Critical varieties in the Grassmannian

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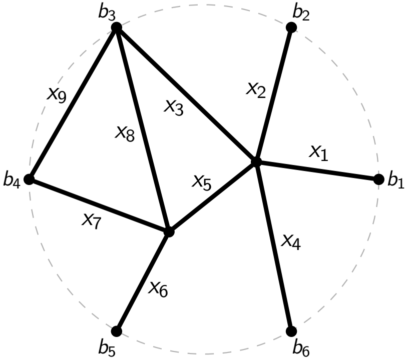
arXiv:2102.13339

Motivation: Ising model

- $(G, \mathbf{x})$: weighted graph embedded in a disk.



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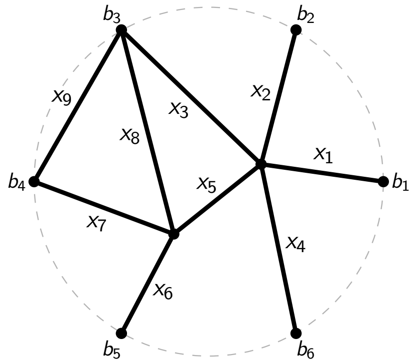

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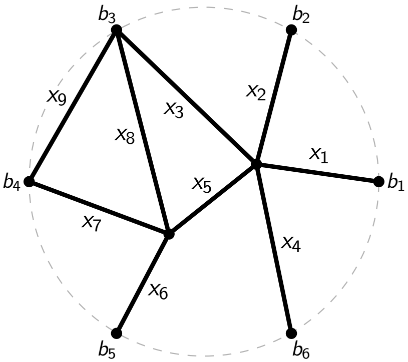

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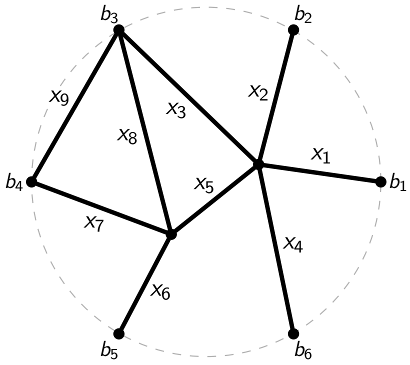
$$\text{Prob}(\sigma) := \frac{1}{Z} \prod_{\substack{\{u,v\} \in E(G): \\ \sigma_u \neq \sigma_v}} x_{\{u,v\}},$$

where Z is such that the total probability is 1.



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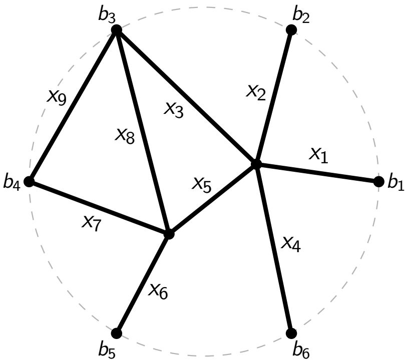
- Boundary correlation:
  $\langle \sigma_i \sigma_j \rangle := \mathrm{Prob}(\sigma_{b_i} = \sigma_{b_j}) - \mathrm{Prob}(\sigma_{b_i} \neq \sigma_{b_j})$.



Phase transition

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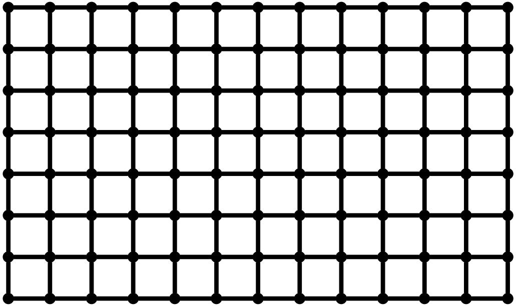
Usually:

- G = large piece of a (e.g. square) lattice;
- $x_e = x$ for all $e \in E(G)$.



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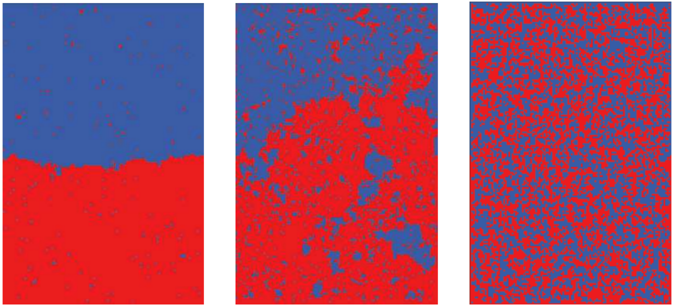
- Get a phase transition at critical temperature x_{crit}.



$x < x_{\text{crit}}$ $\quad$ $x = x_{\text{crit}}$ $\quad$ $x > x_{\text{crit}}$

Picture credit: Dmitry Chelkak

Phase transition

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Usually:

- $G =$ large piece of a (e.g. square) lattice;
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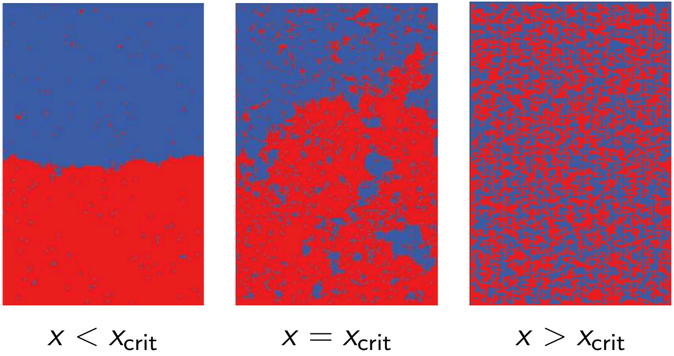
- Get a phase transition at critical temperature x_{crit}.
- Square lattice: $x_{\mathrm{crit}} = \sqrt{2} - 1$.



