The symmetry and Schur expansion of dual stable Grothendieck polynomials

Pavel Galashin

MIT

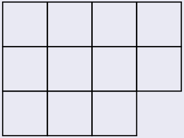
October 7, 2015

Joint work with Gaku Liu and Darij Grinberg

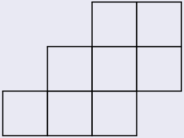
Part 1: Symmetry

Young diagrams, skew-shapes, SSYT

Skew shapes

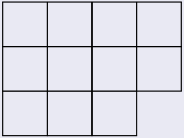


$\lambda = (4,4,3)$

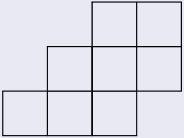


$\lambda/\mu = (4,4,3)/(2,1)$

Skew shapes



$\lambda = (4,4,3)$



$\lambda/\mu = (4,4,3)/(2,1)$

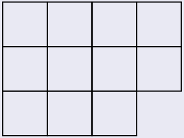
Semi-standard Young tableau (SSYT)

		1	3
	2	2	4
2	6	6	

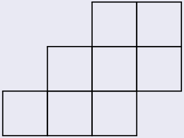
		1	4
	2	2	4
7	6	6	

Young diagrams, skew-shapes, SSYT

Skew shapes



$\lambda = (4,4,3)$



$\lambda/\mu = (4,4,3)/(2,1)$

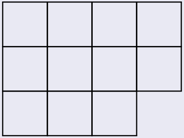
Semi-standard Young tableau (SSYT)

		1	3
	2	2	4
2	6	6	

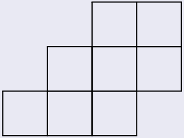
		1	4
	2	2	4
7	6	6	

Young diagrams, skew-shapes, SSYT

Skew shapes



$\lambda = (4,4,3)$



$\lambda/\mu = (4,4,3)/(2,1)$

Semi-standard Young tableau (SSYT)

		1	3
	2	2	4
2	6	6	

		1	**4**
	2	2	**4**
7	**6**	6	

Reverse plane partitions (RPP)

		1	3
	1	2	3
2	2	2	

		1	3
	1	2	2
3	2	2	

Reverse plane partitions (RPP)

		1	3
	1	2	3
2	2	2	

		1	**3**
	1	2	**2**
3	2	2	

Reverse plane partitions (RPP)

		1	3
	1	2	3
2	2	2	

		1	**3**
	1	2	**2**
3	**2**	2	

Reverse plane partitions (RPP)

		1	3
	1	2	3
2	2	2	

		1	**3**
	1	2	**2**
3	**2**	2	

SSYT is a special case of RPP!

Skew-Schur polynomials

Definition

If T is an SSYT then $w(T) := (\#T^{-1}(1), \#T^{-1}(2), \ldots, \#T^{-1}(m))$, where $\#T^{-1}(i) =$ [the number of entries in T equal to i].

Skew-Schur polynomials

Definition

If T is an SSYT then $w(T) := (\#T^{-1}(1), \#T^{-1}(2), \dots, \#T^{-1}(m))$, where $\#T^{-1}(i) = $ [the number of entries in T equal to i].

Example

$$T = \begin{array}{cccc} & & \boxed{1} & \boxed{3} \\ & \boxed{2} & \boxed{2} & \boxed{4} \\ \boxed{2} & \boxed{6} & \boxed{6} & \end{array}, \quad w(T) = (1,3,1,1,0,2), \quad x^{w(T)} = x_1^1 x_2^3 x_3^1 x_4^1 x_5^0 x_6^2.$$

Skew-Schur polynomials

Definition

If T is an SSYT then $w(T) := (\#T^{-1}(1), \#T^{-1}(2), \dots, \#T^{-1}(m))$, where $\#T^{-1}(i) = [\text{the number of entries in } T \text{ equal to } i]$.

Example

$$T = \begin{array}{cccc} & & \boxed{1} & \boxed{3} \\ & \boxed{2} & \boxed{2} & \boxed{4} \\ \boxed{2} & \boxed{6} & \boxed{6} & \end{array}, \quad w(T) = (1,3,1,1,0,2), \quad x^{w(T)} = x_1^1 x_2^3 x_3^1 x_4^1 x_5^0 x_6^2.$$

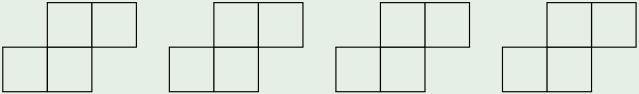
Definition

$$s_{\lambda/\mu}(x_1, \dots, x_m) = \sum_{\substack{T \text{ is a SSYT} \\ \text{of shape } \lambda/\mu \\ \text{with entries } \leq m}} x^{w(T)}.$$

Example

Example

Let $m = 2$, $\lambda = (3, 2)$, $\mu = (1)$.



