Intro to cluster algebras.

Fomin–Zelevinsky 2002

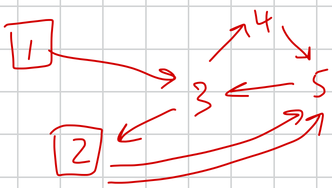
Textbook: Fomin–Williams–Zelevinsky 2024+

① Quivers

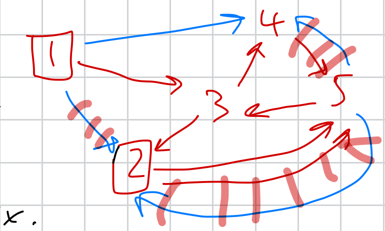
Def. A *quiver* is a directed graph without $\circlearrowleft$ and $\circ \rightleftarrows \circ$.

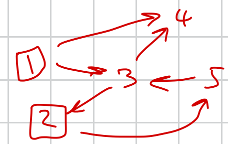
Def. An *ice quiver* is a quiver with vertex set partitioned into *mutable* and *frozen* vertices.

Arrows between frozen vertices are always omitted.

Ex. $Q =$ 

frozen: 1, 2 – boxed
mutable: 3, 4

$\mu_3(Q) =$ 



Def. Let Q – quiver, $j \in V(Q)$ – mutable vertex.

$\mu_j(Q)$ is obtained from Q by:

(1) for each 2-path $i \to j \to k$ in Q, create shortcut $i \to k$.

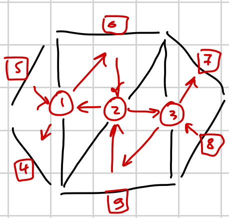
(2) reverse all arrows incident to j.

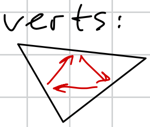
(3) remove all pairs $\circ \rightleftarrows \circ$ one by one.

Exercise: μ_j is an involution: $\mu_j(\mu_j(Q)) = Q$.

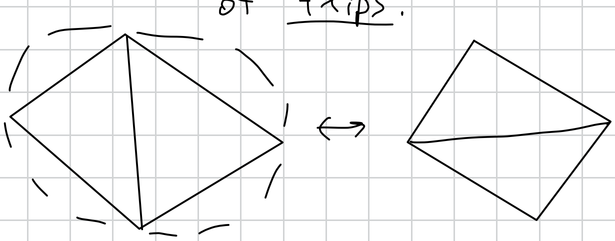
Running ex. T – triangulation of a polygon P.

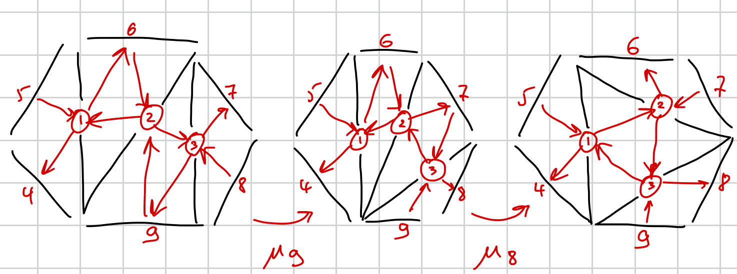
$Q(T)$ – quiver:

$T =$ 

- mutable verts: diagonals in T
- frozen verts: sides of P
- arrows:  for each triangle in T.

Claim. Any two triangulations of P are related by a sequence of *flips*:





Claim. If T_1, T_2 are related by a flip then $Q(T_1), Q(T_2)$ are related by mutation.

Def. Initial seed: (Q, x), $x = (x_j)_{j\in Q}$ – alg. independent
Def. A seed is a pair (Q, f), where $f = (f_j)_{j\in Q}$ $\quad f_j \in \mathbb{C}(x_j \mid j \in Q)$
Def. Seed mutation: (Q, f) – seed, $j \in V(Q)$ – mutable

② Seeds.