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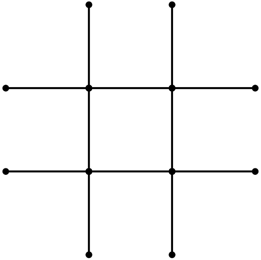

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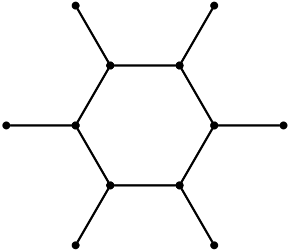
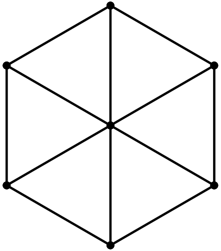


Hexagonal lattice
$x_{\text{crit}} = 2 - \sqrt{3}$



Triangular lattice
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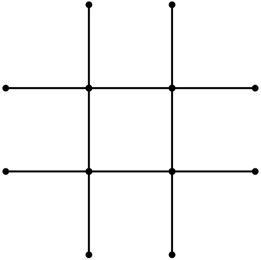
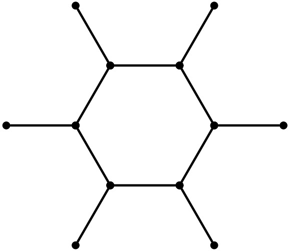


Square lattice

$x_{\text{crit}} = \sqrt{2} - 1$

$x_{\text{crit}} = \tan(\pi/8)$



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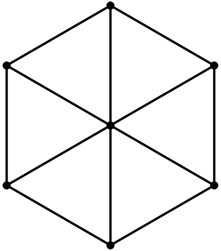

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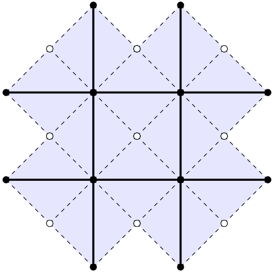

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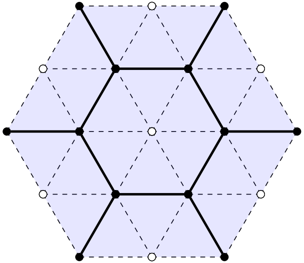

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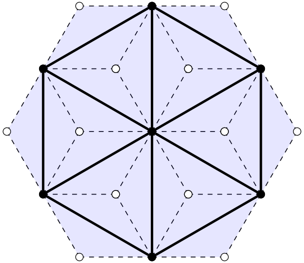

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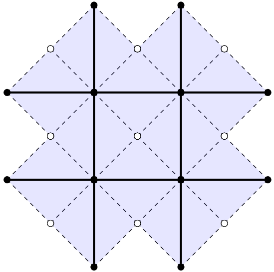

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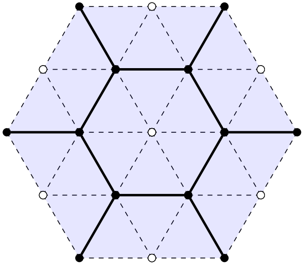

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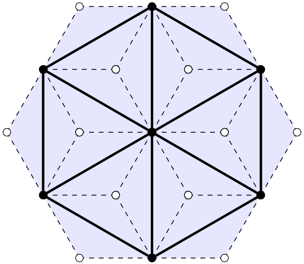

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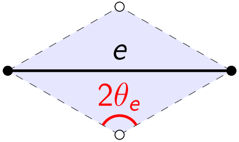

$x_e = \tan(\theta_e/2)$

Critical Z-invariant Ising model

[Bax86] R. J. Baxter. Free-fermion, checkerboard and **Z**-invariant lattice models in statistical mechanics. *Proc. Roy. Soc. London Ser. A*, 404(1826):1–33, 1986.

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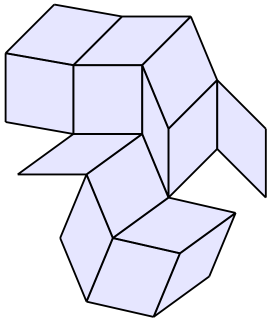

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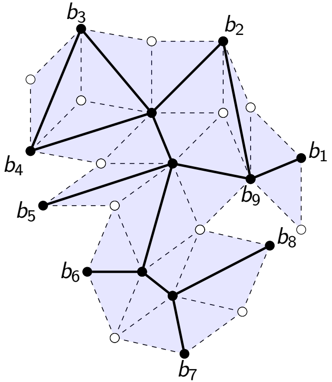

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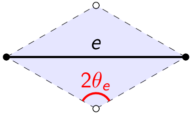

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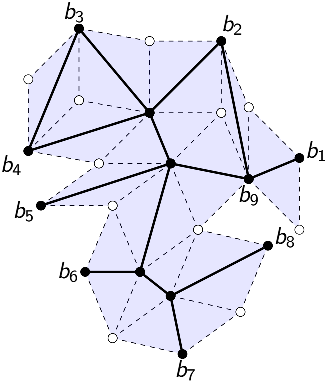

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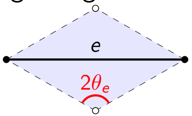

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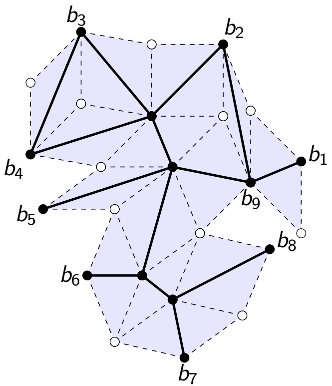

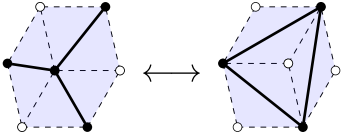

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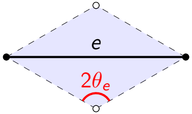

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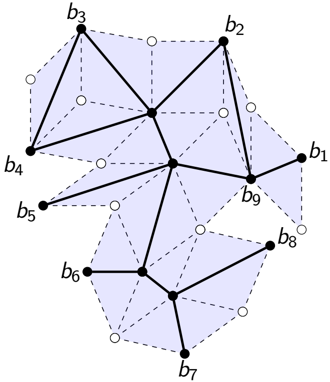
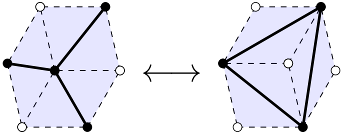
- Conclusion: $\langle \sigma_i \sigma_j \rangle_R$ depends only on the polygonal region R.





