Move-reduced graphs on a torus vs positroid Catalan numbers

Pavel Galashin (UCLA)
based on joint works with Terrence George and Thomas Lam

(scan to play the game)

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- $\mathrm{ncyc}(f) := \#\{\text{cycles of } \bar{f}\}$.

Bipartite graphs on a torus

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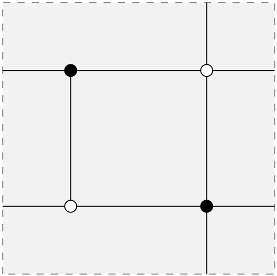

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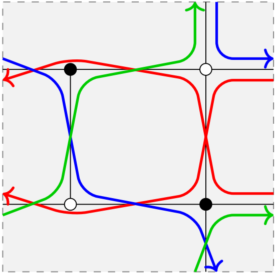

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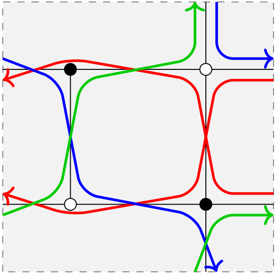

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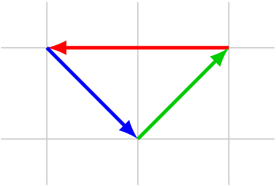

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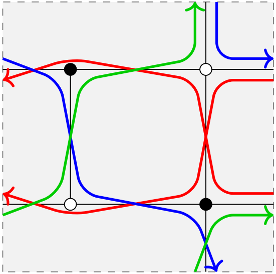

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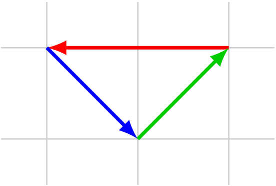

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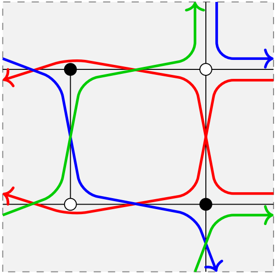
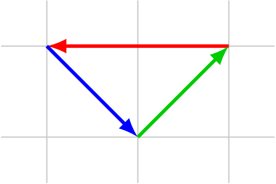


Γ — not minimal!



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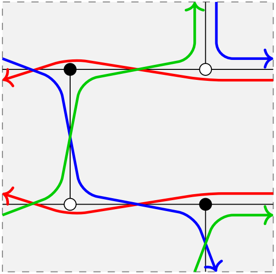

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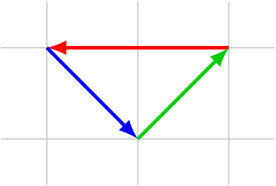

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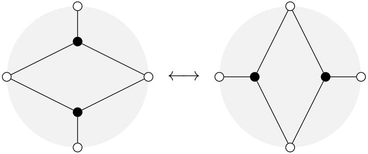
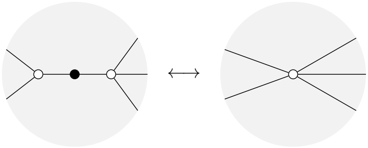


(M1) The spider move.



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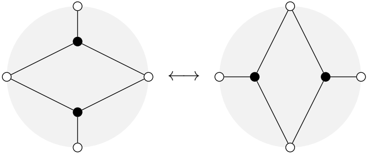

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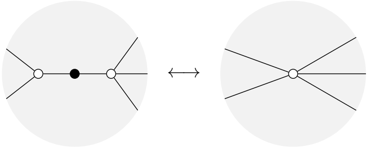

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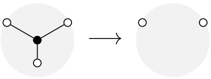
(R1) Parallel edge reduction.



