$\boxed{\text{Thm}}$ (GKL'19) $(G/P)_{\geq 0}$ is a regular CW complex. | joint with: Steven Karp $\boxed{1}$ Thomas Lam arXiv: 1904.00527

① Total positivity.

G = split $/\mathbb{R}$ alg. group. B, B_- = opposite Borel. $T = B \cap B_-$.

$I = \{\text{simple roots}\}$. $W = \mathrm{Norm}_G(T)/T$, gen'd by $\{s_i\}_{i \in I}$.

Bruhat: $G = \bigsqcup_{w \in W} BwB$, Birkhoff: $G = \bigsqcup_{v \in W} B_- vB$. "Richardson cell"

Flag variety: $G/B = \bigsqcup_{v \leq w} R^\circ_{v,w}$, where $R^\circ_{v,w} = (B_- vB \cap BwB)/B$. $\dim = \ell(w) - \ell(v)$

Choose $J \subseteq I$. Parabolic subgroup P gen'd by $B \cup \{s_i\}_{i \in J}$

Partial flag variety: G/P. Projection: $\pi_J : G/B \to G/P$.
$gB \mapsto gP$

U^- = unip. radical of B_-. $y_i : \mathbb{C} \to U^-$ 1-psg for $i \in I$.

$\boxed{\text{Dfn}}$ (Lusztig '94–'98) • $U^-_{\geq 0}$ = submonoid of U^- gen'd by $\left\{ y_i(t) \,\middle|\, \begin{matrix} i \in I \\ t \in \mathbb{R}_{\geq 0} \end{matrix} \right\}$

• $(G/B)_{\geq 0} = \mathrm{Closure}(\{gB \mid g \in U^-_{\geq 0}\})$

• $(G/P)_{\geq 0} = \pi_J((G/B)_{\geq 0}) = \mathrm{Closure}(\{gP \mid g \in U^-_{\geq 0}\})$

$\boxed{\text{Thm}}$ (Rietsch '99 '06) • $(G/B)_{\geq 0} = \bigsqcup_{v \leq w} R^{>0}_{v,w}$, where $R^{>0}_{v,w} = R^\circ_{v,w} \cap (G/B)_{\geq 0}$

• $(G/P)_{\geq 0} = \bigsqcup_{(v,w) \in Q_J} \Pi^{>0}_{v,w}$, where $\Pi^{>0}_{v,w} = \pi_J(R^{>0}_{v,w})$

• $R^{>0}_{v,w} \cong \Pi^{>0}_{v,w} \cong \mathbb{R}^{\ell(w)-\ell(v)}_{>0}$

• Closure relations. E.g. $\mathrm{Closure}(R^{>0}_{v,w}) = \bigsqcup_{v \leq v' \leq w' \leq w} R^{>0}_{v',w'}$.

Let W_J = gen'd by $\{s_i\}_{i \in J}$, $W^J = \{\text{min length coset reps in } W/W_J\}$

$Q_J = \{(v,w) \in W \times W^J \mid v \leq w\}$. For $(v,w) \in Q_J$, $\pi_J : R^\circ_{v,w} \xrightarrow{\sim} \Pi^\circ_{v,w}$ - isomorphism

$\boxed{\text{Thm}}$ (Knutson–Lam–Speyer '13) $G/P = \bigsqcup_{(v,w) \in Q_J} \Pi^\circ_{v,w}$, where $\Pi^\circ_{v,w} = \pi_J(R^\circ_{v,w})$
(Nice properties.)

Case $G/P = Gr(k,n)$: $\binom{[n]}{k} := \{S \subset \underset{\{1,\dots,n\}}{[n]} : |S| = k\}$

$\boxed{\text{Dfn}}$ (Postnikov '06) $Gr_{\geq 0}(k,n) = \{V \in Gr(k,n) \mid \Delta_S(V) \geq 0 \ \forall S \in \binom{[n]}{k}\}$

$\boxed{\text{Prop}}$ (??? '09–'15) if $G/P = Gr(k,n)$ then $(G/P)_{\geq 0} = Gr_{\geq 0}(k,n)$

Warning: False for other G/P!