Example

Example

Let $m = 2$, $\lambda = (3, 2)$, $\mu = (1)$.

$$\begin{ytableau} \none & 1 & \\ & 2 \end{ytableau} \quad \begin{ytableau} \none & 1 & \\ & 2 \end{ytableau} \quad \begin{ytableau} \none & 1 & \\ & 2 \end{ytableau} \quad \begin{ytableau} \none & 1 & \\ & 2 \end{ytableau}$$

Example

Example

Let $m = 2$, $\lambda = (3,2)$, $\mu = (1)$.

$$\ytableaushort{\none 11, 12} \qquad \ytableaushort{\none 12, 12} \qquad \ytableaushort{\none 11, 22} \qquad \ytableaushort{\none 12, 22}$$

Example

Let $m = 2$, $\lambda = (3, 2)$, $\mu = (1)$.

$$w(T) = \underset{(3,1)}{\begin{ytableau} \none & 1 & 1 \\ 1 & 2 \end{ytableau}} \quad \underset{(2,2)}{\begin{ytableau} \none & 1 & 2 \\ 1 & 2 \end{ytableau}} \quad \underset{(2,2)}{\begin{ytableau} \none & 1 & 1 \\ 2 & 2 \end{ytableau}} \quad \underset{(1,3)}{\begin{ytableau} \none & 1 & 2 \\ 2 & 2 \end{ytableau}}$$

Example

Example

Let $m = 2$, $\lambda = (3,2)$, $\mu = (1)$.

$$\begin{array}{ccccc}
 & \begin{array}{ccc} & \boxed{1} & \boxed{1} \\ \boxed{1} & \boxed{2} & \end{array} & \begin{array}{ccc} & \boxed{1} & \boxed{2} \\ \boxed{1} & \boxed{2} & \end{array} & \begin{array}{ccc} & \boxed{1} & \boxed{1} \\ \boxed{2} & \boxed{2} & \end{array} & \begin{array}{ccc} & \boxed{1} & \boxed{2} \\ \boxed{2} & \boxed{2} & \end{array} \\
w(T) = & (3,1) & (2,2) & (2,2) & (1,3) \\
s_{\lambda/\mu}(x_1,x_2) = & x_1^3x_2 & +x_1^2x_2^2 & +x_1^2x_2^2 & +x_1x_2^3.
\end{array}$$

Problem

Example

Let $m = 2$, $\lambda = (3, 2)$, $\mu = (1)$.

$$
\text{“}w(R) = \text{”}
\quad
\underset{(3,0)}{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}}
\quad
\underset{(2,1)}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}}
\quad
\underset{(2,1)}{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}}
\quad
\underset{(1,2)}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}}
\quad
\underset{(0,3)}{\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}}
$$

Example

Let $m = 2$, $\lambda = (3, 2)$, $\mu = (1)$.

$$\text{“}w(R) =\text{”}\quad \begin{array}{c}\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \\ (3,0)\end{array} \quad \begin{array}{c}\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \\ (2,1)\end{array} \quad \begin{array}{c}\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \\ (2,1)\end{array} \quad \begin{array}{c}\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \\ (1,2)\end{array} \quad \begin{array}{c}\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \\ (0,3)\end{array}$$

Dual stable Grothendieck polynomials

Definition

If R is an RPP then $w(R) := (w_1(R), w_2(R), \dots, w_m(R))$, where $w_i(R) = [\text{the number of } \textbf{columns} \text{ in } R \textbf{ containing } i]$.

Dual stable Grothendieck polynomials

Definition

If R is an RPP then $w(R) := (w_1(R), w_2(R), \dots, w_m(R))$, where $w_i(R) = [$the number of **columns** in R **containing** $i]$.

Definition

$$g_{\lambda/\mu}(x_1, \dots, x_m) = \sum_{\substack{R \text{ is a RPP} \\ \text{of shape } \lambda/\mu \\ \text{with entries } \leq m}} x^{w(R)}.$$

Dual stable Grothendieck polynomials

Definition

If R is an RPP then $w(R) := (w_1(R), w_2(R), \ldots, w_m(R))$, where $w_i(R) = [\text{the number of } \textbf{columns} \text{ in } R \textbf{ containing } i]$.

Definition

$$g_{\lambda/\mu}(x_1, \ldots, x_m) = \sum_{\substack{R \text{ is a RPP} \\ \text{of shape } \lambda/\mu \\ \text{with entries} \leq m}} x^{w(R)}.$$

Example

$$\begin{array}{cccccc}
 & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array} \\
w(R) = & (2,0) & (1,1) & (2,1) & (1,2) & (0,2)
\end{array}$$

Properties of $g_{\lambda/\mu}$

Properties of $g_{\lambda/\mu}$

- "represent the classes in K-homology of the ideal sheaves of the boundaries of Schubert varieties" (see [Lam, Pylyavskyy (2007)]);

Properties of $g_{\lambda/\mu}$

- "represent the classes in K-homology of the ideal sheaves of the boundaries of Schubert varieties" (see [Lam, Pylyavskyy (2007)]);
- $\mathrm{SSYT}(\lambda/\mu, \leq m) \subset \mathrm{RPP}(\lambda/\mu, \leq m)$ and the top-degree homogeneous component of $g_{\lambda/\mu}$ is $s_{\lambda/\mu}$;

Properties of $g_{\lambda/\mu}$

- "represent the classes in K-homology of the ideal sheaves of the boundaries of Schubert varieties" (see [Lam, Pylyavskyy (2007)]);
- $\mathrm{SSYT}(\lambda/\mu, \leq m) \subset \mathrm{RPP}(\lambda/\mu, \leq m)$ and the top-degree homogeneous component of $g_{\lambda/\mu}$ is $s_{\lambda/\mu}$;
- $g_{\lambda/\mu}$ are symmetric (see [Lam, Pylyavskyy (2007)]);

Properties of $g_{\lambda/\mu}$

- "represent the classes in K-homology of the ideal sheaves of the boundaries of Schubert varieties" (see [Lam, Pylyavskyy (2007)]);
- $\mathrm{SSYT}(\lambda/\mu, \leq m) \subset \mathrm{RPP}(\lambda/\mu, \leq m)$ and the top-degree homogeneous component of $g_{\lambda/\mu}$ is $s_{\lambda/\mu}$;
- $g_{\lambda/\mu}$ are symmetric (see [Lam, Pylyavskyy (2007)]);
- there exist involutions $B_i : \mathrm{RPP}(\lambda/\mu, \leq m) \to \mathrm{RPP}(\lambda/\mu, \leq m)$ such that
$$w(B_i(R)) = s_i w(R);$$

