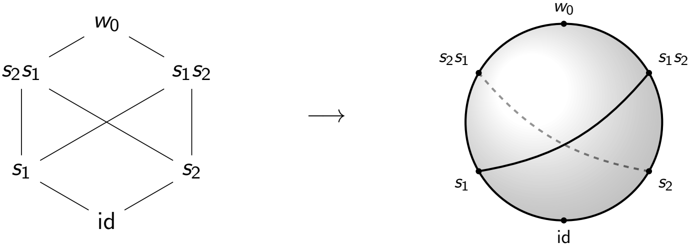
Totally positive spaces: topology and applications

Pavel Galashin

April 26, 2019

Joint work with Steven Karp, Thomas Lam, and Pavlo Pylyavskyy
`arXiv:1707.02010`, `arXiv:1807.03282`, `arXiv:1904.00527`



Part 1. Topology

Regular CW complexes

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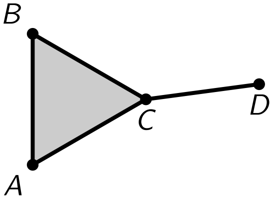

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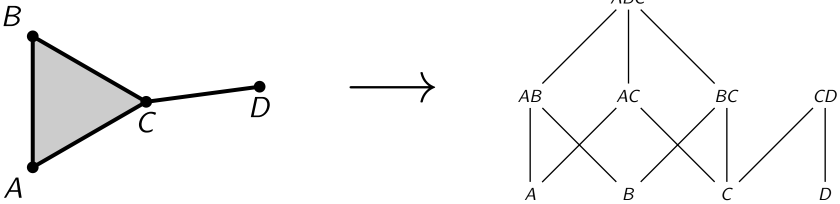

Regular CW complex

Face poset

Face poset → regular CW complex

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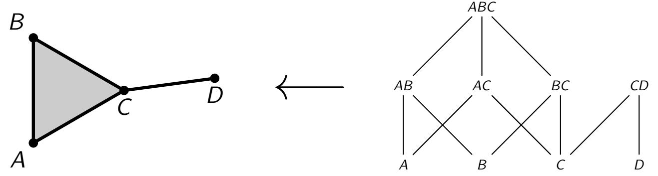

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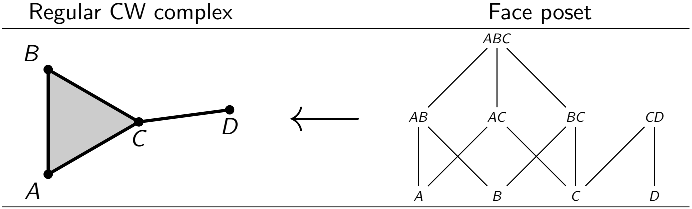

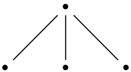

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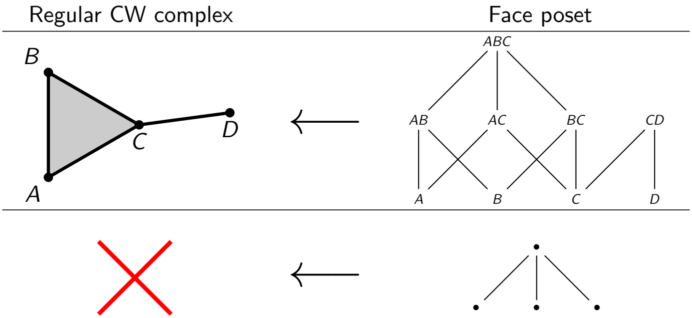

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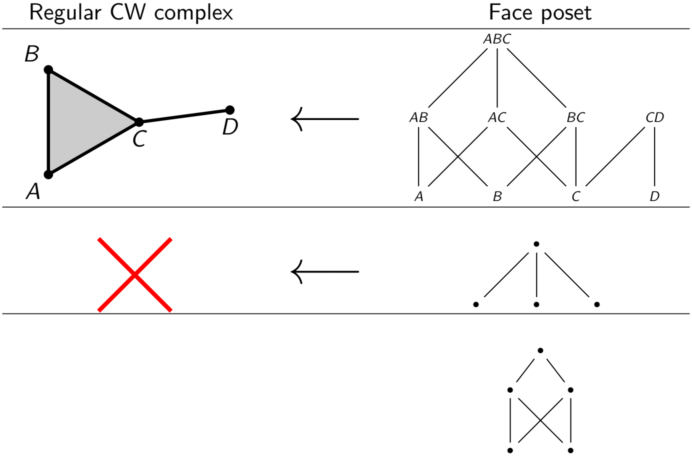

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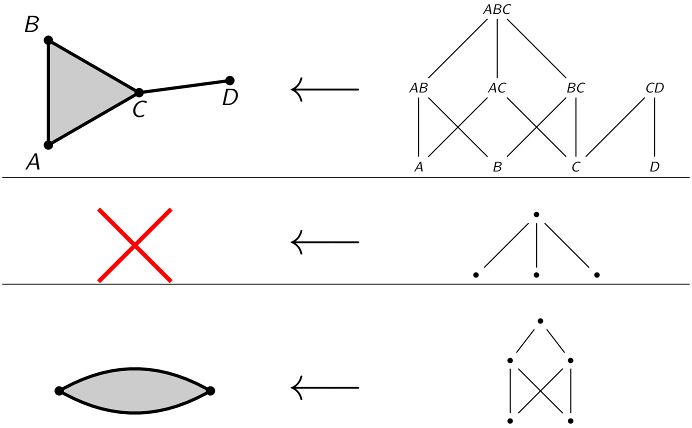

