Let $Q$ be a bipartite recurrent quiver. Then the following are equivalent.*

1.
2.
3.
4.
5. *The $T$-system associated with $Q$ is periodic.*

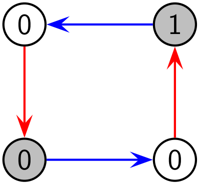
# The result

**Theorem**

*Let $Q$ be a bipartite recurrent quiver. Then the following are equivalent.*

1.
2.
3.
4. **The tropical $T$-system is periodic for any initial value.**
5. *The $T$-system associated with $Q$ is periodic.*

# Example: $A_2 \otimes A_2$



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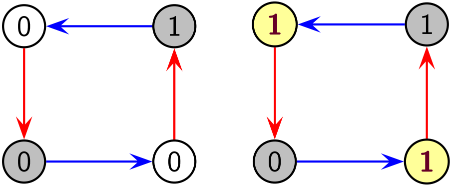

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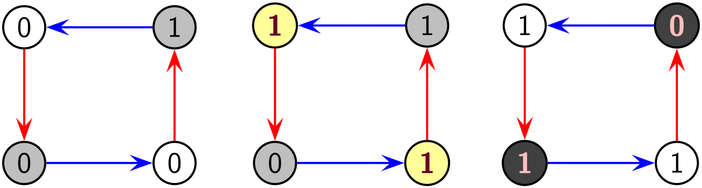

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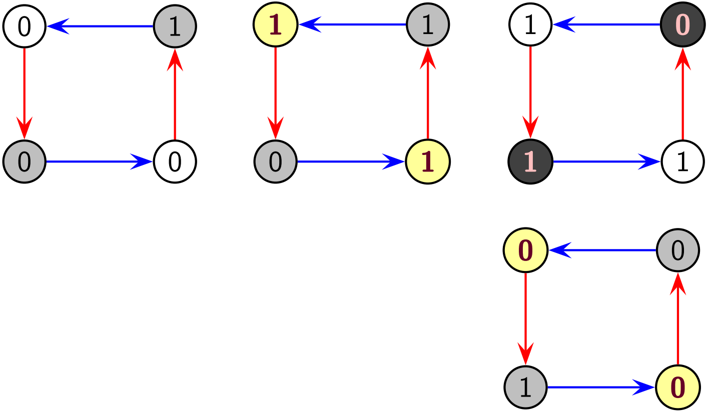
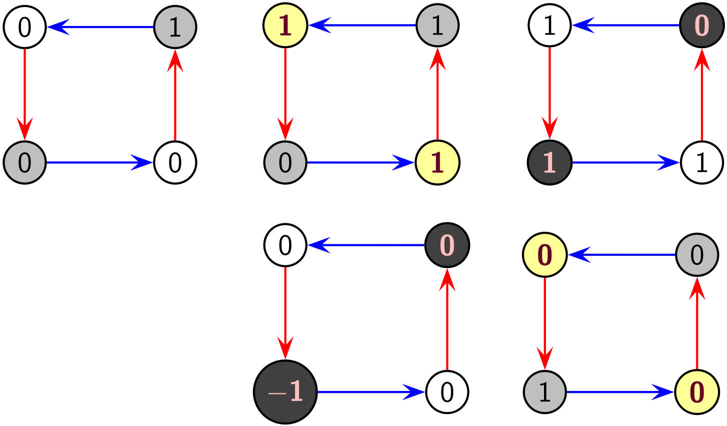
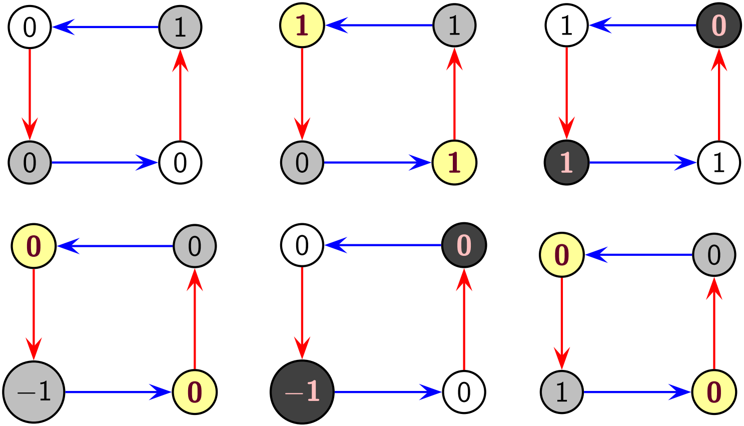