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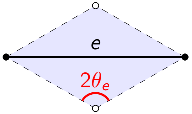

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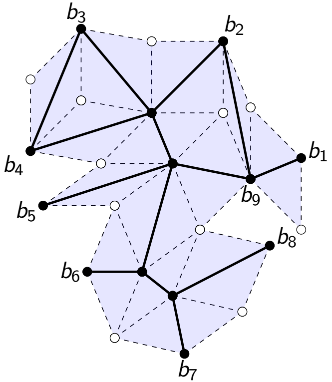
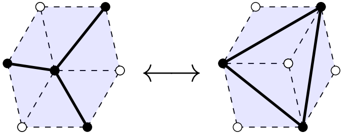
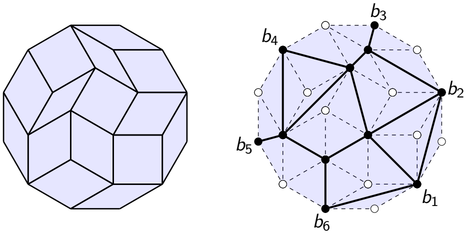
- Formula for $\langle \sigma_i \sigma_j \rangle_R$ in terms of R?





A formula for regular polygons

Let R_N be a regular $2N$-gon and $\langle \sigma_i \sigma_j \rangle_{R_N}$ be the corresponding boundary correlations.



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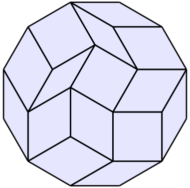

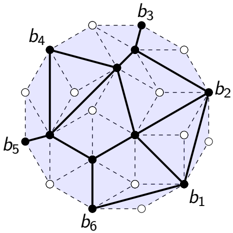

Theorem (G. (2020))

For $1 \leqslant i, j \leqslant N$ and $d := |i - j|$, we have

$$\langle \sigma_i \sigma_j \rangle_{R_N} = \frac{2}{N} \left(\frac{1}{\sin\left((2d-1)\pi/2N\right)} - \frac{1}{\sin\left((2d-3)\pi/2N\right)} + \cdots \pm \frac{1}{\sin\left(\pi/2N\right)} \right) \mp 1.$$

[Gal20] Pavel Galashin. A formula for boundary correlations of the critical Ising model. arXiv:2010.13345, 2020.

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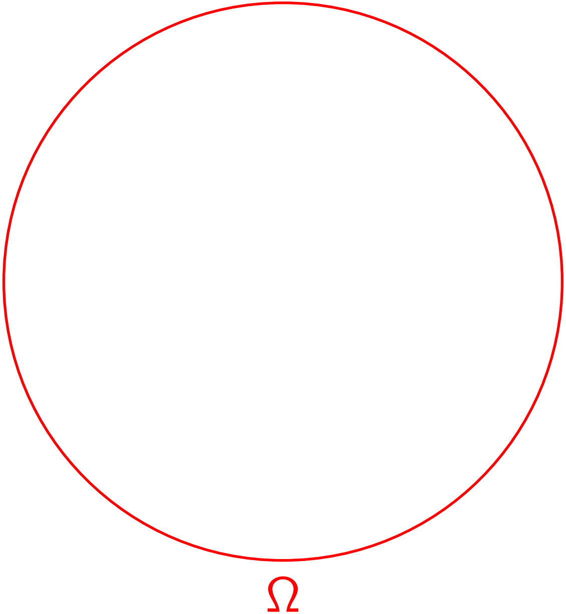
- Q: Does $\langle \sigma_1 \sigma_{d+1} \rangle_{R_N} \to 0$ for $1 \ll d \ll N$?
- A: Yes, by the Leibniz formula for π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

- Choose a domain $\Omega \subseteq \mathbb{C}$;



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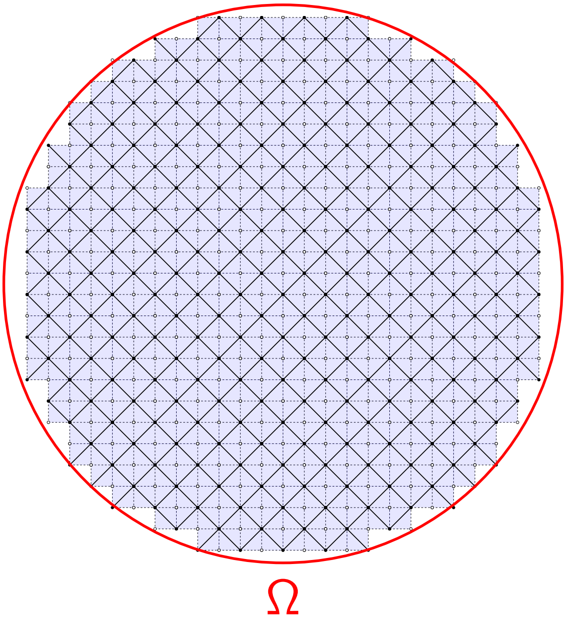

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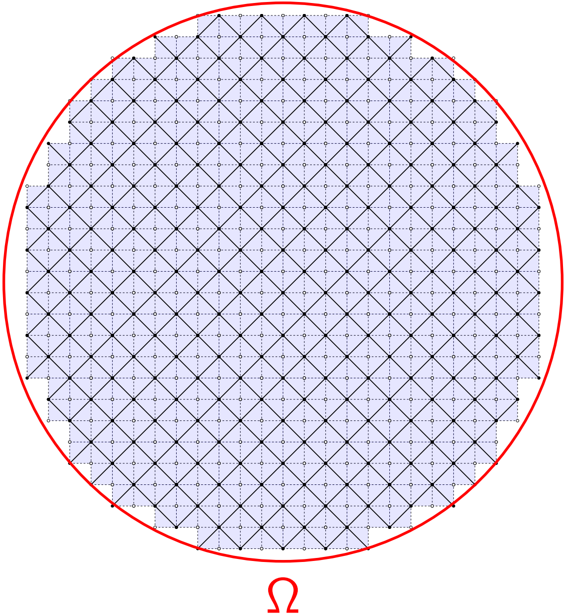