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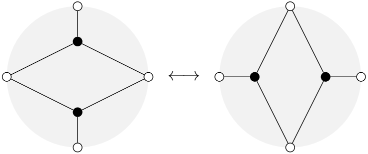

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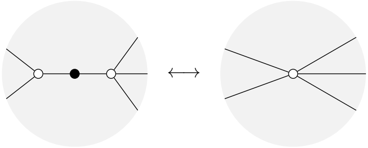

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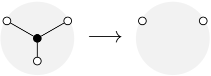

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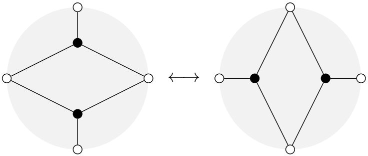

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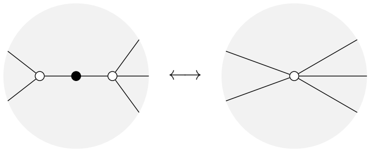

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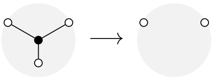

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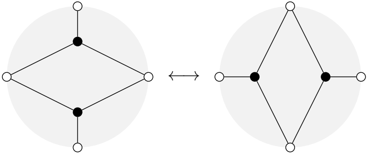
- **Question:** Is minimal the same as move-reduced?



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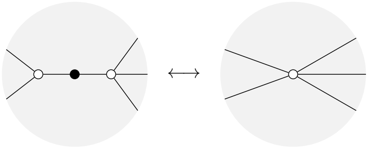
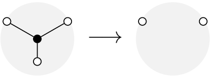


(M2) The contraction-uncontraction move.

(R1) Parallel edge reduction.



(R2) Leaf reduction.

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- Γ, Γ' are move-equivalent ($\Gamma \sim \Gamma'$) if they are related by moves (M1)–(M2).
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- [Goncharov–Kenyon '10] Γ is minimal $\Longrightarrow$ Γ is move-reduced.
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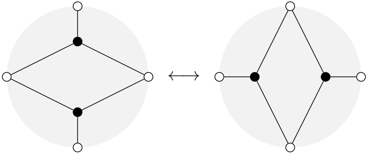

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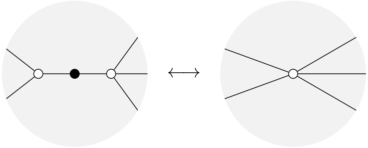

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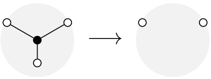

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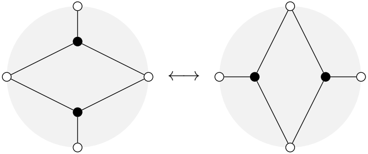

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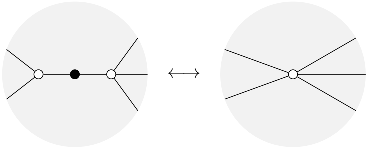

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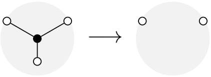

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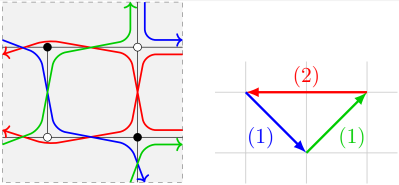

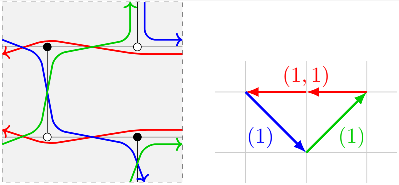

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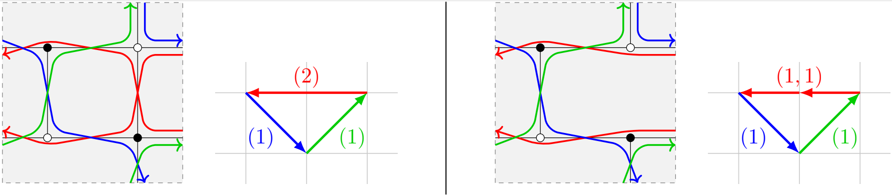