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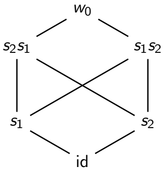

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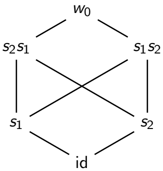

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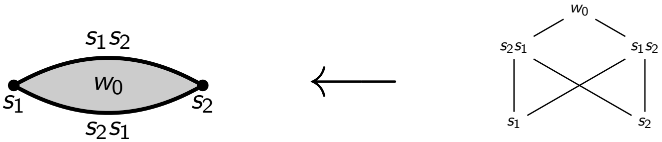

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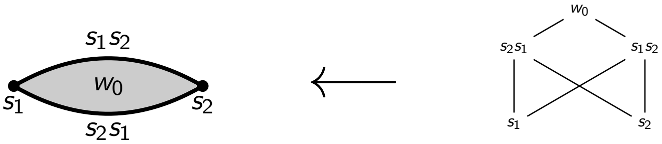
Question (Björner–Bernstein)

Does the corresponding regular CW complex exist "in nature"?



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Theorem (Lusztig (1994))

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Fomin–Shapiro conjecture

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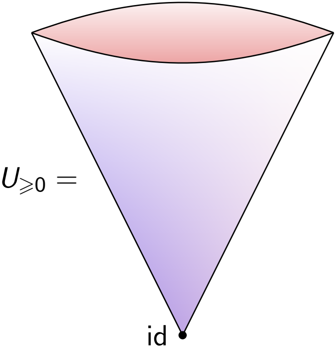

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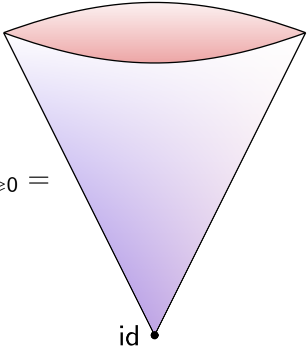

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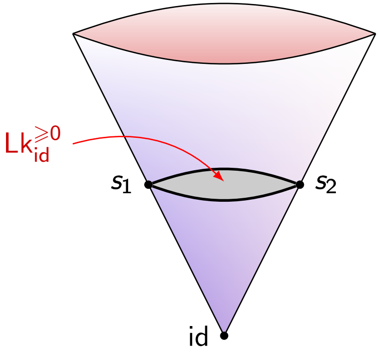

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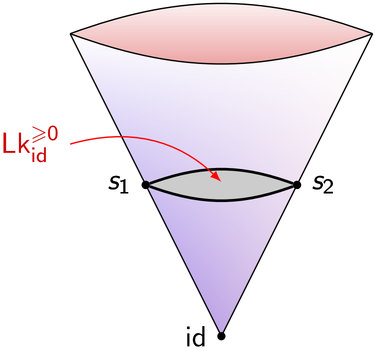

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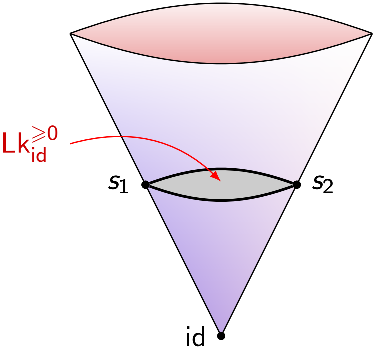

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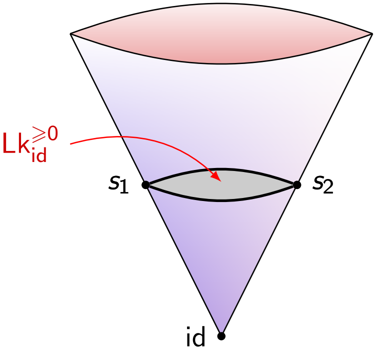

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Theorem (Hersh (2014))

The Fomin–Shapiro Conjecture is true.

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Example

All $n!$ coordinate flags $\{wB \mid w \in S_n\}$ belong to $(G/B)_{\geqslant 0}$.

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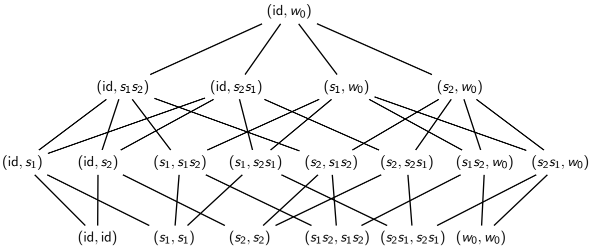
$$(Q, \preceq)$$



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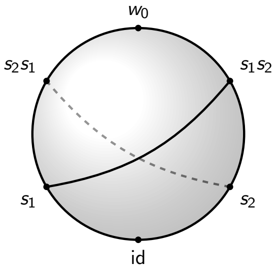

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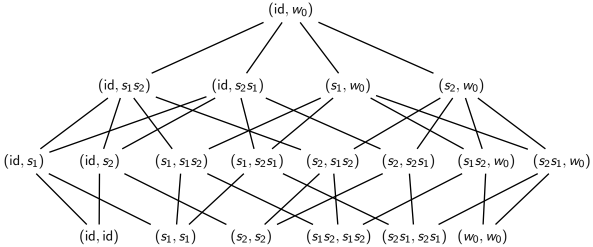

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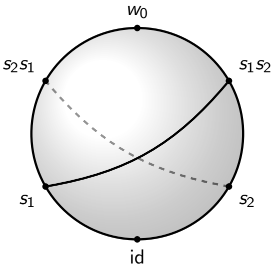