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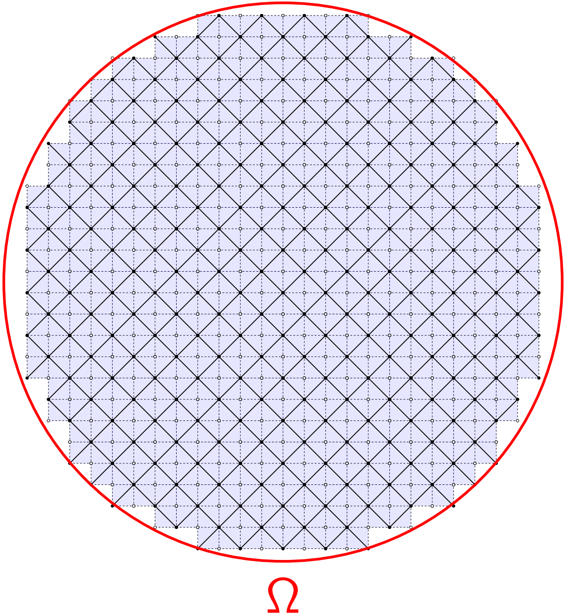

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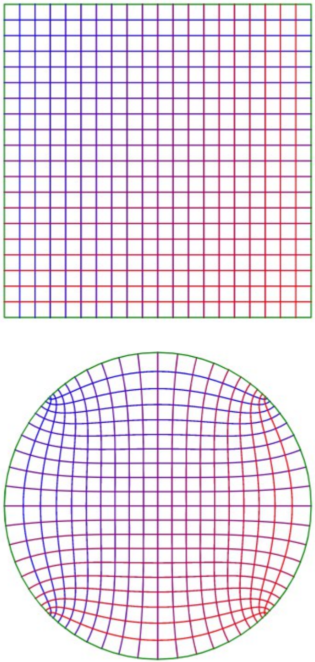

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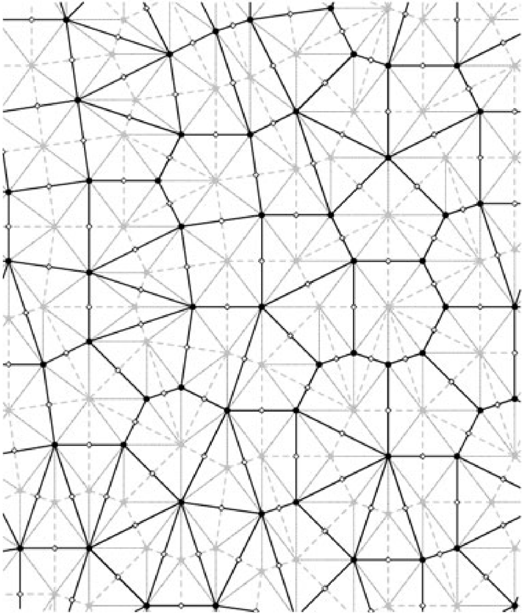

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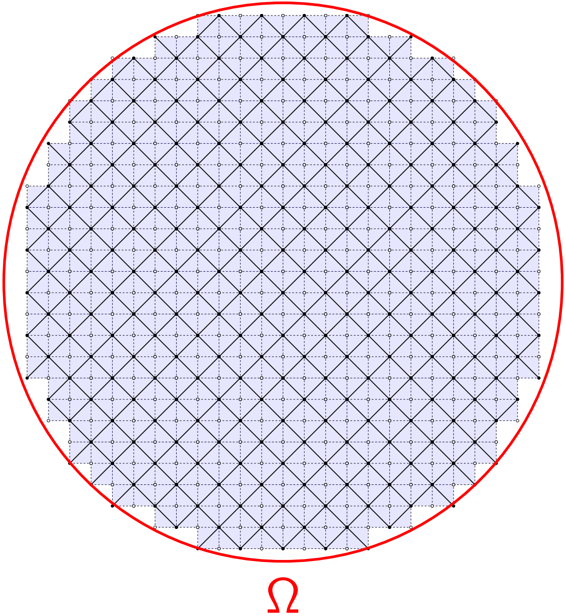

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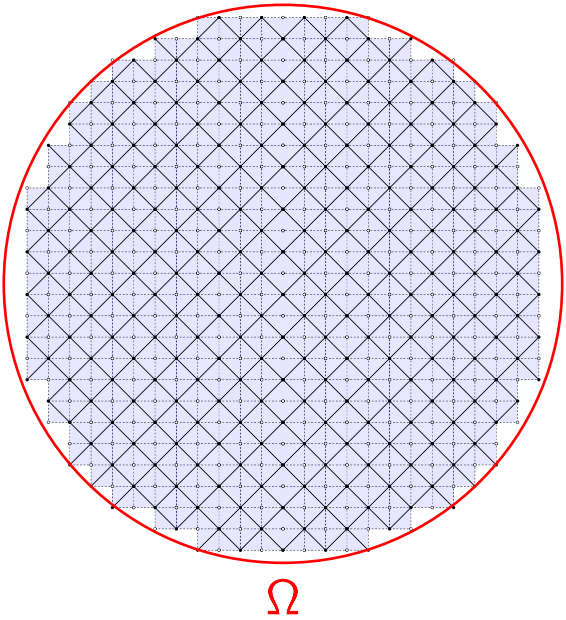

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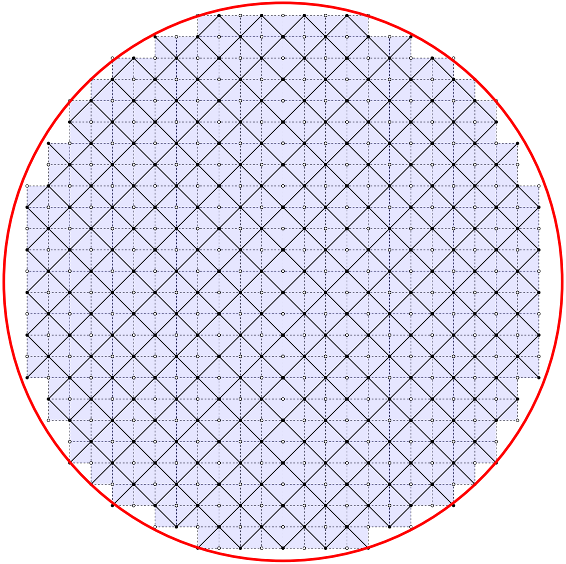