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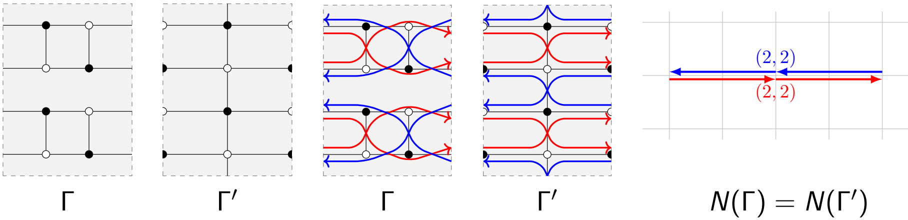

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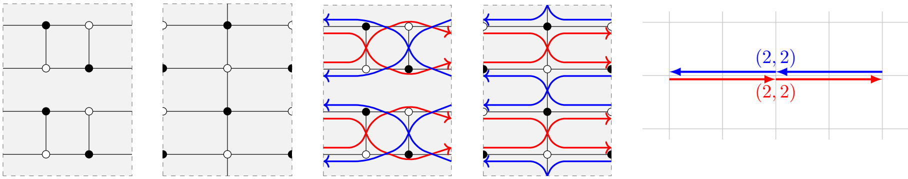

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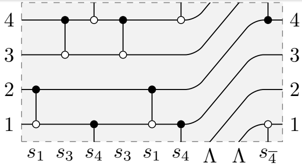

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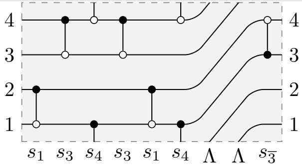

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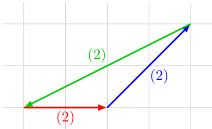

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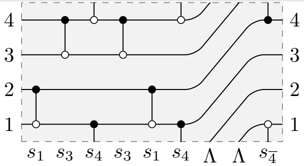

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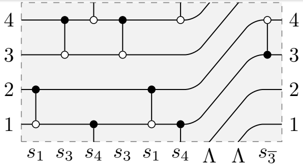

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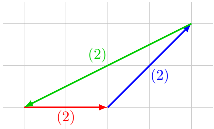

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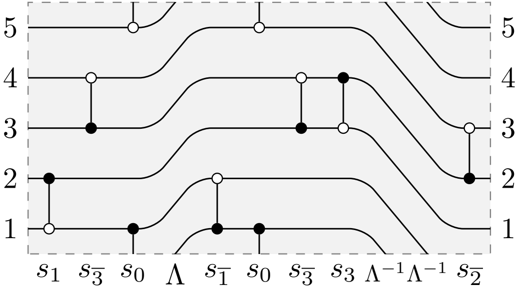

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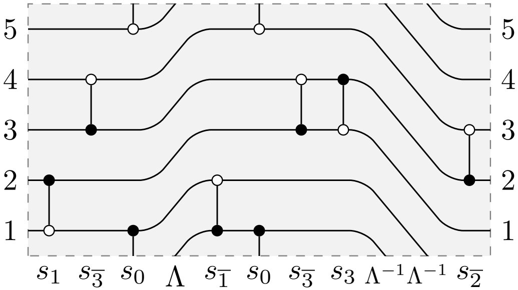

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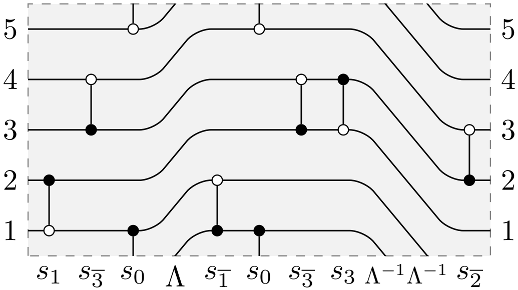

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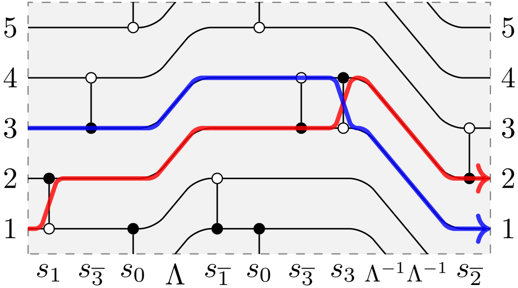