Stratification: $V \in Gr_{\geq 0}(k,n) \longmapsto \{S \in \binom{[n]}{k} \mid \Delta_S(V) > 0\}$ = "positroid of V"

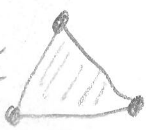
$\boxed{\text{Prop}}$ (Postnikov '06) • positroids $\longleftrightarrow Q_J$

bounded affine permutations

("therefore" stratifications coincide)

• closure relations $\longleftrightarrow$ affine Bruhat order.

Pictures:

$Gr_{\geq 0}(1,3) \cong$ 

$Gr_{\geq 0}(1,n) \cong$ simplex in $\mathbb{RP}^{n-1}$

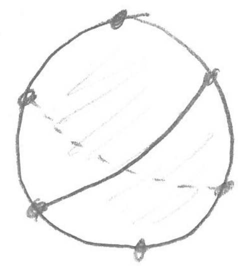
$Gr_{\geq 0}(2,5)$

$\Pi^{\geq 0}_{v,w} \cong$ 

$v = s_1 s_3$

$w = s_2 s_1 s_4 s_3 s_2$

$G = SL_3$

$(G/B)_{\geq 0} \cong$ 

$Gr_{\geq 0}(k,n)$ has $\dim = k(n-k)$, but only n "facets" (cells of codim=1)

② Main result.

Regular CW complex = cell complex: closure of each cell is homeomorphic to a ball.

$\boxed{\text{Dfn}}$ Regular CW complex homeomorphic to a ball

$\boxed{\text{Thm}}$ (GKL'19) $(G/P)_{\geq 0}$ is a regular CW complex

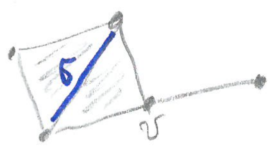
History:
- Björner–Bernstein '84: $(W, \leq)$ is nice
- Lusztig '98: $(G/P)_{\geq 0}$ – contractible
- Fomin–Shapiro '00
- Postnikov '06
- Williams '07

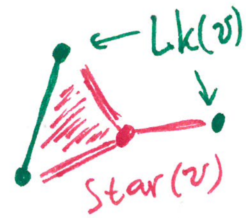
  (Fomin–Shapiro '00, Postnikov '06, Williams '07:) regularity conj. $U^-_{\geq 0}$, $Gr_{\geq 0}(k,n)$, $(G/P)_{\geq 0}$
- W '07: $(Q_J, \preceq)$ is nice
- P-Speyer-W '09, Rietsch-W '08: CW complex
- RW '10: closure contractible
- Hersh '14: proof for $U^-_{\geq 0}$
- GKL '17: $Gr_{\geq 0}(k,n)$ is a ball
- GKL '18: $(G/P)_{\geq 0}$ is a ball

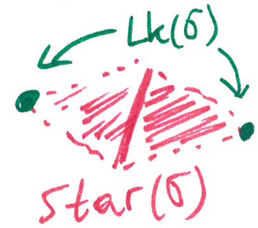
Follows from GKL'19.

③ Proof overview.

Abstract simplicial complex:







$\boxed{\text{Prop}}$ $\text{Star}(\sigma) \cong \sigma \times \text{Cone}^\circ(\text{Lk}(\sigma))$, where $\text{Cone}^\circ(A) = A \times \mathbb{R}_{\geq 0} / A \times \{0\}$

Recall: $(G/P)_{\geq 0} = \bigsqcup_{g \in Q_J} \Pi_g^{>0}$. Rietsch '06: $\text{Closure}(\Pi_g^{>0}) = \bigsqcup_{f \preceq g} \Pi_f^{>0}$

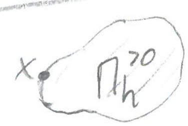
$\boxed{(Q_J, \preceq) \text{ face poset}}$