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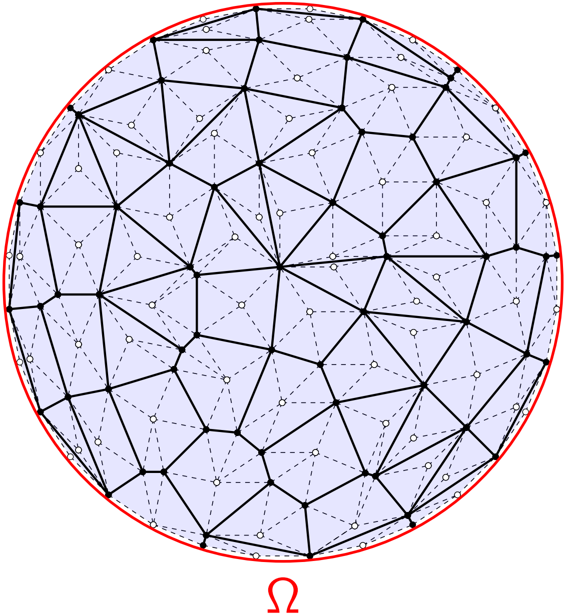

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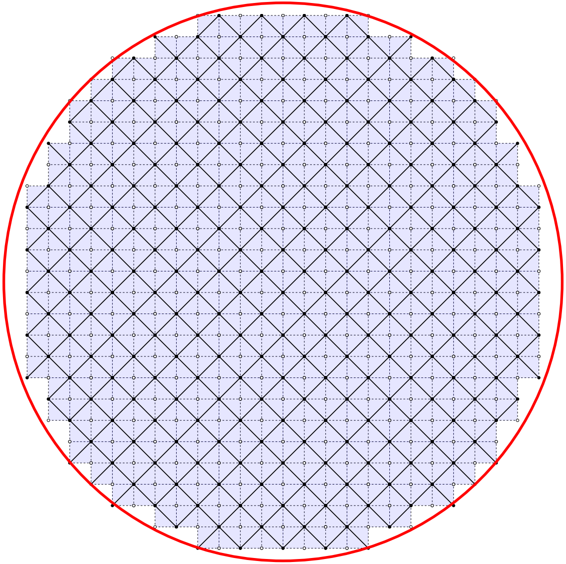

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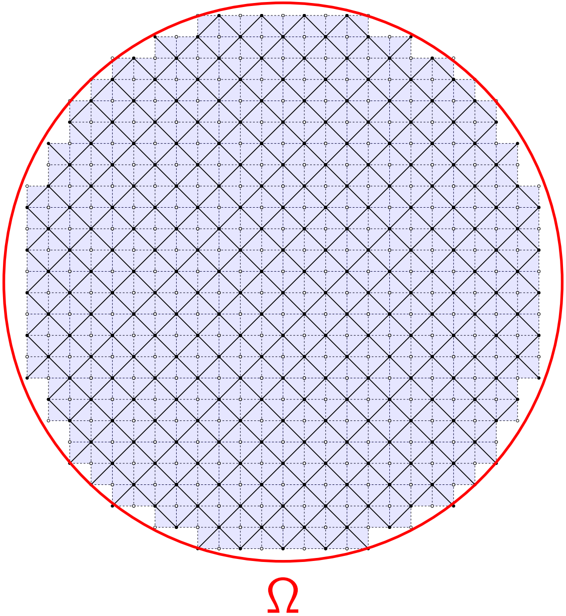

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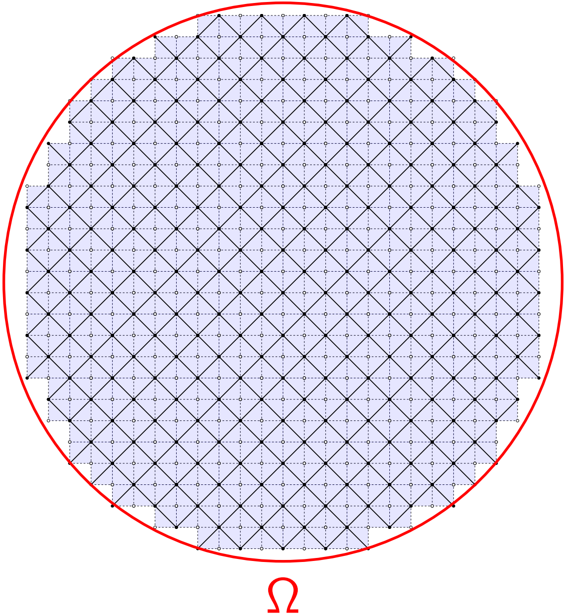

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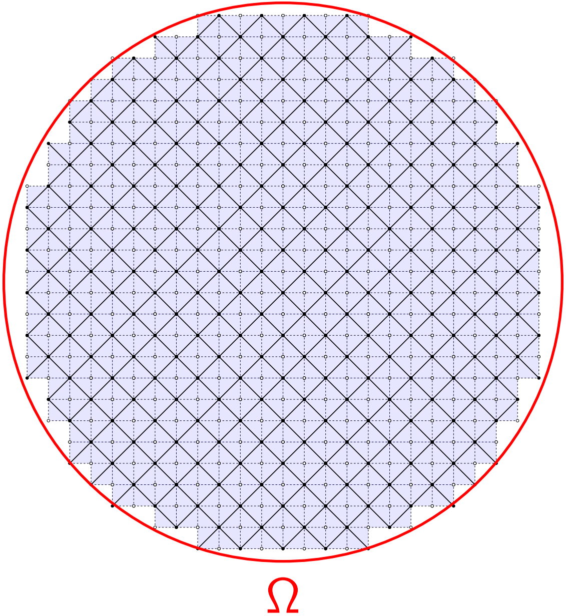