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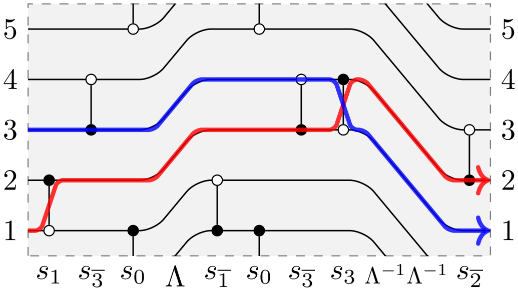

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Theorem (G.–George; see also Fock–Marshakov '16)

- *If f, f' are c-reduced then Γ is move-reduced.*
- *Any move-reduced graph Γ can be obtained in this way.*

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Recall: $\tilde{S}_{k,n} := \{f \in \tilde{S}_n \mid \frac{1}{n}\sum_{i=1}^{n}(f(i) - i) = k\}$.

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$$C_f = \frac{1}{n}\binom{n}{k} = \#\{\textit{Dyck paths from } (0,0) \textit{ to } (n-k,k) \textit{ above diagonal}\}.$$

Definition (G.–Lam)

Let $f \in \mathcal{B}_{k,n}$ be such that $\mathrm{ncyc}(f) = 1$. The positroid Catalan number is

$$C_f := (\text{Euler characteristic of } \Pi_f^\bullet(\mathbb{C})) = \#\Pi_f^\bullet(\mathbb{F}_q)\big|_{q=1}.$$

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Assume $\mathrm{ncyc}(f) = 1$.

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- *If* $\ell(s_i f s_i) = \ell(f) + 2$ *then* $C_f = C_{g_1} \cdot C_{g_2} + C_{s_i f s_i}$*, where* g_1, g_2 *are the two cycles of* $s_i f \overset{c}{\sim} f s_i$*.*

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- *If $n = 1$ then $C_f = 1$. If $f(i) \in \{i, i+n\}$, removing i preserves C_f.*
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- *This recurrence computes C_f for any $f \in \mathcal{B}_{k,n}$ with* $\mathrm{ncyc}(f) = 1$.

Question

What does this have to do with Dyck paths?

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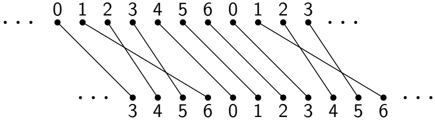

$$\delta(f) = (4, 3)$$

For $g \in \tilde{S}_{k,n}$, denote $\delta(g) := (n-k, k)$.

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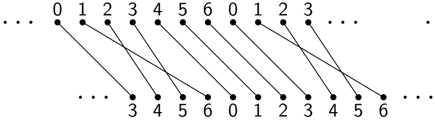
Let $f \in \mathcal{B}_{k,n}$ with $\mathrm{ncyc}(f) = 1$.

- For each crossing point x of f, resolving x gives two cycles g_1^x, g_2^x entering x from the top-left and top-right, respectively.



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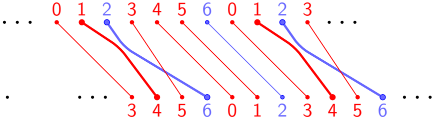
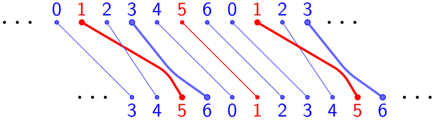


$\delta(g_1^{x_1}) = (3, 2)$



$\delta(g_1^{x_2}) = (1, 1)$

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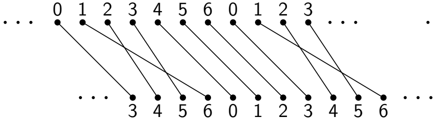
- Let $\Delta(f)$ be the multiset of $\delta(g_1^x)$ for all crossings x of f.