# Example: $A_2 \otimes A_2$



# Example: $A_2 \otimes A_2$



# The result

**Theorem**

*Let $Q$ be a bipartite recurrent quiver. Then the following are equivalent.*

1. 
2. 
3. 
4. *The tropical $T$-system is periodic for any initial value.*
5. *The $T$-system associated with $Q$ is periodic.*

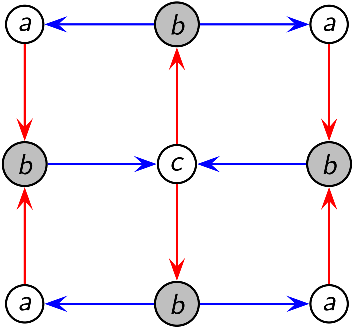
# The result

**Theorem**

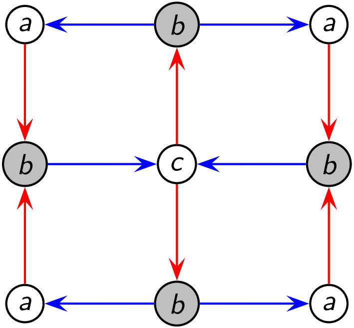
*Let $Q$ be a bipartite recurrent quiver. Then the following are equivalent.*

1.
2.
3. $Q$ **has a fixed point.**
4. *The tropical $T$-system is periodic for any initial value.*
5. *The $T$-system associated with $Q$ is periodic.*

# Fixed point

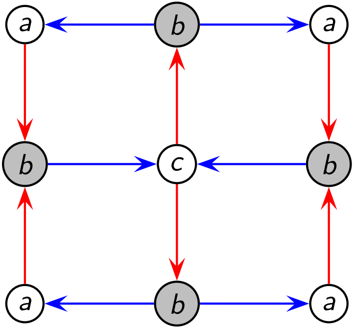


# Fixed point




$$a^2 = b + b; \quad b^2 = a^2 + c; \quad c^2 = b^2 + b^2.$$

# Fixed point




$$a^2 = b + b; \quad b^2 = a^2 + c; \quad c^2 = b^2 + b^2.$$

$$a = \sqrt{4 + 2\sqrt{2}}; \quad b = 2 + \sqrt{2}; \quad c = 2 + 2\sqrt{2}.$$

# The result

**Theorem**

*Let $Q$ be a bipartite recurrent quiver. Then the following are equivalent.*

1.
2.
3. *$Q$ has a fixed point.*
4. *The tropical $T$-system is periodic for any initial value.*
5. *The $T$-system associated with $Q$ is periodic.*

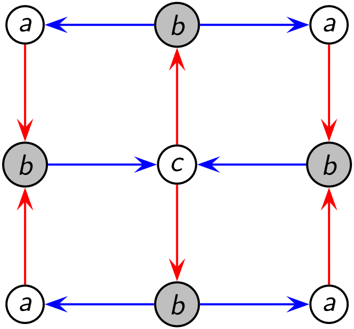
# The result

**Theorem**

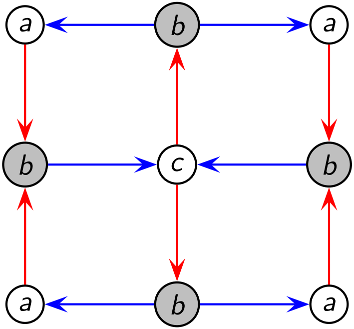
*Let $Q$ be a bipartite recurrent quiver. Then the following are equivalent.*

1. 
2. $Q$ **has a strictly subadditive labeling.**
3. *$Q$ has a fixed point.*
4. *The tropical $T$-system is periodic for any initial value.*
5. *The $T$-system associated with $Q$ is periodic.*