Properties of $g_{\lambda/\mu}$

- "represent the classes in K-homology of the ideal sheaves of the boundaries of Schubert varieties" (see [Lam, Pylyavskyy (2007)]);
- $\mathrm{SSYT}(\lambda/\mu, \leq m) \subset \mathrm{RPP}(\lambda/\mu, \leq m)$ and the top-degree homogeneous component of $g_{\lambda/\mu}$ is $s_{\lambda/\mu}$;
- $g_{\lambda/\mu}$ are symmetric (see [Lam, Pylyavskyy (2007)]);
- there exist involutions $B_i : \mathrm{RPP}(\lambda/\mu, \leq m) \to \mathrm{RPP}(\lambda/\mu, \leq m)$ such that
$$w(B_i(R)) = s_i w(R);$$
- B_i restricted to $\mathrm{SSYT}(\lambda/\mu, \leq m)$ are classical Bender-Knuth involutions.

$s_{\lambda/\mu}$ and $g_{\lambda/\mu}$ are symmetric!

Bender-Knuth involutions

Want to construct $B_i : \mathrm{RPP}(\lambda/\mu, \leq m) \to \mathrm{RPP}(\lambda/\mu, \leq m)$. Note that it is enough to consider the case $i = 1, m = 2$:

$s_{\lambda/\mu}$ and $g_{\lambda/\mu}$ are symmetric!

Bender-Knuth involutions

Want to construct $B_i : \mathrm{RPP}(\lambda/\mu, \leq m) \to \mathrm{RPP}(\lambda/\mu, \leq m)$. Note that it is enough to consider the case $i = 1, m = 2$:

Reduction to the case $m = 2$

Let $i = 5$.

					1	5	5
					2	6	7
			1	3	3	7	8
	1	1	5	6	6	8	9
5	5	6	6	9			
6	7	7	8				

$s_{\lambda/\mu}$ and $g_{\lambda/\mu}$ are symmetric!

Bender-Knuth involutions

Want to construct $B_i : \mathrm{RPP}(\lambda/\mu, \leq m) \to \mathrm{RPP}(\lambda/\mu, \leq m)$. Note that it is enough to consider the case $i = 1, m = 2$:

Reduction to the case $m = 2$

Let $i = 5$.

					1	**5**	**5**
					2	**6**	7
			1	3	3	7	8
	1	1	**5**	**6**	**6**	8	9
5	**5**	**6**	**6**	9			
6	7	7	8				

$s_{\lambda/\mu}$ and $g_{\lambda/\mu}$ are symmetric!

Bender-Knuth involutions

Want to construct $B_i : \mathrm{RPP}(\lambda/\mu, \leq m) \to \mathrm{RPP}(\lambda/\mu, \leq m)$. Note that it is enough to consider the case $i = 1$, $m = 2$:

Reduction to the case $m = 2$

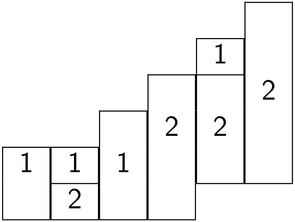
Let $i = 5$.

					1	**5**	**5**
					2	**6**	7
			1	3	3	7	8
	1	1	**5**	**6**	**6**	8	9
5	**5**	**6**	**6**	9			
6	7	7	8				

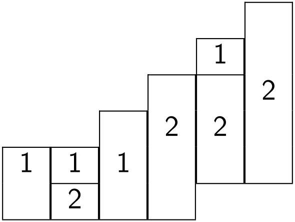
$\longrightarrow$

						5	5
						6	
			5	6	6		
5	5	6	6				
6							

Three types of columns



Three types of columns

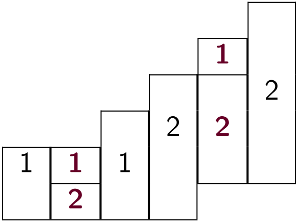


Definition

Let $R \in \mathrm{RPP}(\lambda/\mu, 2)$. A column of R is called

- *mixed*, if it contains a 1 and a 2;

Three types of columns

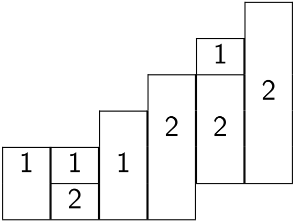


Definition

Let $R \in \mathrm{RPP}(\lambda/\mu, 2)$. A column of R is called

- *mixed*, if it contains a 1 and a 2;

Three types of columns

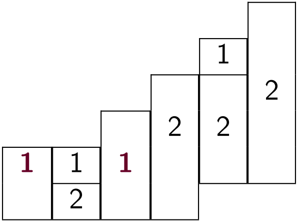


Definition

Let $R \in \mathrm{RPP}(\lambda/\mu, 2)$. A column of R is called

- *mixed*, if it contains a 1 and a 2;
- 1-*pure*, if it contains a 1 and not a 2;

Three types of columns

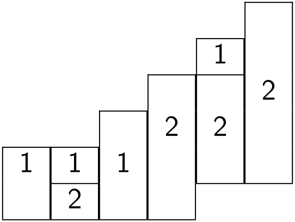


Definition

Let $R \in \mathrm{RPP}(\lambda/\mu, 2)$. A column of R is called

- *mixed*, if it contains a 1 and a 2;
- 1-*pure*, if it contains a 1 and not a 2;