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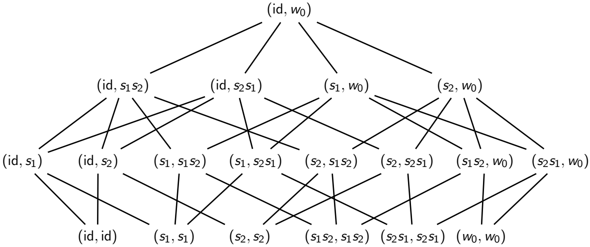

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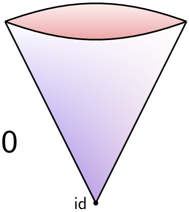

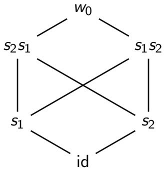

w_0

s_2s_1 s_1s_2

s_1 s_2

id

$(S_n \leq)$

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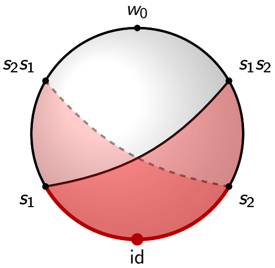

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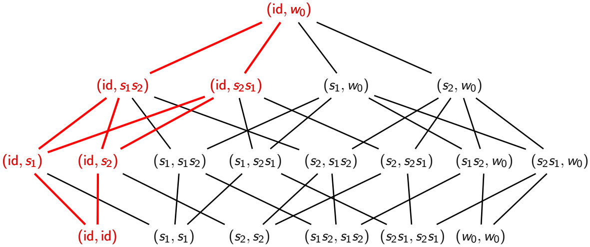

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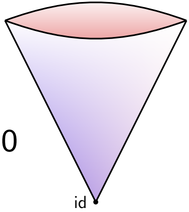

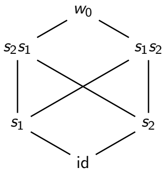

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In this case $G/P = \mathrm{Gr}(k,n)$, and the projection is

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Let $P \supset B$ be a parabolic subgroup of G. We get a projection

$$\pi : G/B \to G/P \qquad \overset{\text{flag}}{(V_0, V_1, \dots, V_{n-1}, V_n)} \mapsto \overset{\text{partial flag}}{(V_0, V_{j_1}, \dots, V_{j_m}, V_n)}.$$

Lusztig (1994): $(G/P)_{\geqslant 0} := \pi\left((G/B)_{\geqslant 0}\right)$.

Example

Maximal parabolic subgroup: $P = \begin{pmatrix} \mathrm{GL}_k(\mathbb{R}) & * \\ 0 & \mathrm{GL}_{n-k}(\mathbb{R}) \end{pmatrix}$.

In this case $G/P = \mathrm{Gr}(k,n)$, and the projection is

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Postnikov (2006):

$$\mathrm{Gr}_{\geqslant 0}(k,n) := \{V_k \in \mathrm{Gr}(k,n) \mid \Delta_I(V_k) \geqslant 0 \text{ for all } I \subset [n] \text{ of size } k\}.$$

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Surprising fact: When $G/P = \mathrm{Gr}(k, n)$, we have $(G/P)_{\geqslant 0} = \mathrm{Gr}_{\geqslant 0}(k, n)$.

Regularity theorem

Conjecture (Postnikov (2006), Williams (2007))

- $\mathrm{Gr}_{\geqslant 0}(k,n)$ *is a regular CW complex homeomorphic to a ball.*

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Corollary of proof (Hersh (2014)): $\mathrm{Lk}_{\mathrm{id}}^{\geqslant 0} \subset U_{\geqslant 0}$ is a regular CW complex.

Proof idea

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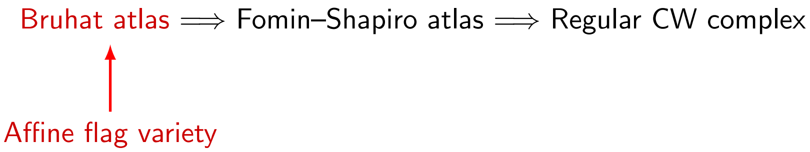

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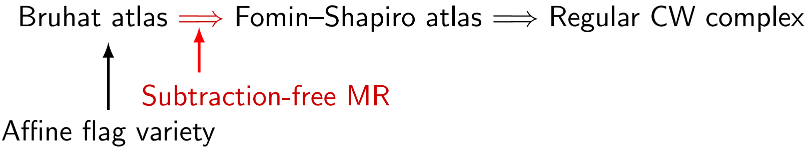

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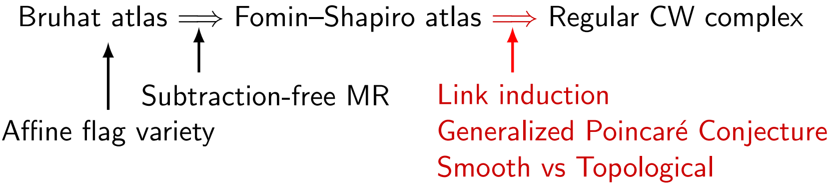

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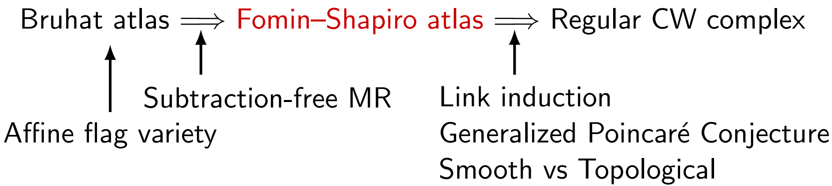