Let Q – quiver with mutable verts $1, \dots, n$, frozens $n+1, \dots, n+m$.
$\mathcal{F} = \mathbb{C}(x_1, x_2, \dots, x_{n+m})$ – ambient field.

Fix $\mathcal{F}$ – ambient field. Usually, $\mathcal{F} = \mathbb{C}(x_1, \dots, x_{|V(Q)|})$.

Def. A **seed** is a pair (Q, f), where $f = (f_j)_{j\in Q}$ $\quad f_j \in \mathcal{F}$ – algebraically independent.

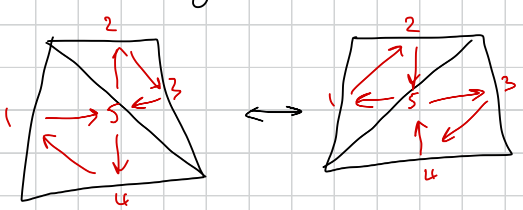
Def. **Seed mutation:** (Q, f) – seed, $j \in V(Q)$ – mutable

$\Rightarrow$ $\mu_j(Q, f) = (Q', f')$ is obtained from (Q, f) as follows:

- $Q' = \mu_j(Q)$;
- $f'_i = f_i$ for $i \neq j$
- $$f'_j = \frac{\prod_{i\to j} f_i + \prod_{j\to k} f_k}{f_j}.$$

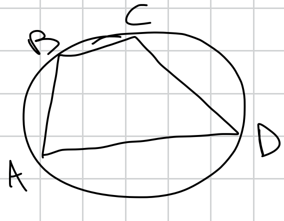
μ_j is still an involution!

Ex. Triangulation flip:



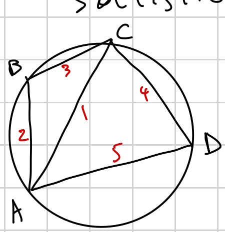
$$x_5' = \frac{x_1 x_3 + x_2 x_4}{x_5}$$

Reminder. **Ptolemy's thm.** If $ABCD$ inscribed quadrilateral, then



$$|AC|\cdot|BD| = |AB|\cdot|CD| + |AC|\cdot|BD|$$

Cor. If P – inscribed polygon, then setting x_j = length of j-th diagonal satisfies cluster mutation relations.



$x_1 = |AC|$ $\quad x_2 = |AB|$ $\quad x_3 = |BC|$ $\quad x_4 = |CD|$ $\quad x_5 = |AD|$

Problem: $(x_1, x_2, \dots, x_5)$ – not alg. independent!

Def. $M \in Mat_{2\times n}$ $\quad \Delta_{AB} = \det\begin{bmatrix} | & | \\ M_A & M_B \\ | & | \end{bmatrix}$. $\quad 1 \le A < B \le n$

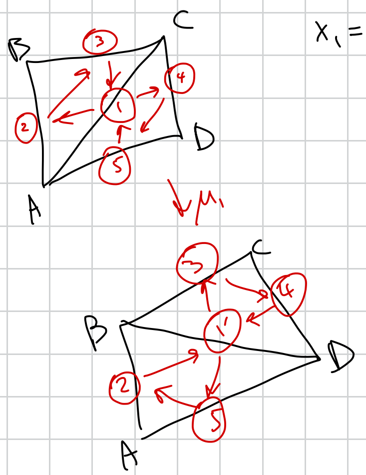
$$M = \begin{bmatrix} | & | & & | \\ M_1 & M_2 & \cdots & M_n \\ | & | & & | \end{bmatrix}$$

Plücker relation: $\Delta_{AC}\,\Delta_{BD} = \Delta_{AB}\,\Delta_{CD} + \Delta_{AD}\,\Delta_{BC}$.