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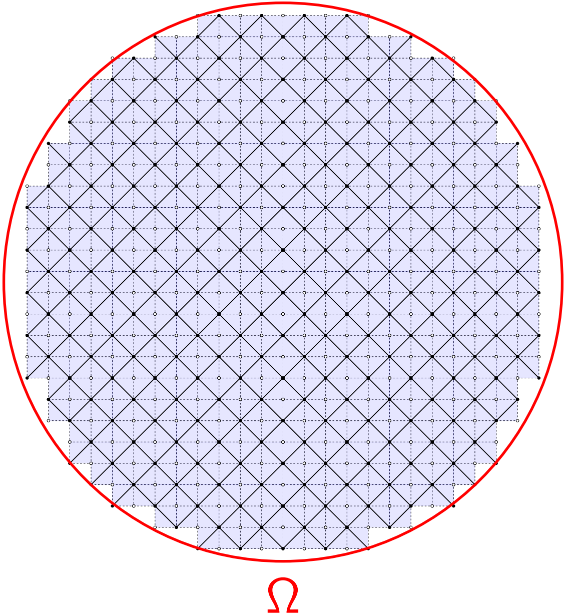

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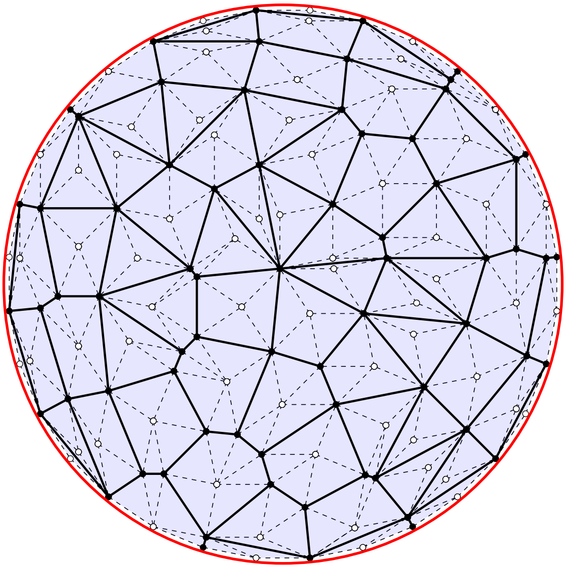

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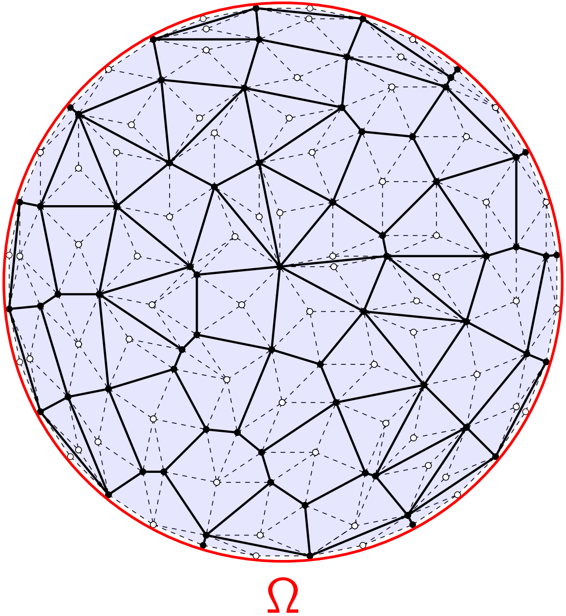

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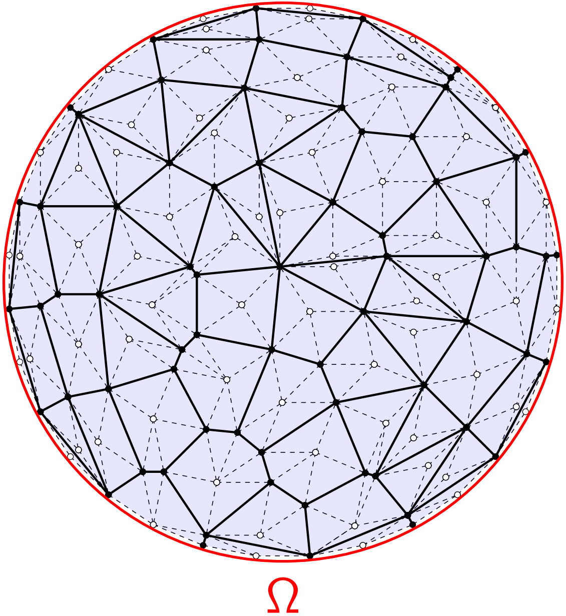

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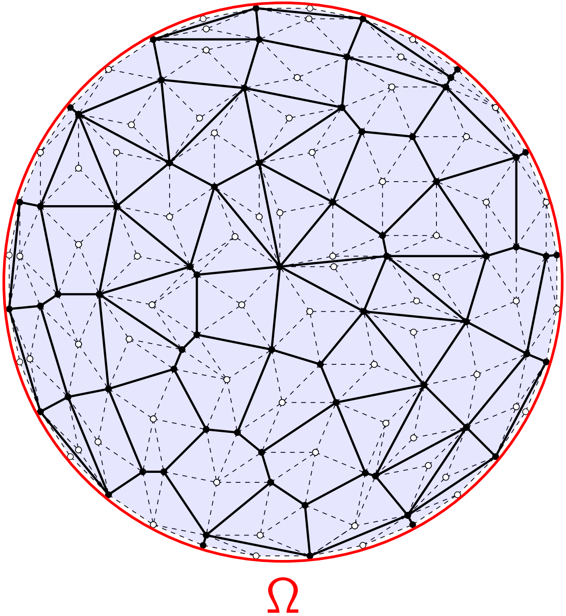