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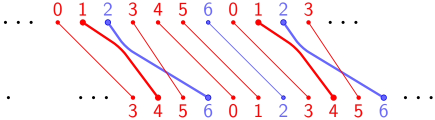

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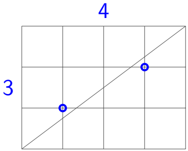
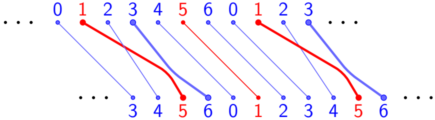


$\Delta(f) = \{(1,1), (3,2)\}$



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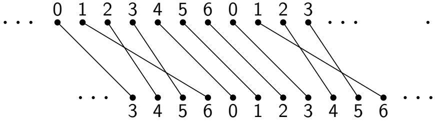
- f is called repetition-free if all points in $\Delta(f)$ are distinct.



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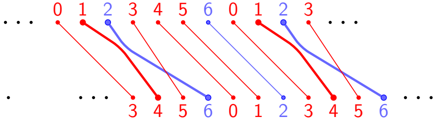

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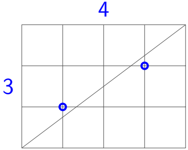

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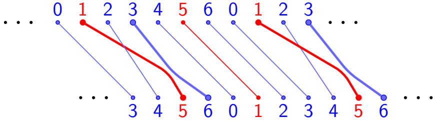

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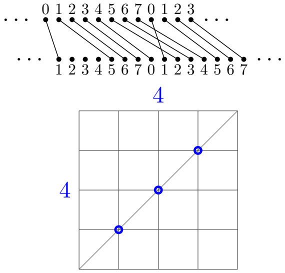

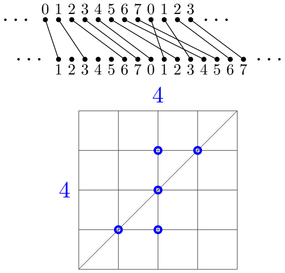

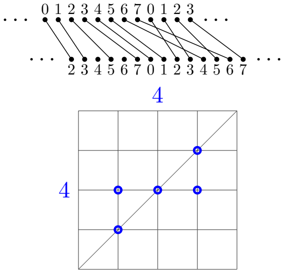

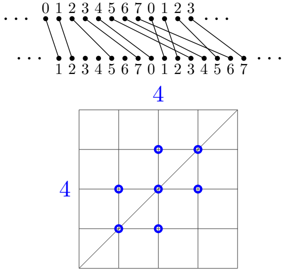

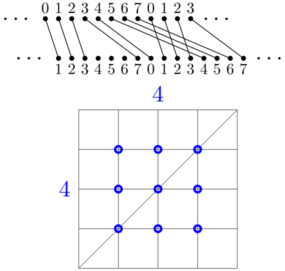

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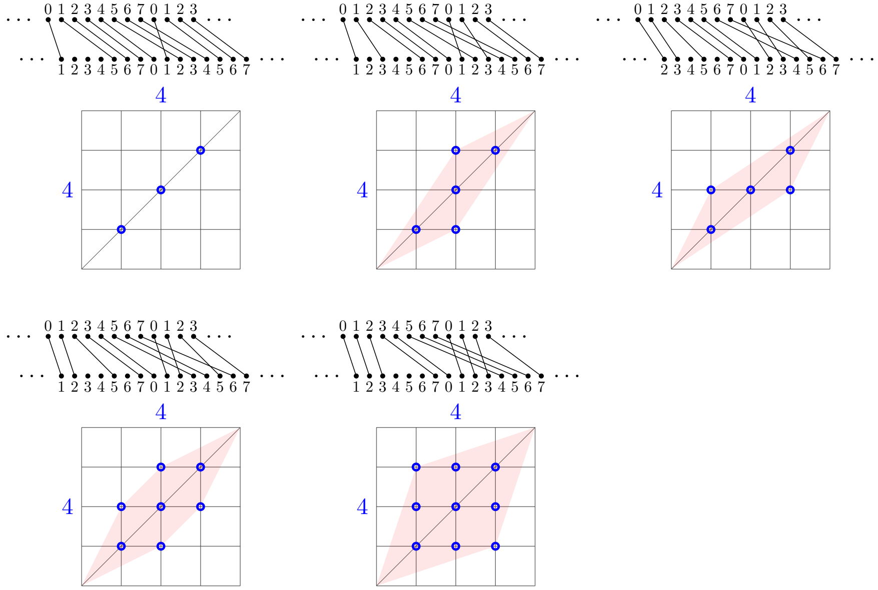