Three types of columns

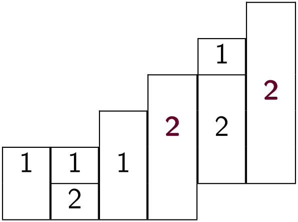


Definition

Let $R \in \mathrm{RPP}(\lambda/\mu, 2)$. A column of R is called

- *mixed*, if it contains a 1 and a 2;
- 1-*pure*, if it contains a 1 and not a 2;
- 2-*pure*, if it contains a 2 and not a 1;

Three types of columns



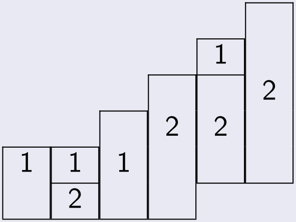
Definition

Let $R \in \mathrm{RPP}(\lambda/\mu, 2)$. A column of R is called

- *mixed*, if it contains a 1 and a 2;
- 1-*pure*, if it contains a 1 and not a 2;
- 2-*pure*, if it contains a 2 and not a 1;

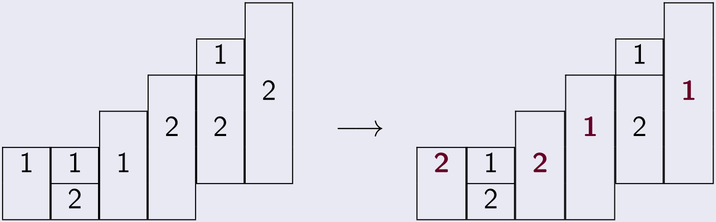
Construction of Bender-Knuth involutions

Flip map



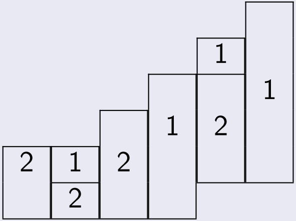
Construction of Bender-Knuth involutions

Flip map



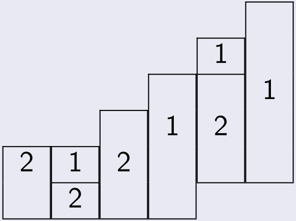
Descent-resolution step

A lot of descents



Descent-resolution step

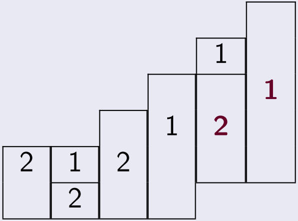
A lot of descents



(M1) mixed vs. 1-pure;

Descent-resolution step

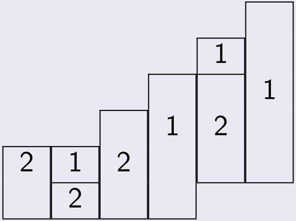
A lot of descents



(M1) mixed vs. 1-pure;

Descent-resolution step

A lot of descents

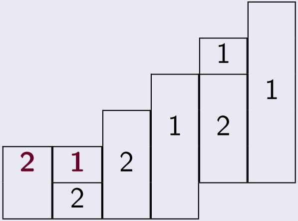


(M1) mixed vs. 1-pure;

(2M) 2-pure vs. mixed;

Descent-resolution step

A lot of descents

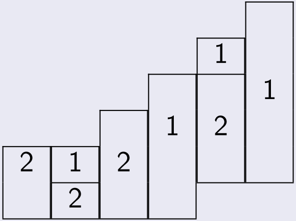


(M1) mixed vs. 1-pure;

(2M) 2-pure vs. mixed;

Descent-resolution step

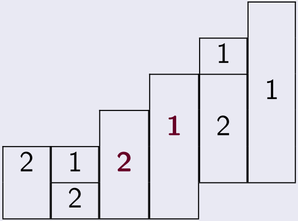
A lot of descents



(M1) mixed vs. 1-pure;
(2M) 2-pure vs. mixed;
(21) 2-pure vs. 1-pure.

Descent-resolution step

A lot of descents



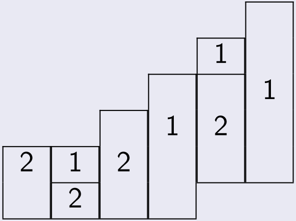
(M1) mixed vs. 1-pure;

(2M) 2-pure vs. mixed;

(21) 2-pure vs. 1-pure.

Descent-resolution step

A lot of descents



(M1) mixed vs. 1-pure;

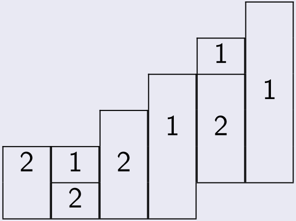
(2M) 2-pure vs. mixed;

(21) 2-pure vs. 1-pure.

(MM) mixed vs. mixed

Descent-resolution step

A lot of descents



(M1) mixed vs. 1-pure;

(2M) 2-pure vs. mixed;

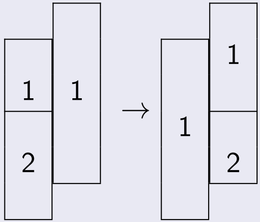
(21) 2-pure vs. 1-pure.