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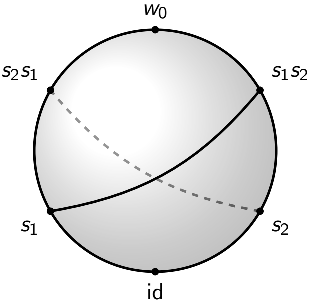

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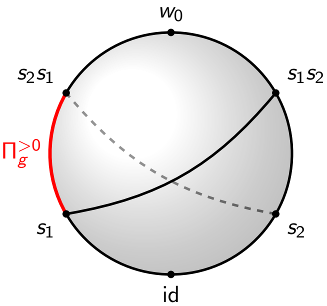

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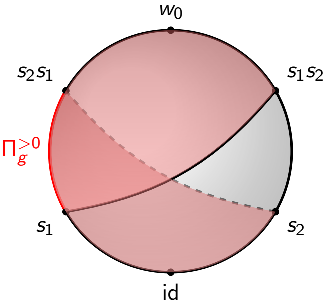

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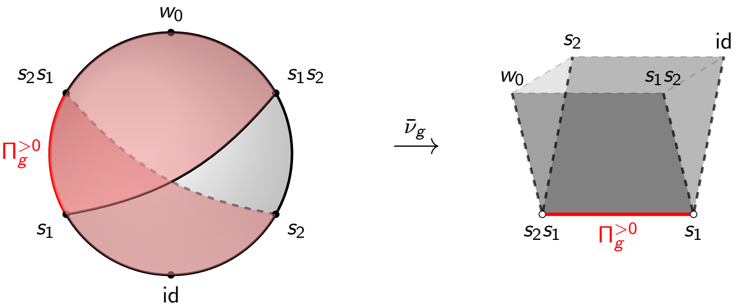

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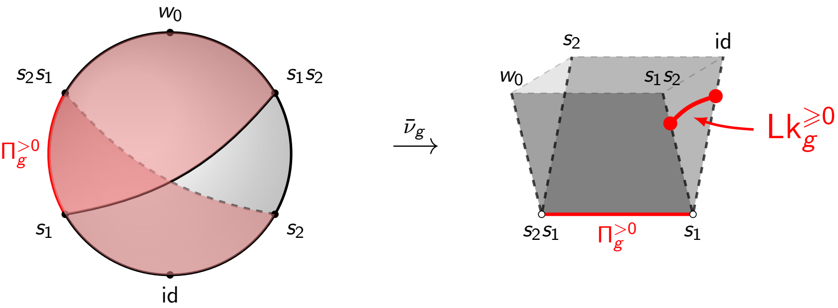
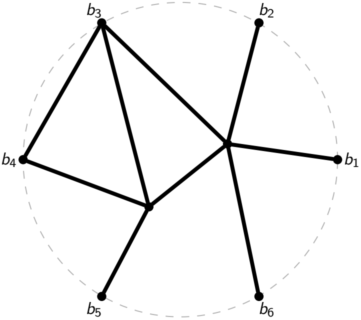


Part 2. Applications

Ising model

Definition

- Planar Ising network: planar weighted graph embedded in a disk.



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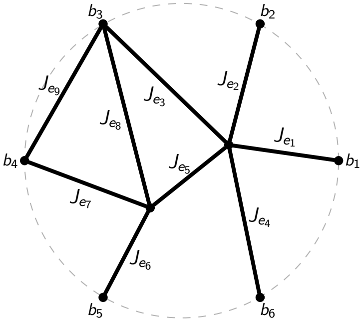

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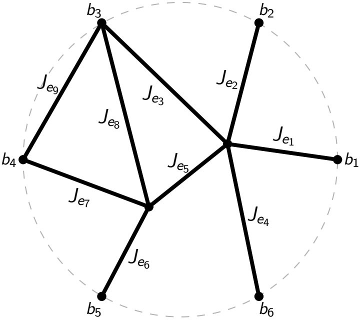

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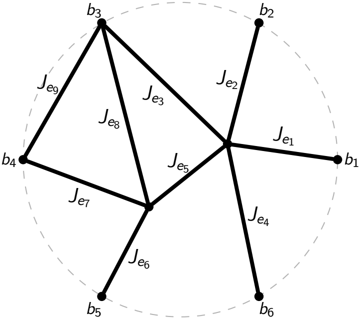

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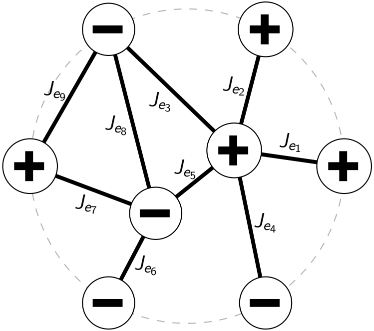

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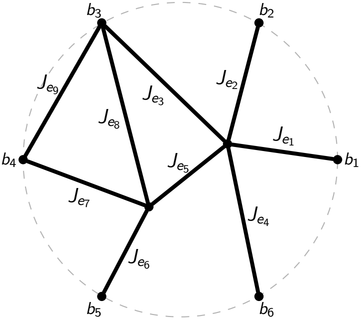

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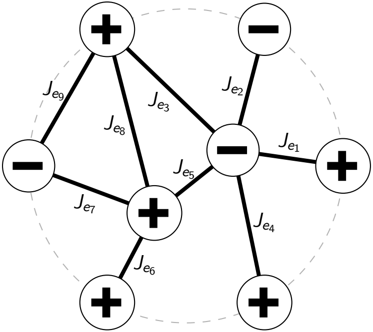