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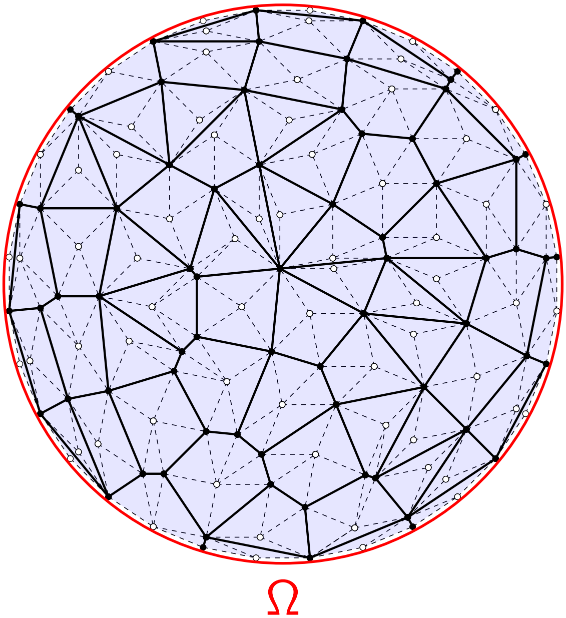
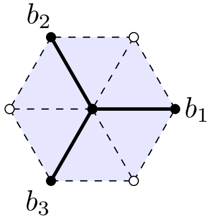


Electrical networks

- Treat each edge of G as a resistor.



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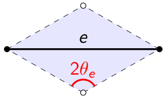

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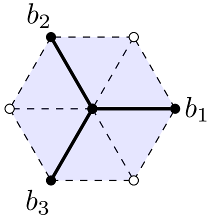

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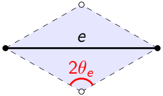

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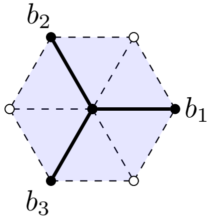

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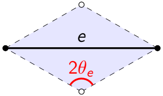
- Resistance R_e = ratio of diagonals:  $R_e = \tan(\theta_e)$
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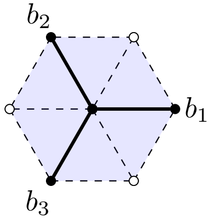

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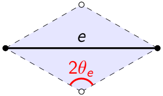
- Resistance R_e = ratio of diagonals: 

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If R is a regular $2N$-gon then for $1 \leqslant i, j \leqslant N$ and $d := |i - j|$, we have

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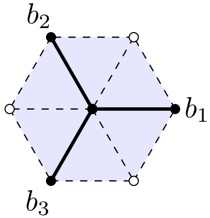

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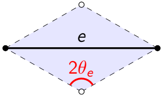

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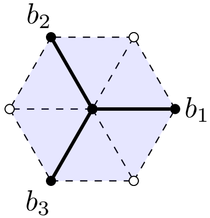

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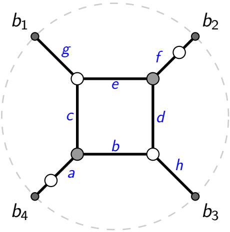
Critical varieties

- (G, wt) – a weighted planar bipartite graph, with n black boundary vertices $b_1, b_2, \ldots, b_n$ of degree 1.



[Pos06] Alexander Postnikov. Total positivity, Grassmannians, and networks. Preprint, arXiv:math/0609764, 2006.

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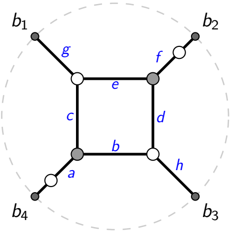

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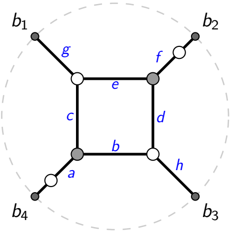

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- A strand is a path in G that makes a sharp right turn at each black vertex and a sharp left turn at each white vertex.
- Strand permutation: $f_G \in S_n$. (aka "loopless bounded affine permutation")

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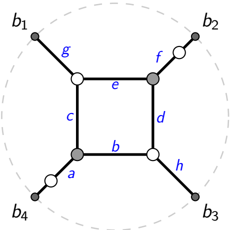

Critical dimer model

[Ken02] R. Kenyon. The Laplacian and Dirac operators on critical planar graphs. *Invent. Math.*, 150(2):409–439, 2002.

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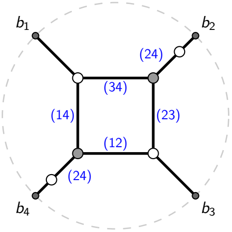

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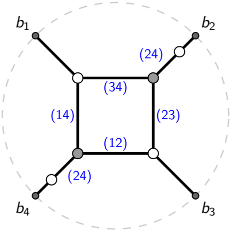

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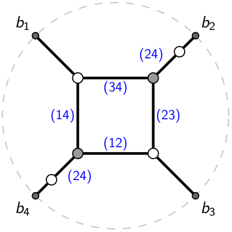

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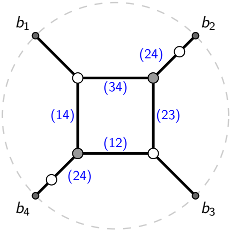
[Ken02] R. Kenyon. The Laplacian and Dirac operators on critical planar graphs. *Invent. Math.*, 150(2):409–439, 2002.

[OPS15] Suho Oh, Alexander Postnikov, and David E. Speyer. Weak separation and plabic graphs. *Proc. Lond. Math. Soc. (3)*, 110(3):721–754, 2015.