# Strictly subadditive labeling

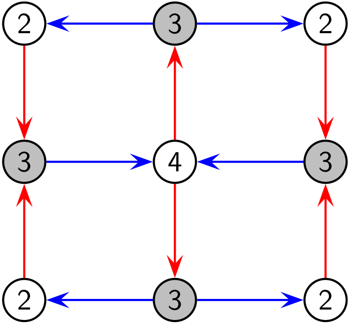


# Strictly subadditive labeling




$$2a > \max(b, b); \quad 2b > \max(a + a, c); \quad 2c > \max(b + b, b + b).$$

# Strictly subadditive labeling




$$2a > \max(b, b); \quad 2b > \max(a + a, c); \quad 2c > \max(b + b, b + b).$$

$$a = 2; \quad b = 3; \quad c = 4.$$

# The result

**Theorem**

*Let $Q$ be a bipartite recurrent quiver. Then the following are equivalent.*

1. 
2. *$Q$ has a strictly subadditive labeling.*
3. *$Q$ has a fixed point.*
4. *The tropical $T$-system is periodic for any initial value.*
5. *The $T$-system associated with $Q$ is periodic.*

# The result

**Theorem**

*Let $Q$ be a bipartite recurrent quiver. Then the following are equivalent.*

1. **The red and blue components of $Q$ are finite Dynkin diagrams**
2. *$Q$ has a strictly subadditive labeling.*
3. *$Q$ has a fixed point.*
4. *The tropical $T$-system is periodic for any initial value.*
5. *The $T$-system associated with $Q$ is periodic.*

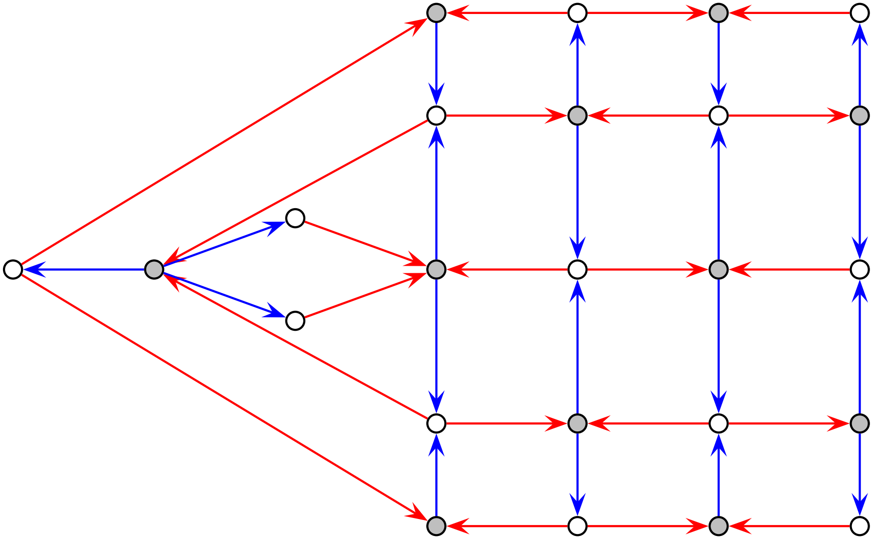
# The result

**Theorem**

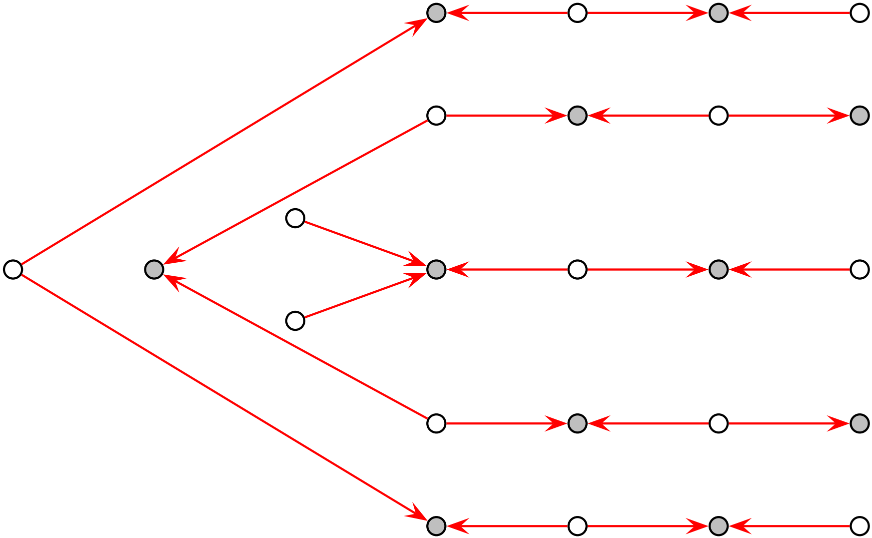
*Let $Q$ be a bipartite recurrent quiver. Then the following are equivalent.*