(MM) mixed vs. mixed – **never happens!**

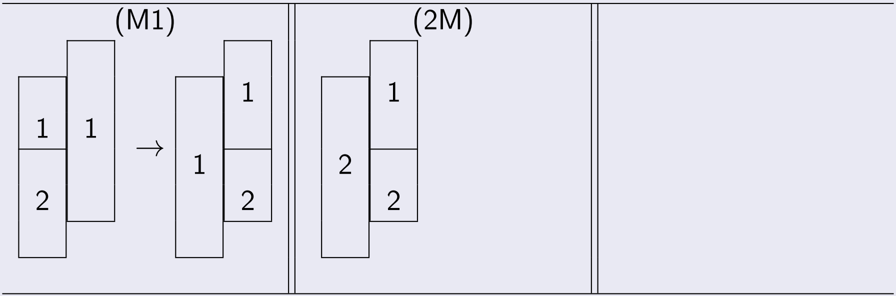
Descent-resolution steps

Descent-resolution steps

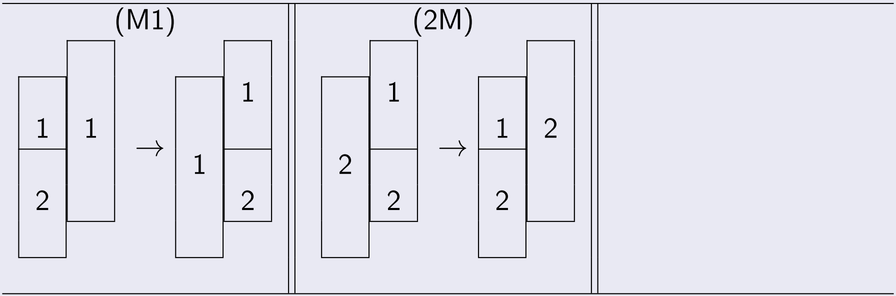
(M1)



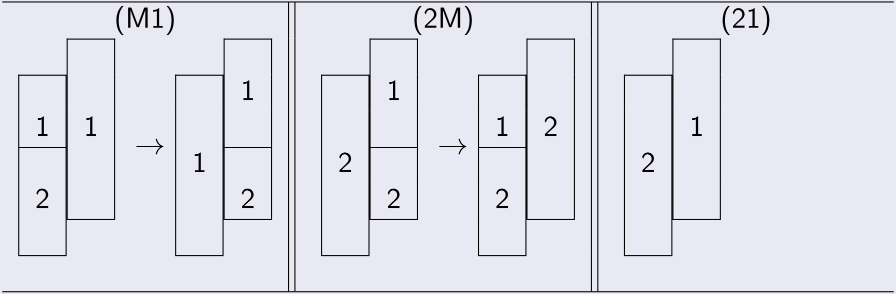
Descent-resolution steps



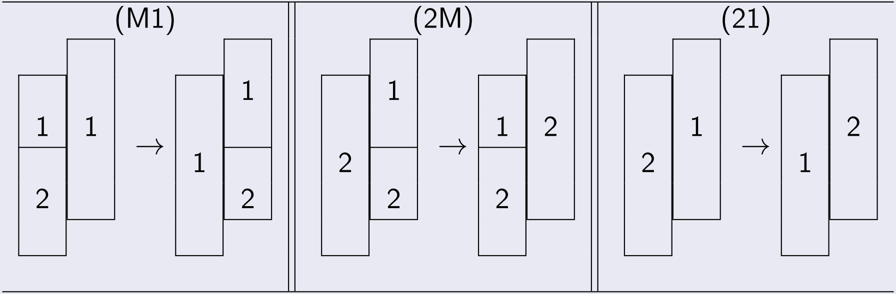
Descent-resolution steps



Descent-resolution steps



Descent-resolution steps



Properties

The descent-resolution process

Properties

The descent-resolution process

- ends after a finite number of steps;

Properties

The descent-resolution process

- ends after a finite number of steps;
- the result does not depend on the order!

Properties

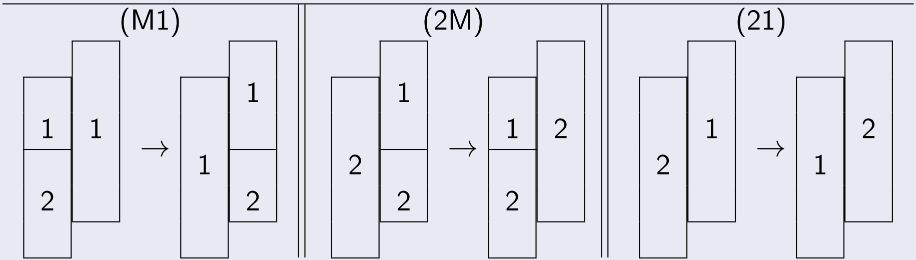
The descent-resolution process

- ends after a finite number of steps;
- the result does not depend on the order!

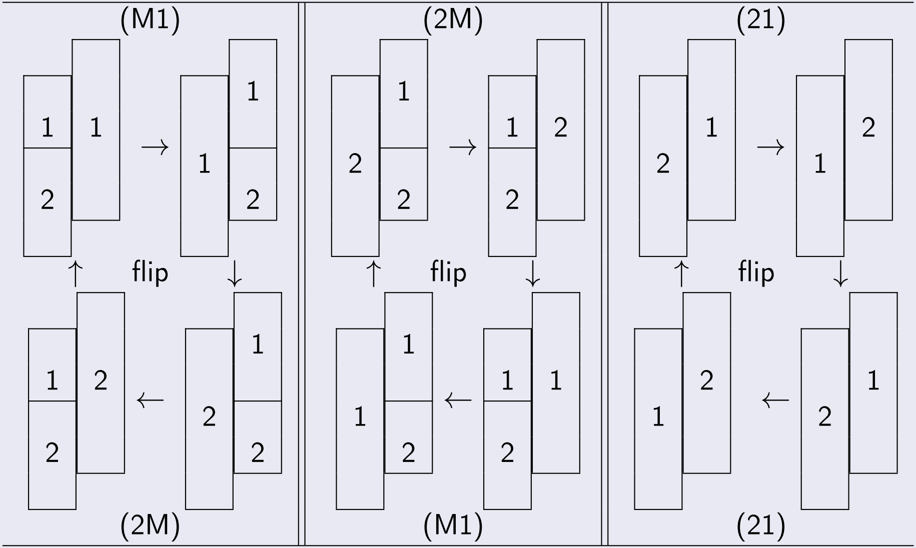
Corollary

B_1 is an **involution** *on $\mathrm{RPP}(\lambda/\mu, 2)$ that switches the number of 1-pure columns with the number of 2-pure columns.*

Why is B_i an involution?