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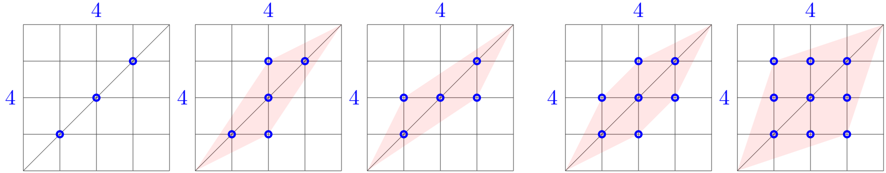

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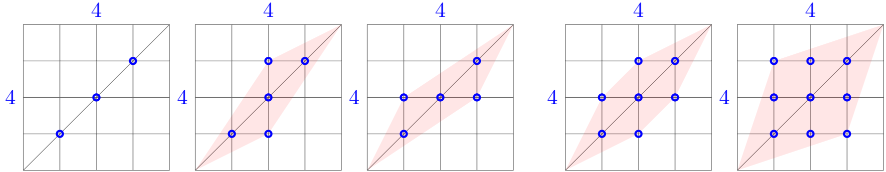

Theorem (G.–Lam)

- *$\Delta \subseteq [n-k-1] \times [k-1]$ arises as $\Delta(f)$ for some repetition-free $f \in \mathcal{B}_{k,n}$ if and only if*

$$\Delta \sqcup \{(0,0), (n-k, k)\} = P \cap \mathbb{Z}^2$$

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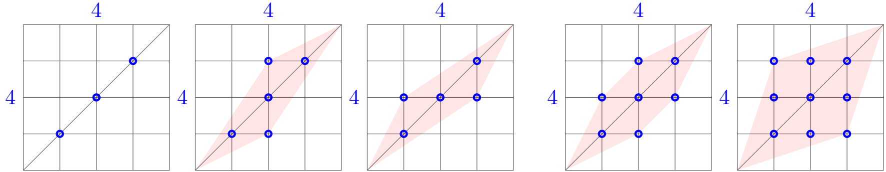

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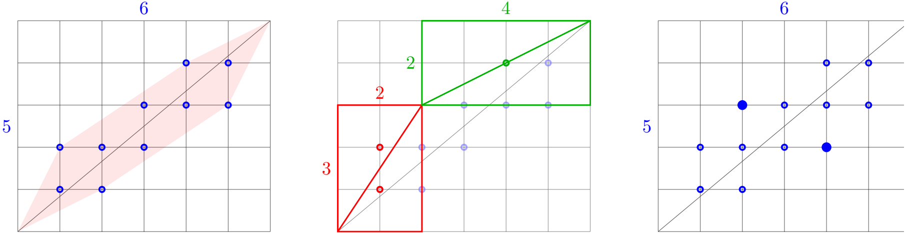

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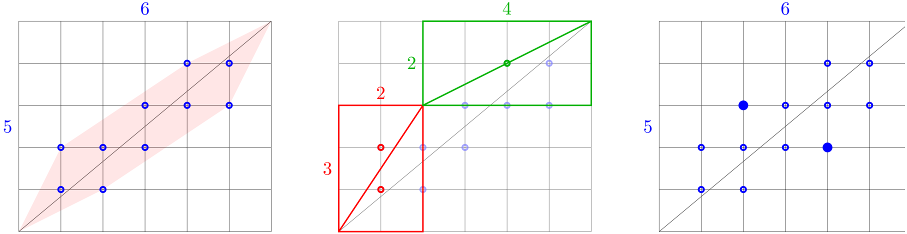

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