1. **$Q$ is a finite $\boxtimes$ finite quiver.**
2. *$Q$ has a strictly subadditive labeling.*
3. *$Q$ has a fixed point.*
4. *The tropical $T$-system is periodic for any initial value.*
5. *The $T$-system associated with $Q$ is periodic.*

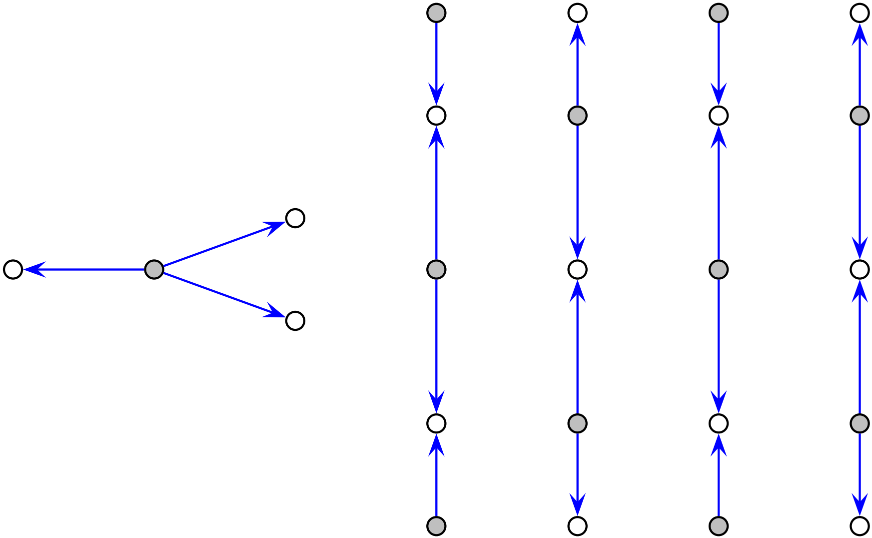
# Finite ⊠ finite quivers



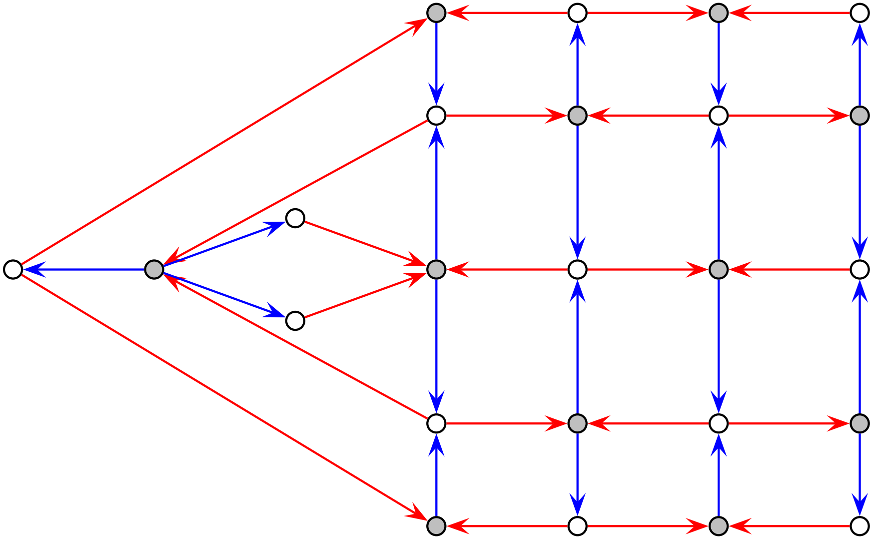
# Finite ⊠ finite quivers



# Finite ⊠ finite quivers



# Finite ⊠ finite quivers



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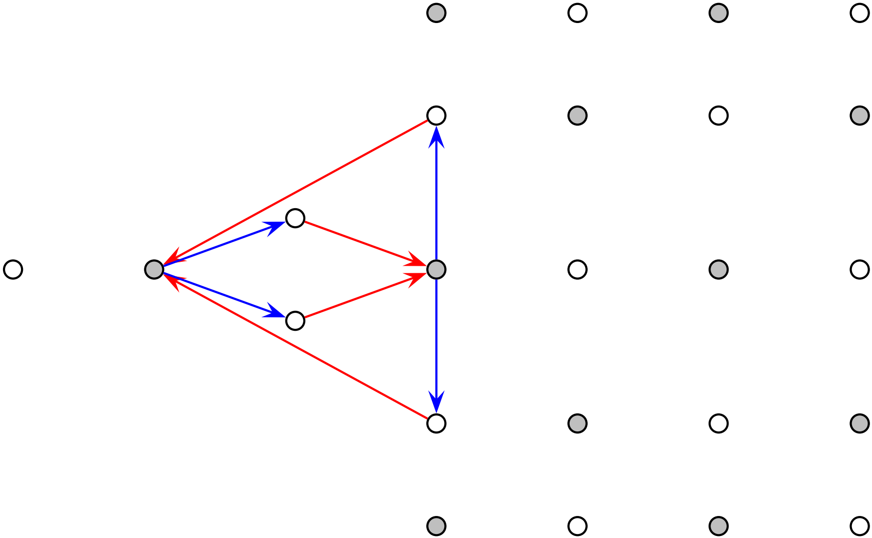