Ex.

$$\begin{array}{cccc} A & B & C & D \end{array}$$
$$\begin{bmatrix} p & x & 0 & -y \\ 0 & z & 1 & w \end{bmatrix}$$

$$p\cdot(xw + yz) = pz\cdot y + pw\cdot x$$



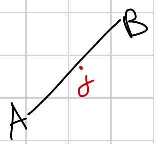
$x_1 = \Delta_{AC} = p \qquad x_2 = \Delta_{AB} = pz \qquad x_3 = \Delta_{BC} = x$

$x_4 = \Delta_{CD} = y \qquad x_5 = \Delta_{AD} = pw$ — alg. independent!

$$x_1' = \frac{x_2x_4 + x_1x_3}{x_1} = \frac{pzy + pxw}{p} = xw + zy = \Delta_{BD}.$$

Summary. Let $\mathcal{F} = \mathbb{C}(m_{11}, m_{12}, \dots, m_{2n})$, P - polygon, $V(P) = 1, 2, \dots, n$.

For each triangulation T of P, get a seed (Q_T, x_T)

For each diagonal  of T (or side of P), set $x_j = \Delta_{AB}(M)$.

If T_1, T_2 are related by flip then seeds (Q_{T_1}, x_{T_1}), (Q_{T_2}, x_{T_2}) are related by a mutation.

Questions:

(1) How to generalize this from $2\times n$ matrices to $k \times n$?

(2) What does it have to do with dimers?

(3) What is a cluster algebra?

3 Cluster algebras.

Let Q be a quiver, mut. verts $1, 2, \dots, n$
frozen verts $n+1, \dots, n+m$.

Set $\mathcal{F} = \mathbb{C}(x_1, x_2, \dots, x_{n+m})$

Let (Q, x) – initial seed, $x = (x_1, \dots, x_{n+m})$.

Write down <u>all</u> other seeds obtained from (Q, x) via a seq. of mutations

usually ∞-many

Elements of $\mathcal{F}$ obtained this way are called <u>cluster variables</u>.

If $(Q, x) \xrightarrow[\mu_{j_1}]{} \cdots \xrightarrow[\mu_{j_N}]{} (Q', x')$, $x'_1, \dots, x'_{n+m}$ are said to form a <u>cluster</u>.

$x_{n+1}, \dots, x_{n+m}$ are called frozen variables.

<u>Def</u>. Cluster algebra $\mathcal{A}(Q) \subseteq \mathcal{F}$ is the subalgebra generated by all cluster variables.

<u>Alternative Def</u>: $\mathcal{A}^0(Q) \subseteq \mathcal{F}$ is generated by all cluster variables and inverses of frozen vars.

$$\mathcal{A}^0(Q) \cong \mathbb{C}[x_1, x_1', x_2, x_2^{-1}]/(x_1 x_1' = x_2 + 1)$$

<u>Ex</u>. $Q = 1 \longrightarrow \boxed{2}$ $\quad (Q, x) = x_1 \longrightarrow \boxed{x_2}$ $\quad \mu_1(Q, x) = \frac{x_2+1}{x_1} \longleftarrow \boxed{x_2}$

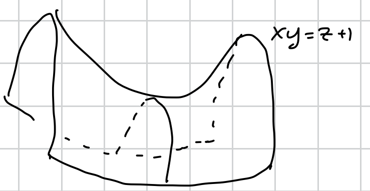
initial seed $\qquad$ other seed

cluster vars: $x_1, x_2, \frac{x_2+1}{x_1}$

clusters: (x_1, x_2), $\left(\frac{x_2+1}{x_1}, x_2\right)$

$$\mathcal{A}(Q) \cong \mathbb{C}[x_1, x_1', x_2]/(x_1 x' = x_2 + 1)$$

ring of polynomial fcns on hyperbolic paraboloid



<u>Ex</u>. T – triang. of an m-gon. m frozen, $n = m-3$ mutable verts.