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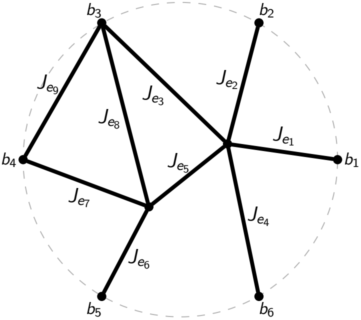

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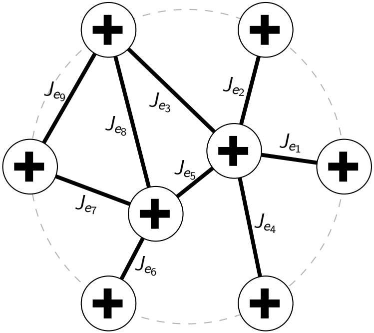

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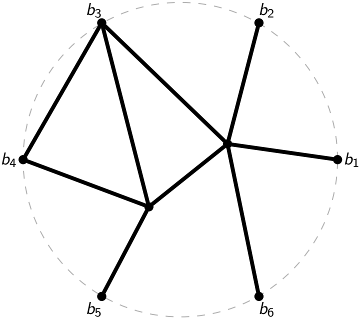

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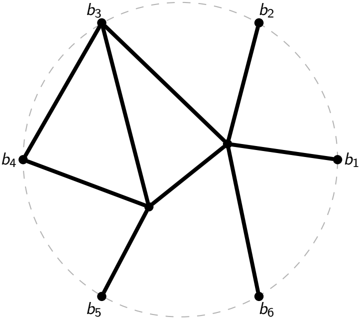

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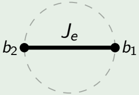
Example ($n = 2$)



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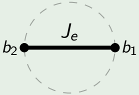

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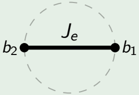

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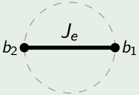

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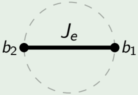

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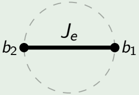

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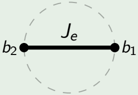

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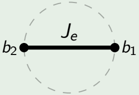

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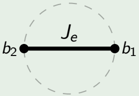

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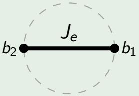

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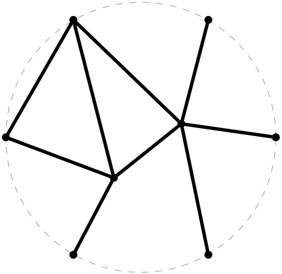
Boundary stratification

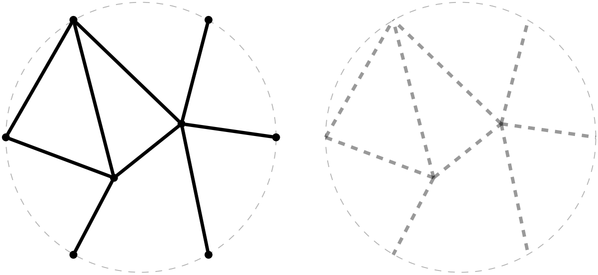
Planar Ising network $\to$ medial graph $\to$ matching on $[2n]$.



G

Boundary stratification

Planar Ising network → medial graph → matching on $[2n]$.



G Medial graph of G

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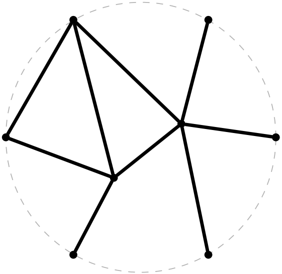

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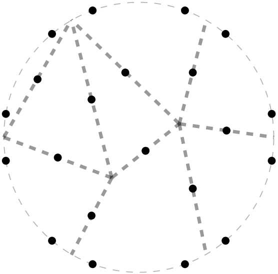

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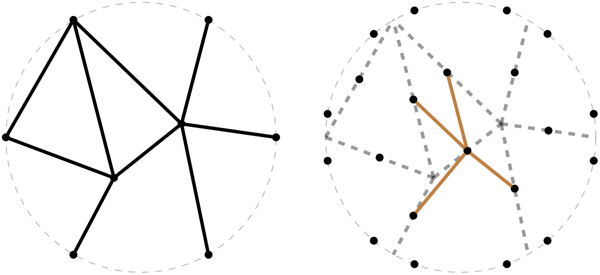

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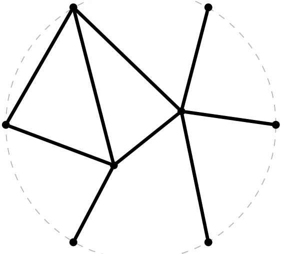

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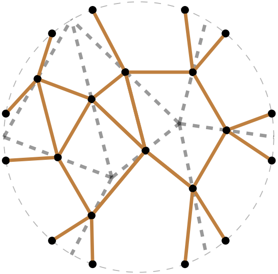

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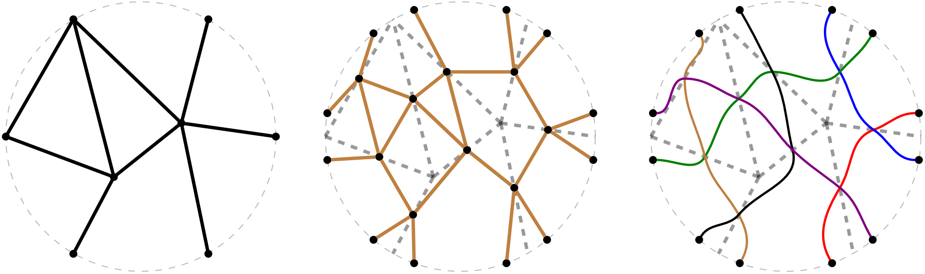