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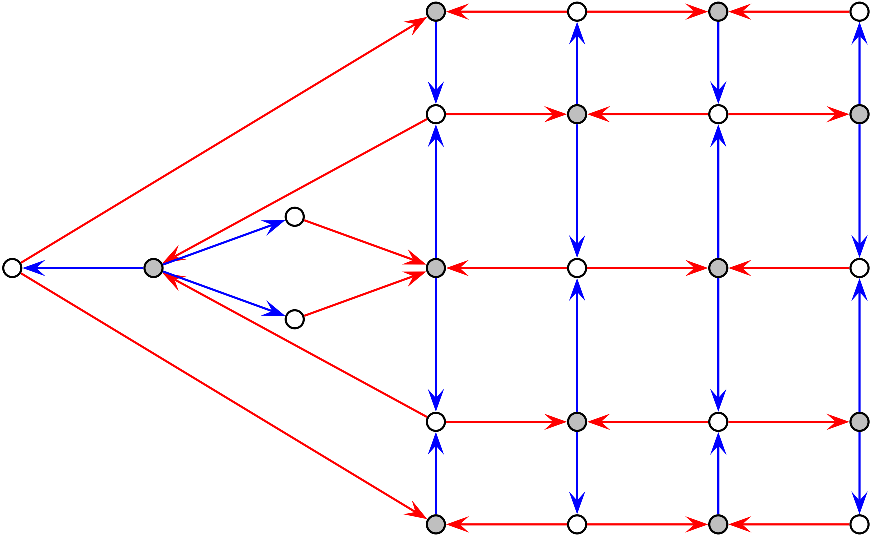

# The classification of Zamolodchikov periodic quivers

**Theorem (G.-Pylyavskyy, 2016)**

*Let $Q$ be a bipartite recurrent quiver. Then the following are equivalent.*

1. *$Q$ is a finite $\boxtimes$ finite quiver.*
2. *$Q$ has a strictly subadditive labeling.*
3. *$Q$ has a fixed point.*
4. *The tropical $T$-system is periodic for any initial value.*
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$$2(h + h').$$