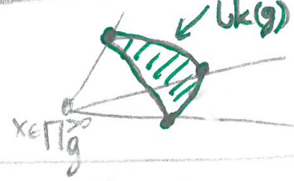
For $g \in Q_J$, let $\text{Star}(g) = \bigsqcup_{h \succeq g} \Pi_h^{>0}$

"Fomin-Shapiro atlas"

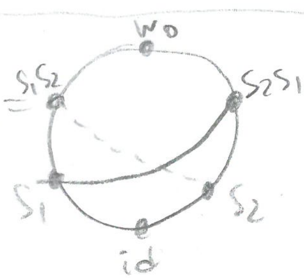
Goal: for $g \in Q_J$, show $\boxed{\text{Star}(g) \cong \Pi_g^{>0} \times \text{Cone}^\circ(\text{"Lk"}(g))}$

Topology: this is enough.





Ex. $(G/B)_{\geq 0}$ =



- $g = (id, id)$ $\quad$ Star(g) =

Lk(g) =

- $g = (s_1, s_1 s_2)$ $\quad$ Star(g) =

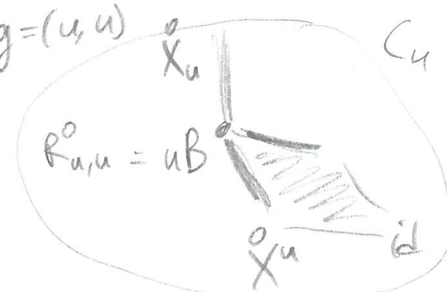
Lk(g) =

④ Proof: algebraic part.

For $u \in W$, let $C_u = (uB_-B)/B$ (open dense), $\mathring{X}_u = (B_-uB)/B$ (opp. Schubert), $\mathring{X}^u = (BuB)/B$ (Schubert)

$R^\circ_{v,w} = \mathring{X}_v \cap \mathring{X}^w$, $\quad \mathring{X}_u, \mathring{X}^u \subset C_u$, $\quad \mathring{X}_u = \bigsqcup_{w \geq u} R^\circ_{u,w}$, $\quad \mathring{X}^u = \bigsqcup_{v \leq u} R^\circ_{v,u}$, $\quad C_u \subsetneq \bigsqcup_{v \leq u \leq w} R^\circ_{v,w}$

$C_u^{\geq 0} = C_u \cap (G/B)_{\geq 0} \Rightarrow C_u^{\geq 0} = \bigsqcup_{v \leq u \leq w} R^{>0}_{v,w} = \mathrm{Star}(g)$ for $g = (u,u)$

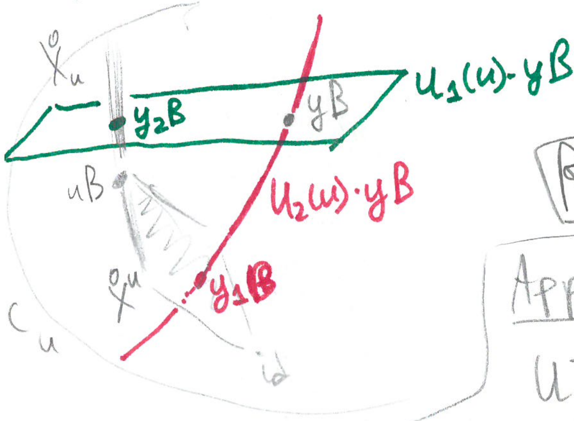


$\boxed{\text{Lemma}}$. For $u \in W$, $\exists$ isomorphism $\psi : C_u \xrightarrow{\sim} \mathring{X}^u \times \mathring{X}_u$
which restricts to $(C_u \cap R^\circ_{v,w}) \xrightarrow{\sim} R^\circ_{v,u} \times R^\circ_{u,w}$
and to $R^{>0}_{v,w} \xrightarrow{\sim} R^{>0}_{v,u} \times R^{>0}_{u,w}$ $\forall\, v \leq u \leq w$.