Why is B_i an involution?



Part 2: Schur expansion

Knuth equivalence

Definition

Two words are *Knuth equivalent* if they can be obtained from each other by moves

$$yzx \leftrightarrow yxz, \text{ if } x < y \leq z;$$

$$xzy \leftrightarrow zxy, \text{ if } x \leq y < z.$$

Knuth equivalence

Definition

Two words are *Knuth equivalent* if they can be obtained from each other by moves

$$yzx \leftrightarrow yxz, \text{ if } x < y \leq z;$$

$$xzy \leftrightarrow zxy, \text{ if } x \leq y < z.$$

Definition

Reading word: concatenate rows from bottom to top.

Knuth equivalence

Definition

Two words are *Knuth equivalent* if they can be obtained from each other by moves

$$yzx \leftrightarrow yxz, \text{ if } x < y \le z;$$

$$xzy \leftrightarrow zxy, \text{ if } x \le y < z.$$

Definition

Reading word: concatenate rows from bottom to top.

$$T = \begin{array}{|c|c|c|} \hline \multicolumn{1}{c|}{} & 1 & 3 \\ \hline 2 & 2 & 4 \\ \hline \end{array}$$

Knuth equivalence

Definition

Two words are *Knuth equivalent* if they can be obtained from each other by moves

$$yzx \leftrightarrow yxz, \text{ if } x < y \leq z;$$

$$xzy \leftrightarrow zxy, \text{ if } x \leq y < z.$$

Definition

Reading word: concatenate rows from bottom to top.

$$T = \begin{array}{ccc} & \boxed{1} & \boxed{3} \\ \boxed{2} & \boxed{2} & \boxed{4} \end{array} \qquad \mathrm{rw}(T) = (2, 2, 4, 1, 3).$$

Knuth equivalence

Definition

Two words are *Knuth equivalent* if they can be obtained from each other by moves

$$yzx \leftrightarrow yxz, \text{ if } x < y \le z;$$

$$xzy \leftrightarrow zxy, \text{ if } x \le y < z.$$

Definition

Reading word: concatenate rows from bottom to top.

$$T = \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 1 & 3 \\ \hline 2 & 2 & 4 \\ \hline\end{array} \qquad \mathrm{rw}(T) = (2,2,4,1,3).$$

Proposition

Every word is Knuth equivalent to **exactly one** *word which is a reading word of a SSYT of* **straight shape** *($\mu = ()$).*

Crystal operators

Definition

$E_i : [m]^r \to [m]^r \cup \{0\}$ is defined as follows. For $u \in [m]^r$,

Assume $i = 3$.

$u =$ `1,4,1,3,5,4,4,3,3,1,3,1,4,5,3,1,3,3,4,1,4`

Crystal operators

Definition

$E_i : [m]^r \to [m]^r \cup \{0\}$ is defined as follows. For $u \in [m]^r$,

- ignore all letters of u except for i and $i+1$;

Assume $i = 3$.

$u =$ `1,4,1,3,5,4,4,3,3,1,3,1,4,5,3,1,3,3,4,1,4`

Crystal operators

Definition

$E_i : [m]^r \to [m]^r \cup \{0\}$ is defined as follows. For $u \in [m]^r$,

- ignore all letters of u except for i and $i+1$;

Assume $i = 3$.

$u =$ 1,4,1,3,5,4,4,3,3,1,3,1,4,5,3,1,3,3,4,1,4

Crystal operators

Definition

$E_i : [m]^r \to [m]^r \cup \{0\}$ is defined as follows. For $u \in [m]^r$,

- ignore all letters of u except for i and $i+1$;
- label each i by $)$ and each $i+1$ by $($;

Assume $i = 3$.

$u =$ 1,4,1,3,5,4,4,3,3,1,3,1,4,5,3,1,3,3,4,1,4

Crystal operators

Definition

$E_i : [m]^r \to [m]^r \cup \{0\}$ is defined as follows. For $u \in [m]^r$,

- ignore all letters of u except for i and $i+1$;
- label each i by) and each $i+1$ by (;

Assume $i = 3$.

$u =$ `1,4,1,3,5,4,4,3,3,1,3,1,4,5,3,1,3,3,4,1,4`

`1,(,1,),5,(,(,),),1,),1,(,5,),1,),),(,1,(`

Crystal operators

Definition

$E_i : [m]^r \to [m]^r \cup \{0\}$ is defined as follows. For $u \in [m]^r$,

- ignore all letters of u except for i and $i+1$;
- label each i by $)$ and each $i+1$ by $($;
- ignore all pairs of matching parentheses;

Assume $i = 3$.

$u =$ 1,4,1,3,5,4,4,3,3,1,3,1,4,5,3,1,3,3,4,1,4

1,(,1,),5,(,(,),),1,),1,(,5,),1,),),(,1,(

Crystal operators

Definition

$E_i : [m]^r \to [m]^r \cup \{0\}$ is defined as follows. For $u \in [m]^r$,

- ignore all letters of u except for i and $i+1$;
- label each i by) and each $i+1$ by (;
- ignore all pairs of matching parentheses;

Assume $i = 3$.