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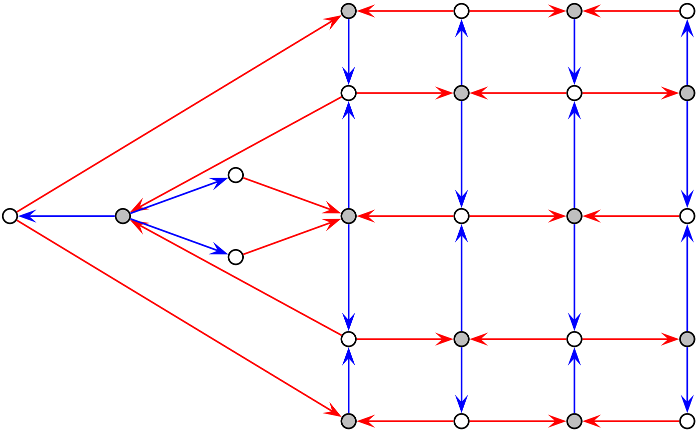

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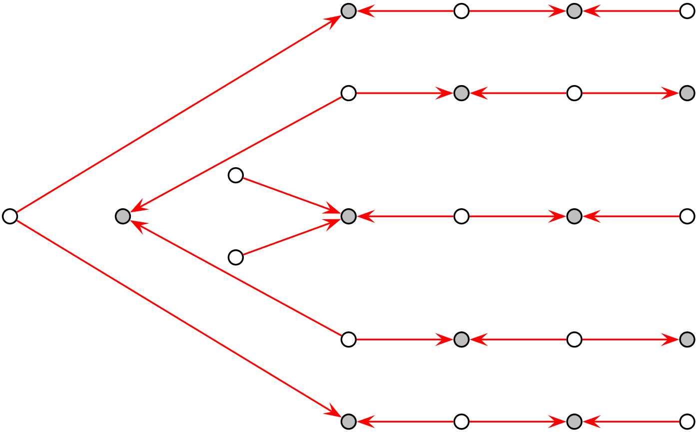

$\mathbf{h} = \mathbf{9} + \mathbf{1} = \mathbf{12} - \mathbf{2} = \mathbf{10}$;

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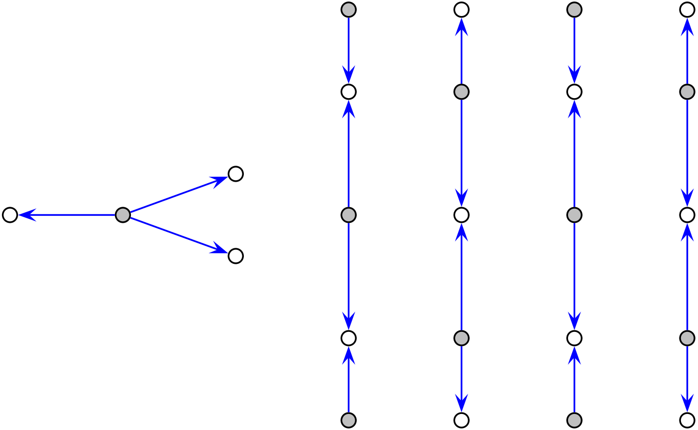

$h = 9 + 1 = 12 - 2 = 10$; $\mathbf{h' = 5 + 1 = 8 - 2 = 6}$;

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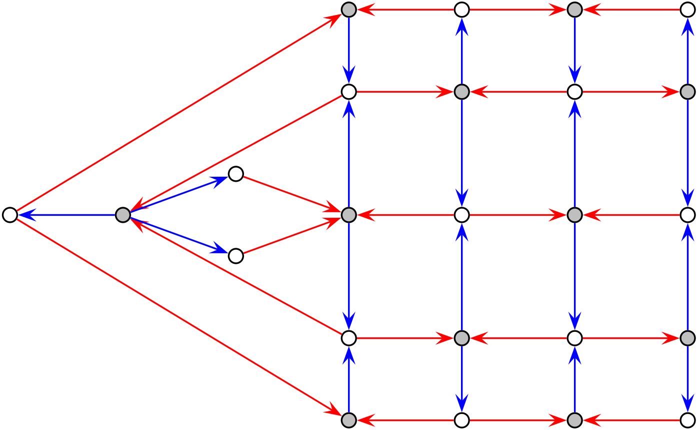