(KL'79
FS'00
KWY'13)

Proof sketch: Let $U_1(u) := uU_-u^{-1} \cap U$, $\dim = \ell(u)$
$U_2(u) := uU_-u^{-1} \cap U_-$, $\operatorname{codim} = \ell(u)$

multiplication gives $U_1(u) \times U_2(u) \xrightarrow{\sim} uU_-u^{-1}$, $\quad U_2(u) \times U_1(u) \xrightarrow{\sim} uU_-u^{-1}$.



$$\psi(yB) := (y_1B, y_2B). \quad \square$$

$\boxed{\text{Rmk}}$ This lemma holds in Kac-Moody generality.

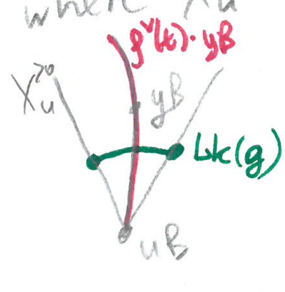
Application: FS conjecture.

$U^-_{\geq 0} \hookrightarrow (G/B)_{\geq 0}$ $\quad$ image $= C_{id}^{\geq 0} = \bigsqcup_{u \in W} R^{>0}_{id,u}$
$y \longmapsto yB$

Let $\Pi_g^{>0} = R^{>0}_{id,u} \Rightarrow \mathrm{Star}(g) = \bigsqcup_{w \geq u} R^{>0}_{id,w} \subset C_u^{\geq 0} \subset C_u$

$$\begin{tikzcd}
\mathrm{Star}(g) \arrow[r, "?"] & \Pi_g^{>0} \times \mathrm{Cone}^\circ(\mathrm{Lk}(g)) \\
\bigsqcup_{w \geq u} (R^{>0}_{id,w}) \arrow[d, "\psi"] & \\
\bigsqcup_{w \geq u} (R^{>0}_{id,u} \times R^{>0}_{u,w}) \arrow[r] & R^{>0}_{id,u} \times X_u^{>0} \arrow[uu]
\end{tikzcd}$$

, where $X_u^{>0} = \bigsqcup_{w \geq u} R^{>0}_{u,w} = \mathring{X}_u \cap (G/B)_{\geq 0}$



$X_u^{>0}$ has action of $T_{>0} \ni \rho^\vee(t) = \begin{pmatrix} t^{n-1} & & \\ & t^{n-2} & \\ & & 1 \end{pmatrix}$

5 Proof: affine part.

$\widetilde{W} = Q^\vee \rtimes W$ affine Weyl group

$\mathcal{G}$ = affine Kac-Moody group

$\mathcal{B}, \mathcal{B}_- \subset \mathcal{G}$. $\tilde{h} \le \tilde{f} \in \widetilde{W} \Rightarrow \mathcal{R}^\circ_{\tilde{h},\tilde{f}} = (\mathcal{B}_- \tilde{h} \mathcal{B} \cap \mathcal{B} \tilde{f} \mathcal{B})/\mathcal{B} \subset \mathcal{G}/\mathcal{B}$

dim $= \ell(\tilde{f}) - \ell(\tilde{h})$

$\lambda \in Q^\vee$ coroot $\Rightarrow \tau_\lambda \in \widetilde{W}$ transl. el't. Fix $\lambda \in Q^\vee$: dominant, fixed by W_J.

- order-reversing bijection: $Q_J \longrightarrow$ subposet of $(\widetilde{W}, \le)$ [He-Lam '15]
  $g \longmapsto \tilde{g}$
- For each $u \in W^J$, $\gamma: C_u^J \hookrightarrow \widetilde{\mathring{X}}^{\tau_{u\lambda}}$

$C_u^J = uB_-P/P$

$\left(\widetilde{\mathring{X}}^{\tau_{u\lambda}} = \mathcal{B}\tau_{u\lambda}\mathcal{B}/\mathcal{B}\right)$

$\forall g \succeq (u,u)$, $\Pi^\circ_g \cap C_u^J \xrightarrow{\sim} \mathcal{R}^\circ_{\tilde{g},\tau_{u\lambda}}$