$u =$ 1,4,1,3,5,4,4,3,3,1,3,1,4,5,3,1,3,3,4,1,4

1,(,1,),5,(,(,),),1,),1,(,5,),1,),),(,1,(

1,(,1,),5,(,(,),),1,**)**,1,(,5,),1,**)**,**)**,**(**,1,**(**

Crystal operators

Definition

$E_i : [m]^r \to [m]^r \cup \{0\}$ is defined as follows. For $u \in [m]^r$,

- ignore all letters of u except for i and $i+1$;
- label each i by) and each $i+1$ by (;
- ignore all pairs of matching parentheses;
- replace the rightmost unmatched) by (.

Assume $i = 3$.

$u =$ 1,4,1,3,5,4,4,3,3,1,3,1,4,5,3,1,3,3,4,1,4

1,(,1,),5,(,(,),),1,),1,(,5,),1,),),(,1,(

1,(,1,),5,(,(,),),1,**)**,1,(,5,),1,**)**,**)**,**(**,1,**(**

Crystal operators

Definition

$E_i : [m]^r \to [m]^r \cup \{0\}$ is defined as follows. For $u \in [m]^r$,

- ignore all letters of u except for i and $i+1$;
- label each i by) and each $i+1$ by (;
- ignore all pairs of matching parentheses;
- replace the rightmost unmatched) by (.

Assume $i = 3$.

$u =$ 1,4,1,3,5,4,4,3,3,1,3,1,4,5,3,1,3,3,4,1,4

1,(,1,),5,(,(,),),1,),1,(,5,),1,),),(,1,(

1,(,1,),5,(,(,),),1,**)**,1,(,5,),1,**)**,**)**,**(**,1,**(**

1,(,1,),5,(,(,),),1,),1,(,5,),1,),(,(,1,(

Crystal operators

Definition

$E_i : [m]^r \to [m]^r \cup \{0\}$ is defined as follows. For $u \in [m]^r$,

- ignore all letters of u except for i and $i+1$;
- label each i by) and each $i+1$ by (;
- ignore all pairs of matching parentheses;
- replace the rightmost unmatched) by (.

Assume $i = 3$.

$u =$ 1,4,1,3,5,4,4,3,3,1,3,1,4,5,3,1,3,3,4,1,4

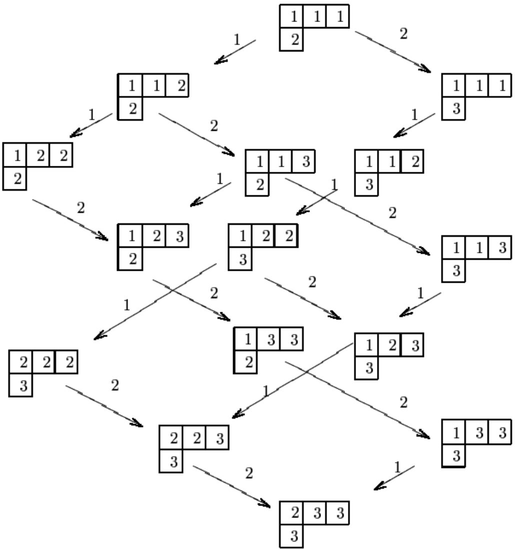
1,(,1,),5,(,(,),),1,),1,(,5,),1,),),(,1,(

1,(,1,),5,(,(,),),1,),1,(,5,),1,),),(,1,(

1,(,1,),5,(,(,),),1,),1,(,5,),1,),(,(,1,(

$E_i(u) =$ 1,4,1,3,5,4,4,3,3,1,3,1,4,5,3,1,3,4,4,1,4

Crystal operators on SSYT



(a picture from [Kashiwara (1995)])

Littlewood-Richardson rule

- Crystal operators on words commute with Knuth equivalence relations;

Littlewood-Richardson rule

- Crystal operators on words commute with Knuth equivalence relations;
- For any straight shape λ and any m, the crystal graph on $\mathrm{SSYT}(\lambda, m)$ is connected;

Littlewood-Richardson rule

- Crystal operators on words commute with Knuth equivalence relations;
- For any straight shape λ and any m, the crystal graph on $\mathrm{SSYT}(\lambda, m)$ is connected;
- Only one tableau $T_\lambda \in \mathrm{SSYT}(\lambda, m)$ satisfies $E_i^{-1}(T_\lambda) = \emptyset$ for all $i < m$:

1	1	1	1
2	2	2	2
3	3	3	

Littlewood-Richardson rule

- Crystal operators on words commute with Knuth equivalence relations;
- For any straight shape λ and any m, the crystal graph on $\mathrm{SSYT}(\lambda, m)$ is connected;
- Only one tableau $T_\lambda \in \mathrm{SSYT}(\lambda, m)$ satisfies $E_i^{-1}(T_\lambda) = \emptyset$ for all $i < m$:

1	1	1	1
2	2	2	2
3	3	3	

- We have $w(T_\lambda) = \lambda$.

Littlewood-Richardson rule

- Crystal operators on words commute with Knuth equivalence relations;
- For any straight shape λ and any m, the crystal graph on $\mathrm{SSYT}(\lambda, m)$ is connected;
- Only one tableau $T_\lambda \in \mathrm{SSYT}(\lambda, m)$ satisfies $E_i^{-1}(T_\lambda) = \emptyset$ for all $i < m$:

1	1	1	1
2	2	2	2
3	3	3	

- We have $w(T_\lambda) = \lambda$.