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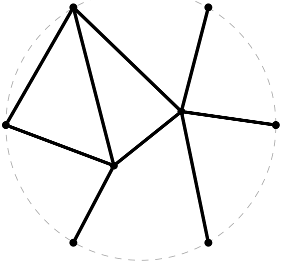

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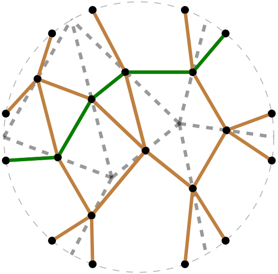

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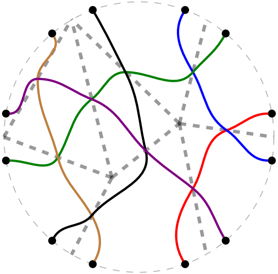

Medial pairing of G

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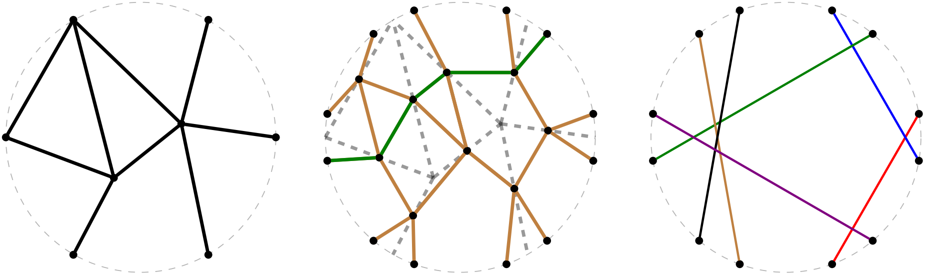
Planar Ising network → medial graph → matching on $[2n]$.



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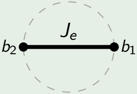

$$\overline{\mathcal{X}}_2 = \left\{ \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix} \middle| m \in [0, 1] \right\}.$$

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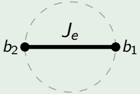

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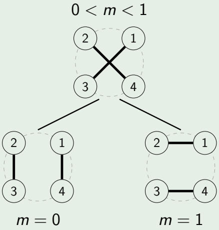
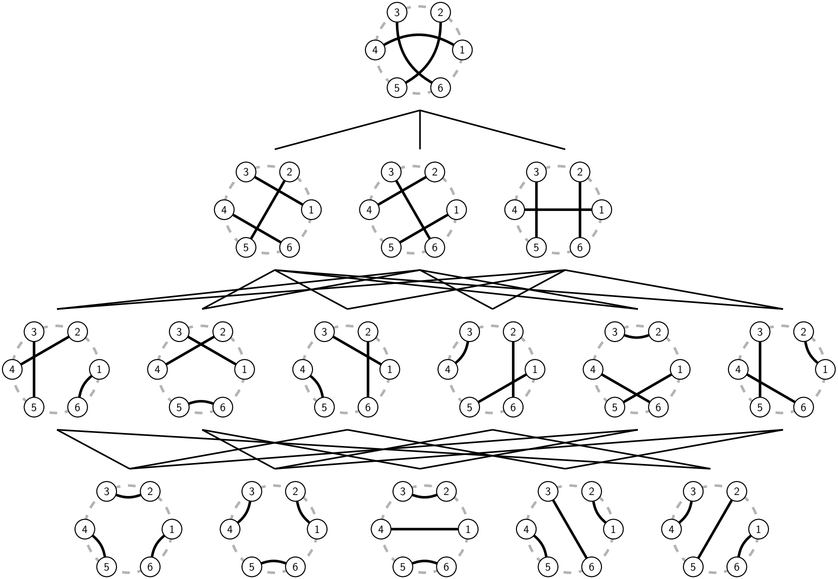


Matchings for $n = 3$



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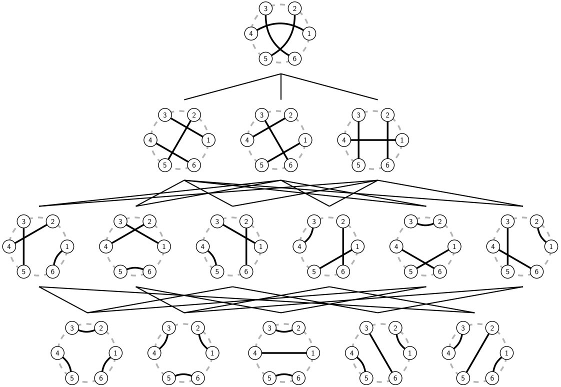

Theorem (Hersh–Kenyon(2018))

The matchings poset is thin and shellable.

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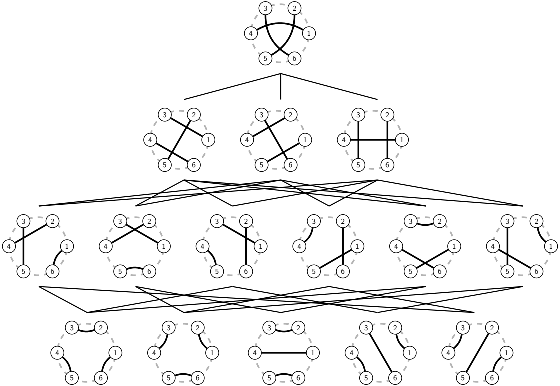

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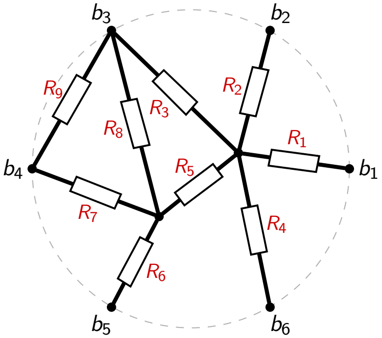
Conjecture (G.–Pylyavskyy (2018))

$\overline{\mathcal{X}}_n \cong \mathrm{OG}_{\geqslant 0}(n, 2n)$ *is a regular CW complex.*

Electrical networks

Let $R : E \to \mathbb{R}_{>0}$ be an assignment of resistances to the edges of G.