$\boxed{\begin{array}{l}\mathrm{Gr}(k,n): \text{Snider '10} \\ G/P: \text{GKL'19}\end{array}}$

Finishing the proof: $\mathrm{Star}(g) \xrightarrow{?} \Pi_g^{>0} \times \mathrm{Cone}^\circ(\mathrm{Lk}(g))$

$g \succeq (u,u) \in Q_J$

$$\begin{tikzcd}
G/P: & \bigsqcup_{h \succeq g} \Pi_h^{>0} \arrow[d, "\gamma"] & & \Pi_g^{>0} \times \widetilde{\mathring{X}}^{\tilde{g}} \\
\mathcal{G}/\mathcal{B}: & \bigsqcup_{h \succeq g} \mathcal{R}^\circ_{\tilde{h},\tau_{u\lambda}} \arrow[rr, dashed, "\Psi"] & & \bigsqcup_{h \succeq g} \mathcal{R}^\circ_{\tilde{g},\tau_{u\lambda}} \times \mathcal{R}^\circ_{\tilde{h},\tilde{g}} \arrow[u, "(\gamma^{-1}, \mathrm{id})"']
\end{tikzcd}$$

$\widetilde{\mathring{X}}^{\tilde{g}} = \mathrm{Cone}^\circ(\ldots)$ — affine torus $\tilde{\rho}^\vee(t) \ldots$ ok.

Problem: Ψ is defined not on $\mathcal{R}^\circ_{\tilde{h},\tau_{u\lambda}}$ but on $\left(\widetilde{C}_{\tilde{g}} \cap \mathcal{R}^\circ_{\tilde{h},\tau_{u\lambda}}\right)$ (open dense)!

HARD: show $\gamma(\Pi_h^{>0}) \subset \widetilde{C}_{\tilde{g}}$ $\forall\, h \succeq g$
(but TRUE)

$Q_J \hookrightarrow \widetilde{W}$

$(v,w) \mapsto v \cdot \tau_\lambda \cdot w^{-1}$

so $(u,u) \mapsto u \cdot \tau_\lambda \cdot u^{-1} = \tau_{u\lambda}$

$C_u^J \hookrightarrow \widetilde{\mathring{X}}^{\tau_{u\lambda}}$

$yuP \mapsto y_1 u \cdot \tau_\lambda \cdot (y_2 u)^{-1} \cdot \mathcal{B} = y_1 \cdot \tau_{u\lambda} \cdot y_2^{-1} \cdot \mathcal{B}$

where $y_1 \in U_2(u) \cdot y$ $\quad yu \in \Pi^\circ_{v,w}$

$y_2 \in U_1(u) \cdot y$ $\quad \underbrace{y_1 u}_{B_- vB} \cdot \tau_\lambda \cdot \underbrace{(y_2 u)^{-1}}_{Bw^{-1}B} \in \mathcal{B}_- v \tau_\lambda w^{-1} \mathcal{B}$

$\boxed{B_- vB \cdot \tau_\lambda Bw^{-1}B \subset \mathcal{B}_- v \tau_\lambda w^{-1} \mathcal{B}}$

(6) $Gr(k,n)$ case

$\boxed{\text{Dfn}}$ (KLS '13) $f \in \text{Bound}(k,n)$ if

1) $f: \mathbb{Z} \to \mathbb{Z}$, $f(i+n) = f(i) + n \;\; \forall i \in \mathbb{Z}$
2) $\frac{1}{n} \sum_{i=1}^{n} (f(i) - i) = k$
3) $\forall i \in \mathbb{Z}$, $i \le f(i) \le i+n$