$h = 9 + 1 = 12 - 2 = 10; \quad h' = 5 + 1 = 8 - 2 = 6; \quad$ Period $=$ **32**

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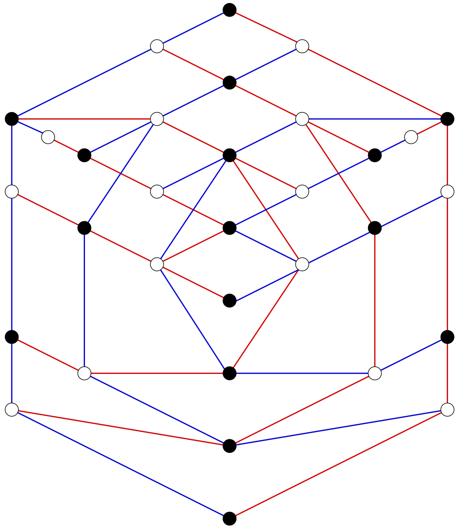
# Finite ⊠ finite classification (Stembridge, 2010)

5 infinite families and 11 exceptional quivers



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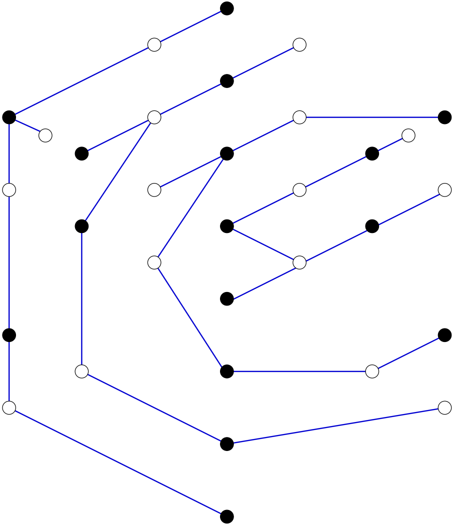

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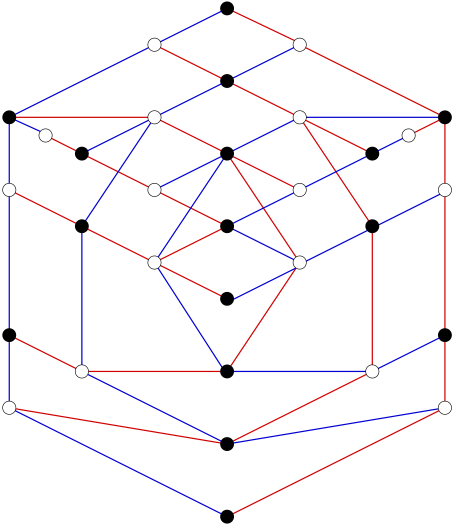
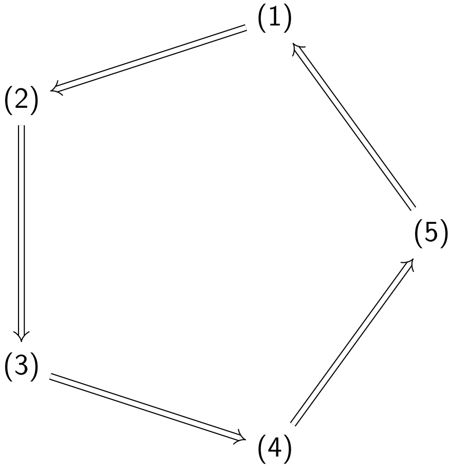


# Plan of the proof



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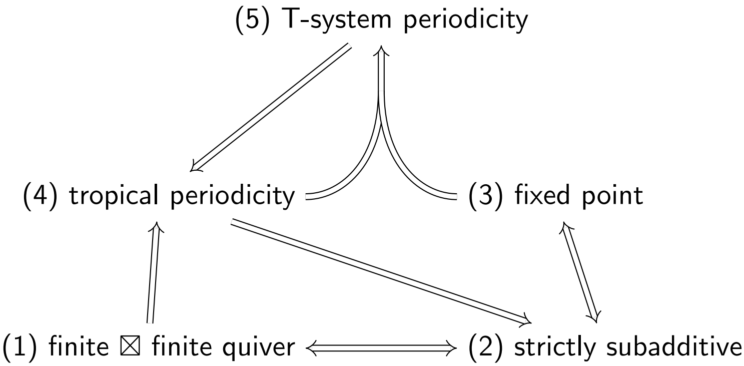