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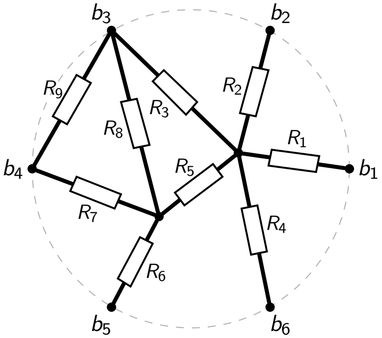
Let $R : E \to \mathbb{R}_{>0}$ be an assignment of resistances to the edges of G.

Definition

Electrical response matrix $\Lambda(G, R) : \mathbb{R}^n \to \mathbb{R}^n$, sending voltages to currents.



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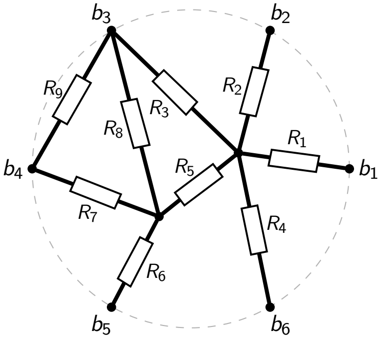

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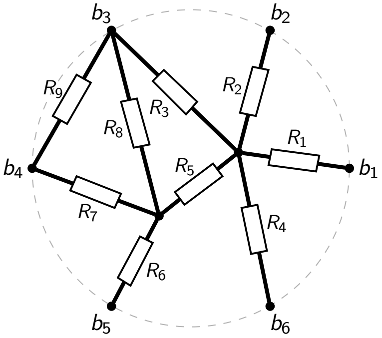

$\overline{E}_n$: compactification of the space of $n \times n$ electrical response matrices [Lam (2014)]

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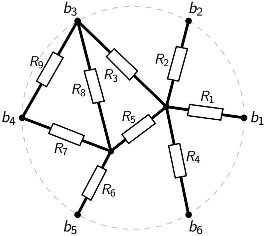

$\overline{E}_n$: compactification of the space of $n \times n$ electrical response matrices [Lam (2014)]

Theorem (G.–Karp–Lam (2017))

$\overline{E}_n$ is homeomorphic to an $\binom{n}{2}$-dimensional closed ball

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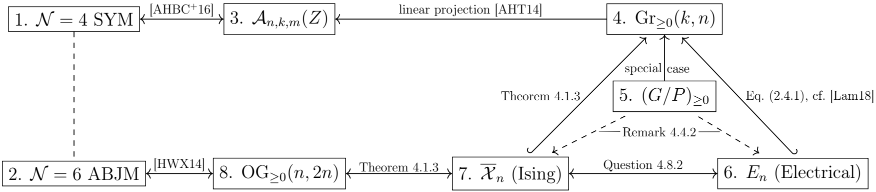
Problem

Construct a stratification-preserving homeomorphism between $\overline{\mathcal{X}}_n$ *and* $\overline{E}_n$.

Connections



1. $\mathcal{N} = 4$ SYM

[AHBC$^+$16]

3. $\mathcal{A}_{n,k,m}(Z)$

linear projection [AHT14]

4. $\mathrm{Gr}_{\geq 0}(k,n)$

special case

5. $(G/P)_{\geq 0}$

Theorem 4.1.3

Eq. (2.4.1), cf. [Lam18]

Remark 4.4.2

2. $\mathcal{N} = 6$ ABJM

[HWX14]

8. $\mathrm{OG}_{\geq 0}(n, 2n)$

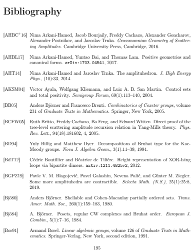
Theorem 4.1.3

7. $\overline{\mathcal{X}}_n$ (Ising)

Question 4.8.2

6. E_n (Electrical)

Bibliography



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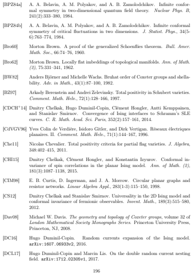

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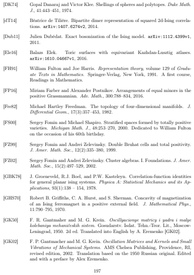

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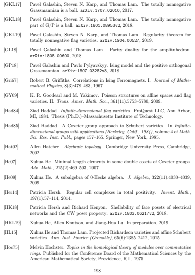

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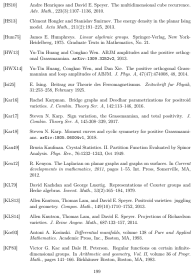

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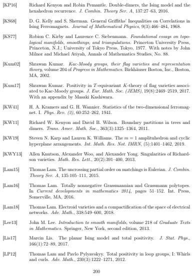

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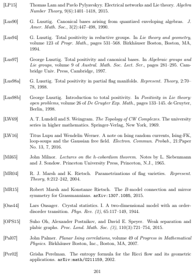