Corollary

If $W \subset [m]^r$ is closed under the action of E_i, then

$$\sum_{u \in W} x^{w(u)} = \sum_{u \in W : E_i^{-1}(u) = \emptyset \ \forall i} s_{w(u)}(x_1, \ldots, x_m).$$

Reading words for RPP

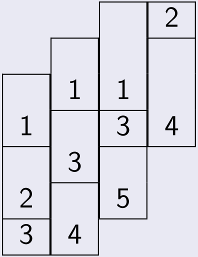
Definition

$$R = \begin{array}{|c|c|c|c|}
\cline{3-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & 2 \\
\cline{2-4}
\multicolumn{1}{c|}{} & 1 & 1 & 4 \\
\hline
1 & 1 & 1 & 4 \\
\hline
1 & 3 & 3 & 4 \\
\hline
2 & 3 & 5 & \multicolumn{1}{c}{} \\
\cline{1-3}
2 & 4 & 5 & \multicolumn{1}{c}{} \\
\cline{1-3}
3 & 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{1-2}
\end{array}$$

Reading words for RPP

Definition

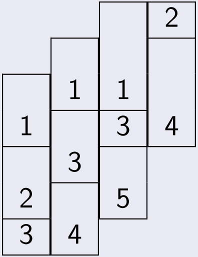
$$R = \begin{array}{cccc} & & 1 & 2 \\ & 1 & 1 & 4 \\ 1 & 1 & 1 & 4 \\ 1 & 3 & 3 & 4 \\ 2 & 3 & 5 & \\ 2 & 4 & 5 & \\ 3 & 4 & & \end{array} \quad \longrightarrow$$



Reading words for RPP

Definition

$$R = \begin{array}{|c|c|c|c|}
\cline{3-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & 2 \\
\cline{2-4}
\multicolumn{1}{c|}{} & 1 & 1 & 4 \\
\hline
1 & 1 & 1 & 4 \\
\hline
1 & 3 & 3 & 4 \\
\hline
2 & 3 & 5 & \multicolumn{1}{c}{} \\
\cline{1-3}
2 & 4 & 5 & \multicolumn{1}{c}{} \\
\cline{1-3}
3 & 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{1-2}
\end{array} \quad \longrightarrow$$



$$\mathrm{rw}(R) = 34253134112.$$

Crystal action on RPP

Definition

For $R \in \mathrm{RPP}(\lambda/\mu, m)$ define $\mathrm{ceq}(R) = (\mathrm{ceq}_1(R), \mathrm{ceq}_2(R), \dots)$ where $\mathrm{ceq}_i(R) :=$ [number of equalities between rows i and $i+1$].

Crystal action on RPP

Definition

For $R \in \mathrm{RPP}(\lambda/\mu, m)$ define $\mathrm{ceq}(R) = (\mathrm{ceq}_1(R), \mathrm{ceq}_2(R), \dots)$ where $\mathrm{ceq}_i(R) :=$ [number of equalities between rows i and $i+1$].

Proposition

If $R \in \mathrm{RPP}(\lambda/\mu, m)$ then there exists a unique $Q \in \mathrm{RPP}(\lambda/\mu, m)$ with

- $\mathrm{rw}(Q) = E_i(\mathrm{rw}(R))$;

Crystal action on RPP

Definition

For $R \in \mathrm{RPP}(\lambda/\mu, m)$ define $\mathrm{ceq}(R) = (\mathrm{ceq}_1(R), \mathrm{ceq}_2(R), \dots)$ where $\mathrm{ceq}_i(R) :=$ [number of equalities between rows i and $i+1$].

Proposition

If $R \in \mathrm{RPP}(\lambda/\mu, m)$ then there exists a unique $Q \in \mathrm{RPP}(\lambda/\mu, m)$ with

- $\mathrm{rw}(Q) = E_i(\mathrm{rw}(R))$;
- $\mathrm{ceq}(Q) = \mathrm{ceq}(R)$.

Crystal action on RPP

Definition

For $R \in \mathrm{RPP}(\lambda/\mu, m)$ define $\mathrm{ceq}(R) = (\mathrm{ceq}_1(R), \mathrm{ceq}_2(R), \dots)$ where $\mathrm{ceq}_i(R) :=$ [number of equalities between rows i and $i+1$].

Proposition

If $R \in \mathrm{RPP}(\lambda/\mu, m)$ then there exists a unique $Q \in \mathrm{RPP}(\lambda/\mu, m)$ with

- $\mathrm{rw}(Q) = E_i(\mathrm{rw}(R))$;
- $\mathrm{ceq}(Q) = \mathrm{ceq}(R)$.

Proposition

The ceq*-statistics is preserved by flips and descent resolution steps.*

Schur expansion of $g_{\lambda/\mu}$

Theorem

$$g_{\lambda/\mu}(x_1,\dots,x_m) = \sum_{\substack{R \text{ is a RPP} \\ \text{of shape } \lambda/\mu \\ \text{with entries} \leq m \\ \text{such that } E_i^{-1}(\mathrm{rw}(R)) = \emptyset \text{ for all } i < m}} s_{w(R)}(x_1,\dots,x_m).$$

Thank you!

- Lam, T., Pylyavskyy, P., Combinatorial Hopf algebras and *K-homology of Grassmanians*, *International Mathematics Research Notices*, Vol. 2007, doi:10.1093/imrn/rnm125.
- Galashin, P. (2014). A Littlewood-Richardson Rule for Dual Stable Grothendieck Polynomials. *arXiv preprint* arXiv:1501.00051.
- Galashin, P., Grinberg, D., and Liu, G. (2015). Refined dual stable Grothendieck polynomials and generalized Bender-Knuth involutions. *arXiv preprint* arXiv:1509.03803.
- Kashiwara, M. (1995). On crystal bases. *Representations of groups* (Banff, AB, 1994), 16, 155-197.