$\#$ clusters = Catalan number C_{m-2} $\qquad C_k = \frac{1}{k+1}\binom{2k}{k}$

$\#$ cluster vars $= \binom{m}{2}$

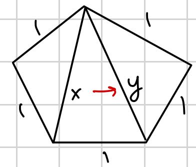
$\mathcal{A}(Q_T)$ depends only on m. ring of polynomial fcns on Grassmannian $Gr_{2,m}$

$\boxed{4'}$

Thm. Any cluster variable is a Laurent polynomial in the initial cluster with positive integer coefficients.

surprising! $\frac{x+y+1}{x+y}$ not Laurent

$\frac{x^3+y^3}{x+y} = x^2-xy+y^2$ Laurent, non-positive.



Ex.

$$x \longrightarrow y$$

$$\frac{y+1}{x} \longleftarrow y$$

$$\frac{y+1}{x} \longrightarrow \frac{x+y+1}{xy} \qquad \frac{\frac{x+y+1}{xy}+1}{\frac{y+1}{x}} = \frac{x+y+1+xy}{y(y+1)} = \frac{x+1}{y}$$

$$\frac{x+1}{y} \longleftarrow \frac{x+y+1}{xy}$$

$$\frac{x+1}{y} \longrightarrow x$$

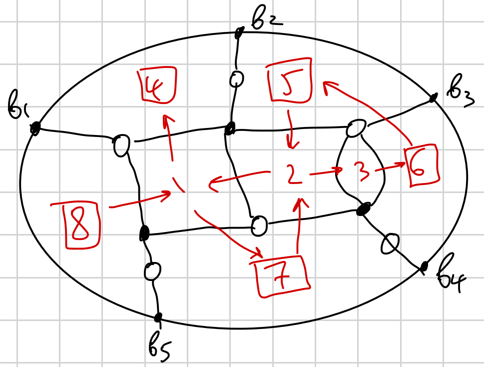
$$y \longleftarrow x$$

Surprising: periodicity!

(4) Planar bipartite graphs

Let G = bipartite graph embedded in a disk

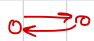
w/ boundary verts $b_1, \dots, b_n$ (clockwise) of degree 1.



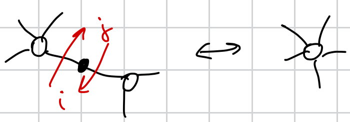
Planar dual of G is naturally a quiver Q_G:

boundary faces of G $\leftrightarrow$ mutable verts of Q_G

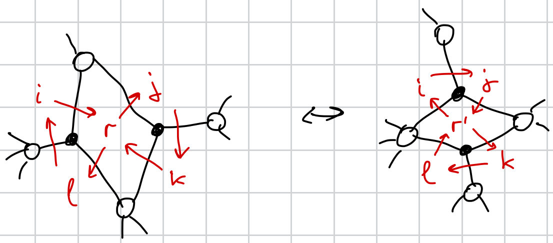
interior —"— $\leftrightarrow$ frozen —"—

(need to omit all )

Local moves for G:

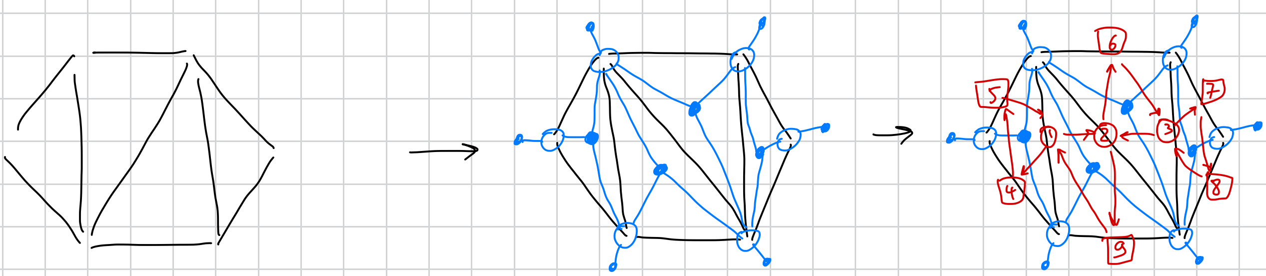
(M1)  preserves Q_G

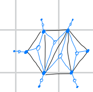
(M2) Spider move:
square move:
urban renewal:



mutates Q_G.