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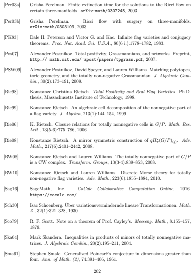

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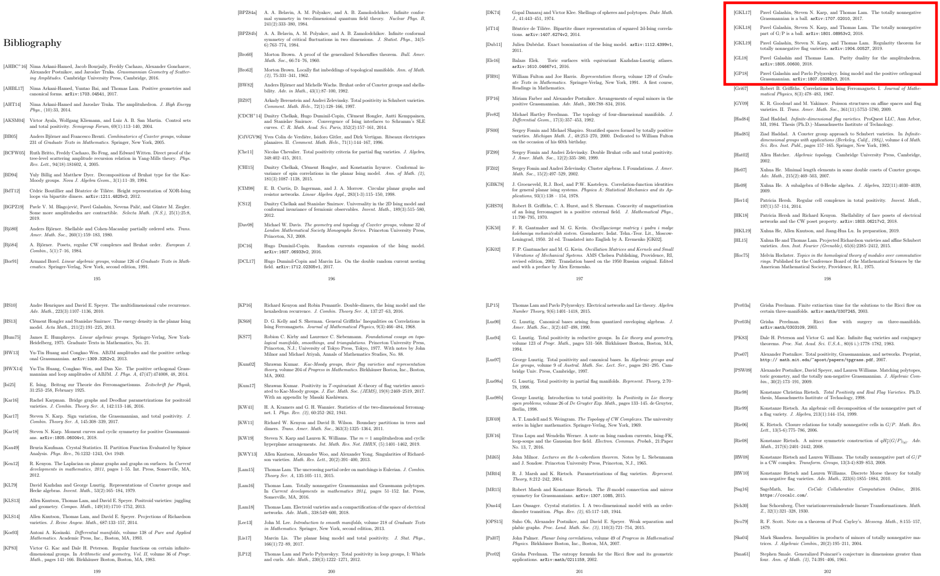

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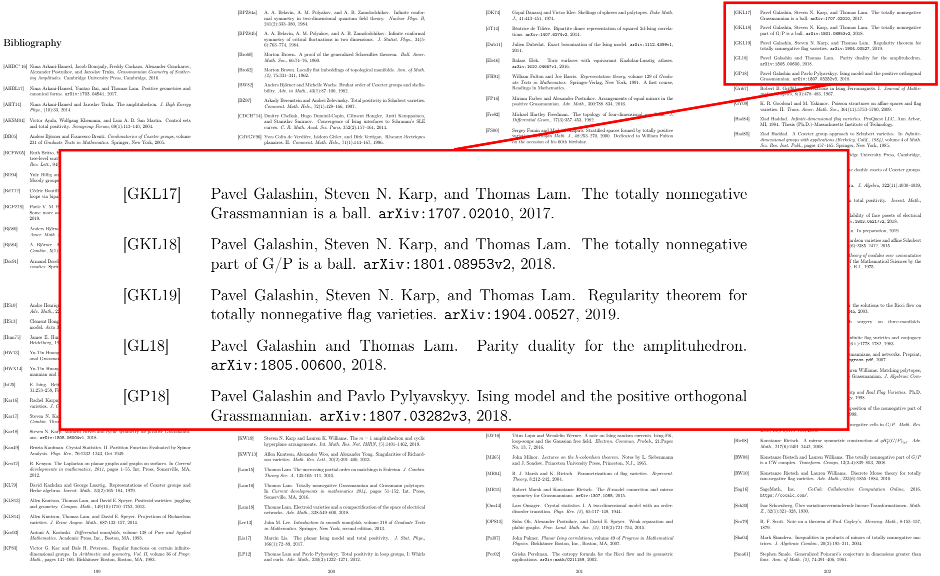

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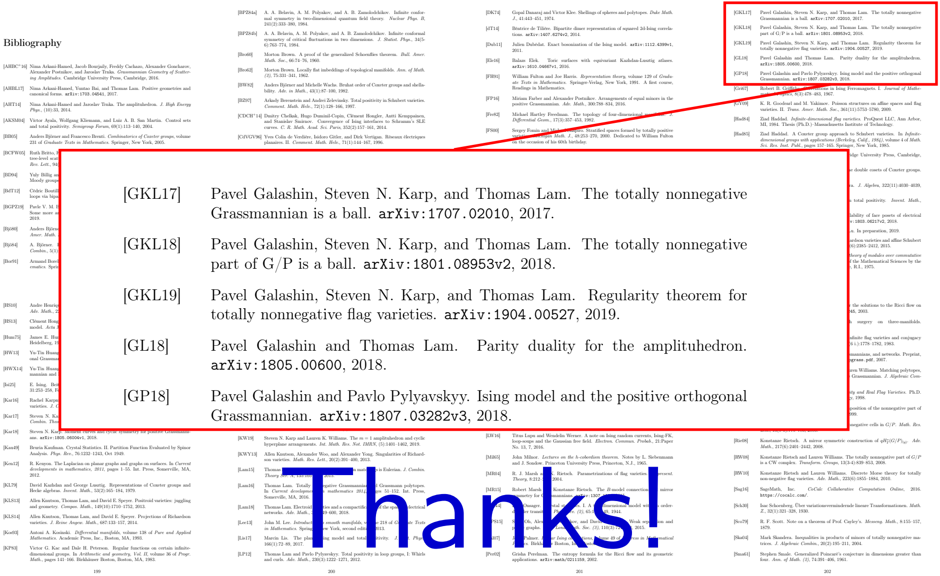

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Thanks!