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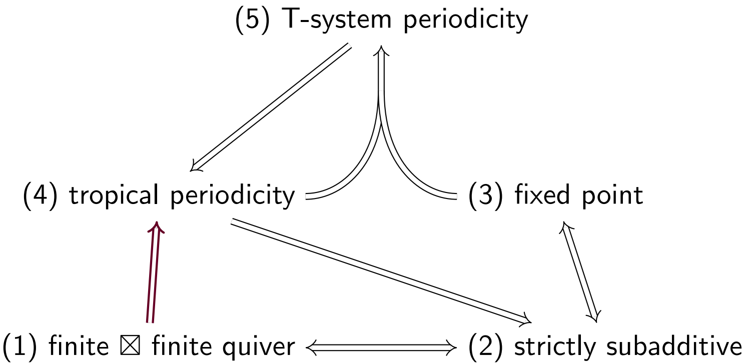
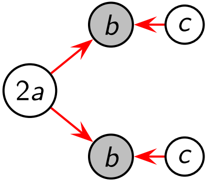
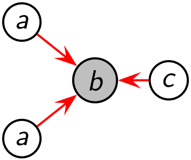


# $A_{2n-1} \leftrightarrow D_{n+1}$ duality



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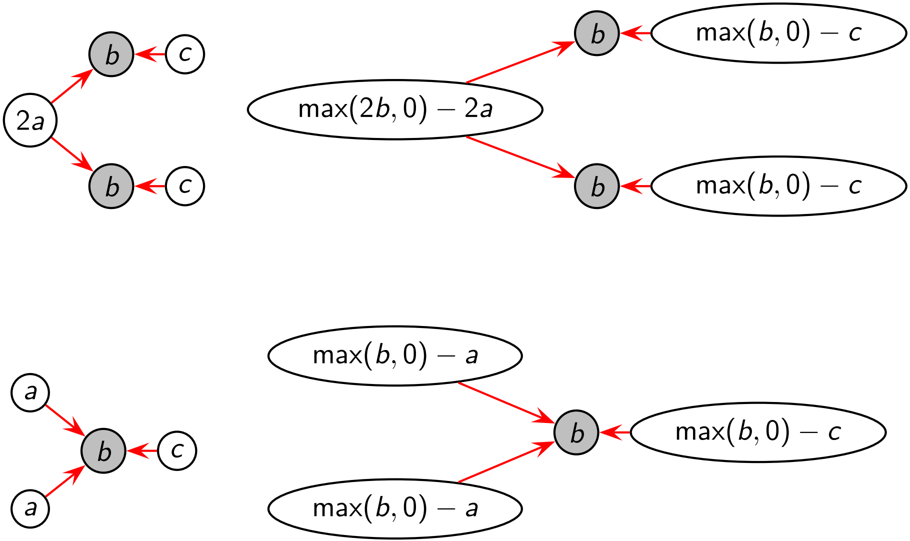
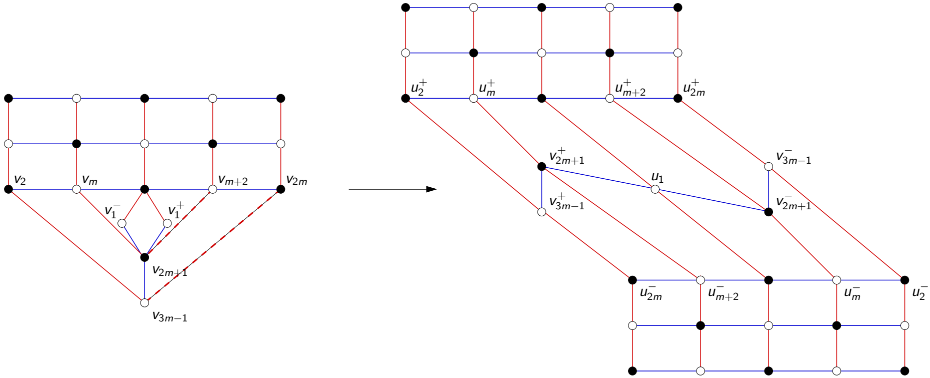


# $(A^{m-1}D)_n \leftrightarrow A_{2n-1} \otimes D_{m+1}$ duality



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Slides: `http://math.mit.edu/~galashin/slides/japan1.pdf`


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Thank you!

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