Triangulation T $\to$ bip. graph G $\to$ quiver $Q_G = Q_T$:





verts of $P \leftrightarrow$ white verts of G connected to bdry

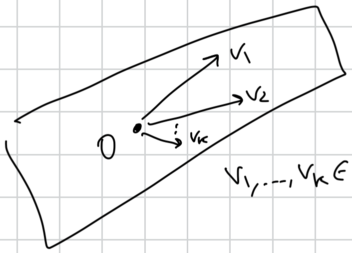
triangles of $T \leftrightarrow$ black trivalent verts of G.

diag's of $T \leftrightarrow$ interior faces of G $\leftrightarrow$ mutable verts of $Q_T = Q_G$.

flips of $T \leftrightarrow$ square moves of G $\leftrightarrow$ mutations of $Q_T = Q_G$

5 Grassmannian

$$Gr(k,n) = \{ V \subseteq \mathbb{R}^n \mid \dim V = k \} = \{\text{full rank } k\times n \text{ matrices}\} / \left(\begin{smallmatrix}\text{row}\\ \text{operations}\end{smallmatrix}\right)$$



$v_1, \dots, v_k \in \mathbb{R}^n$ – basis of V.

$$M = \begin{bmatrix} \text{———} v_1 \text{———} \\ \text{———} v_2 \text{———} \\ \vdots \\ \text{———} v_k \text{———} \end{bmatrix}$$

$$M = \begin{bmatrix} | & | & & | \\ M_1 & M_2 & \cdots & M_n \\ | & | & & | \end{bmatrix}$$

Plücker coordinates:

$I = \{ i_1 < i_2 < \dots < i_k \}$ $\qquad \Delta_I(V) = \det \begin{pmatrix} | & | & & | \\ M_{i_1} & M_{i_2} & \cdots & M_{i_k} \\ | & | & & | \end{pmatrix}$ – defined up to rescaling.

We still have Plücker relation:

for $A < B < C < D$ not in I, $|I| = k-2$,

$$\Delta_{IAC}\,\Delta_{IBD} = \Delta_{IAB}\,\Delta_{ICD} + \Delta_{IAD}\,\Delta_{IBC}$$

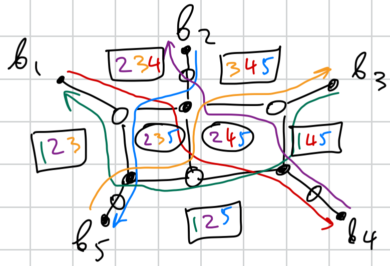
where $\Delta_{IAC} = \Delta_{I \cup \{A,C\}}$, etc.

Want to label each face of a plane bip. graph by a Plücker coord.

Def. A zig-zag path in G is a path that makes a

- sharp right turn @ each black vertex.
- sharp left turn @ each white vertex.

Ex.



$$\pi_G = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$$

Strand permutation $\pi_G \in S_n$:

$\pi_G(i) = j$ if zig-zag path starting @ b_i terminates @ b_j.

Face labels: put j inside each face to the left of zig-zag path $b_i \to b_j$.

7

<u>Def</u>. G is a (k,n)-graph if $\pi_G(i) \equiv i+k \mod n$ $\forall i$,

and G has $\underbrace{k(n-k)+1}$ faces.

↖ minimal possible subject to

<u>Thm</u>. (Postnikov '06) Any two (k,n)-graphs are related by square moves.

<u>Thm</u>. (Scott '06) G (k,n)-graph $\Rightarrow$ $\mathcal{A}(Q_G) \cong$ ring of polynomial functions on $Gr(k,n)$ $= \mathbb{C}[\Delta_I \mid I \subseteq [n], |I|=k] / (\text{Plücker relations})$