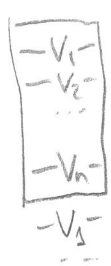
$\ldots 1 \, 2 \quad n$

$1 \, 2 \cdots n \quad n{+}1 \cdots$

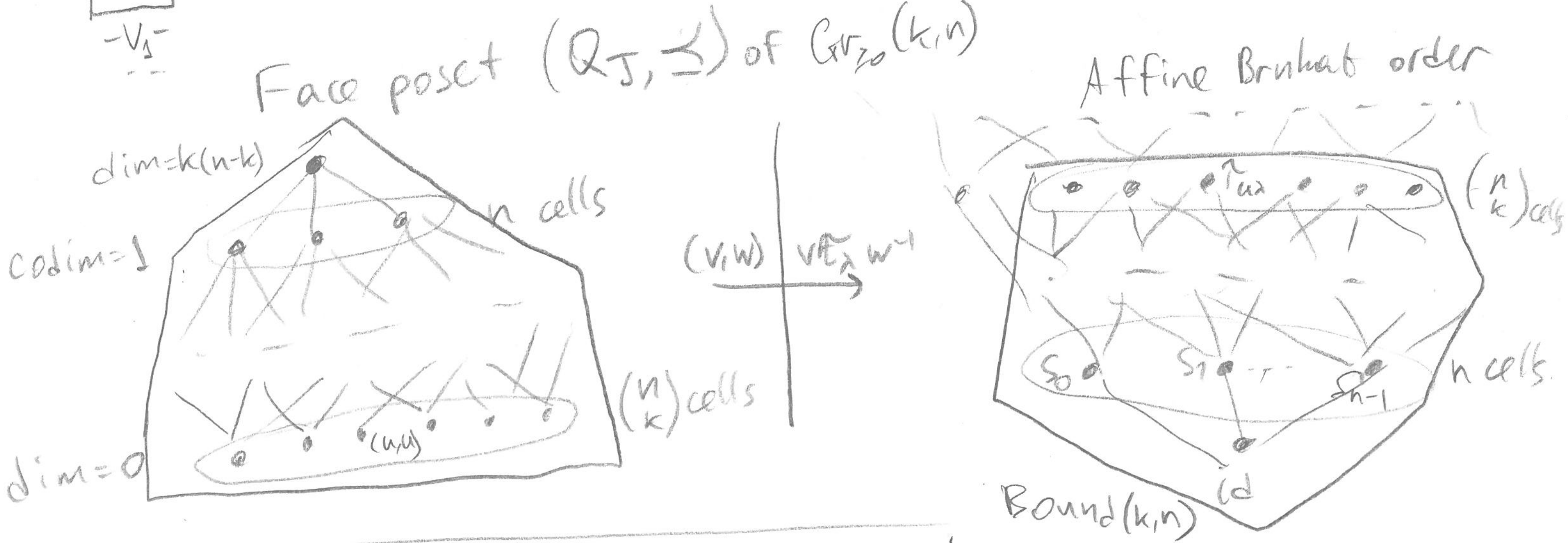
$Gr_{\ge 0}(k,n) \longrightarrow \text{Bound}(k,n)$

$V \mapsto f_V \qquad f_V(i) := \min\{ j \ge i+1 \mid V_i \in \text{span}(V_{i+1}, \ldots, V_j) \}$

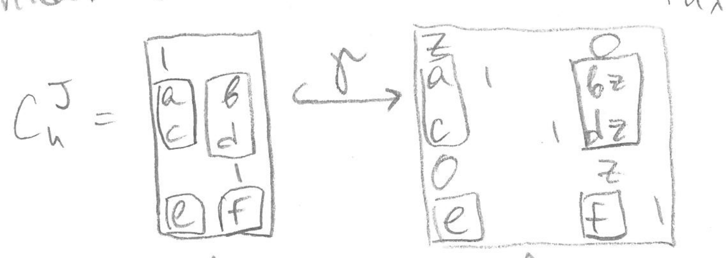
f_V contains same information as positroid of V



$V_{i+n} := V_i \;\; \forall i \in \mathbb{Z}$



Snider '10:



Clearly $\in \mathcal{B} \cdot t_{u\lambda} \cdot \mathcal{B}$

Magic: $yP \in \Pi_g^o \Leftrightarrow \gamma(yP) \in \mathcal{B}_- \cdot \tilde{g} \cdot \mathcal{B}$.