- Fix $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ such that $\theta_1 < \theta_2 < \cdots < \theta_n < \theta_1 + \pi$.
- Each edge e belongs to exactly two strands terminating at b_p and b_q for $1 \leqslant p < q \leqslant n$. Set

$$\mathrm{wt}_{\boldsymbol{\theta}}(e) := \begin{cases} \sin(\theta_q - \theta_p), & \text{if } e \text{ is not a boundary edge,} \\ 1, & \text{otherwise.} \end{cases}$$



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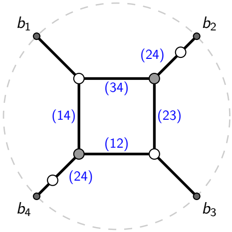
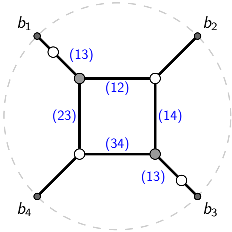
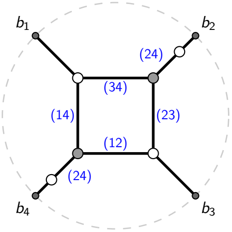


$\leftrightarrow$





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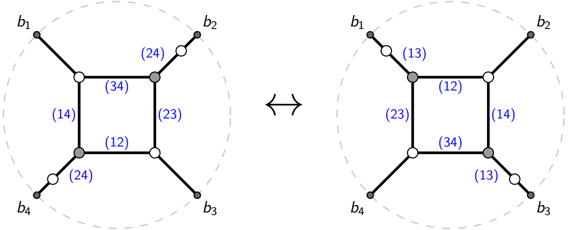
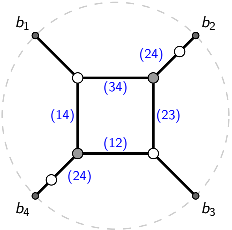


- A graph G is reduced if it has the minimal number of faces among all graphs with the same strand permutation.



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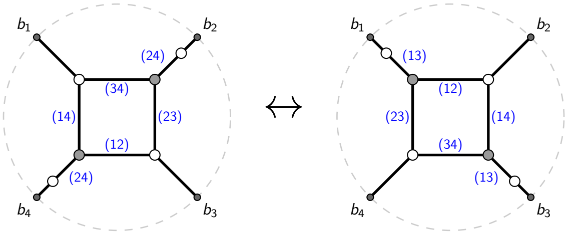

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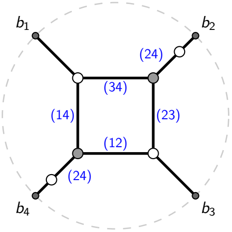

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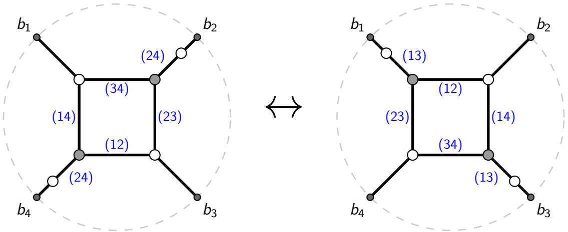

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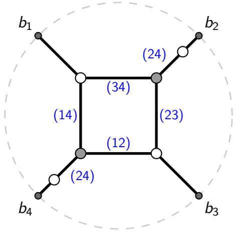

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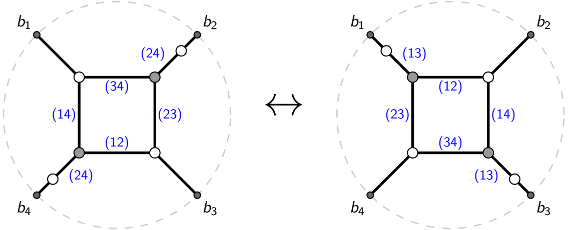

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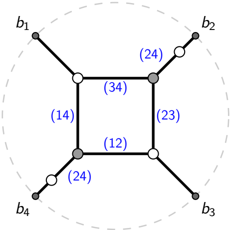
- Formula for $\mathrm{Meas}_f(\boldsymbol{\theta})$ in terms of f and $\boldsymbol{\theta}$?



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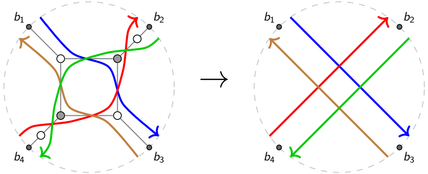
- Recall: $f \in S_n$. The reduced strand diagram of f is obtained by drawing an arrow $b_s^+ \to b_p^-$ if $f(s) = p$.



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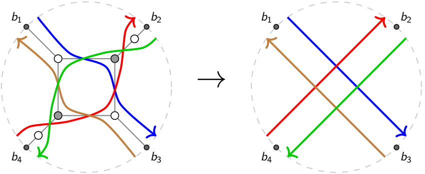

$J_1 = \{2\}$ $J_2 = \{3\}$ $J_3 = \{4\}$ $J_4 = \{1\}$

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- Consider a curve $\gamma_{f,\theta}(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))$:

$$\gamma_r(t) := \epsilon_r \prod_{p \in J_r} \sin(t - \theta_p) \qquad \text{for } r \in [n],$$

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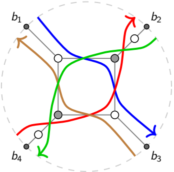
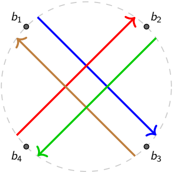


$\to$



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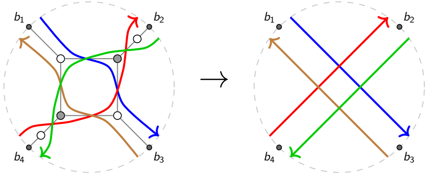
Theorem (G. (2021))

$$\mathrm{Meas}_f(\boldsymbol{\theta}) = \mathrm{Span}(\boldsymbol{\gamma}_{f,\boldsymbol{\theta}}) \quad \textit{inside } \mathrm{Gr}(k,n).$$



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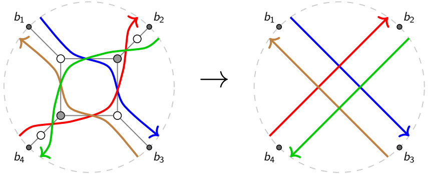

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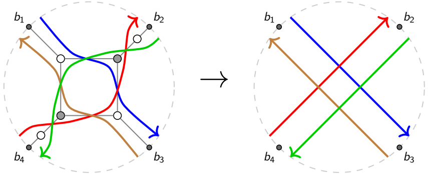

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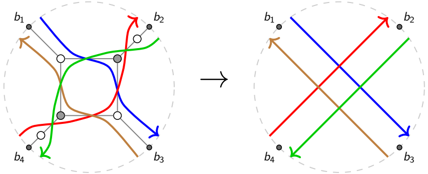
Plücker coordinates Δ_J = maximal $k \times k$ minors



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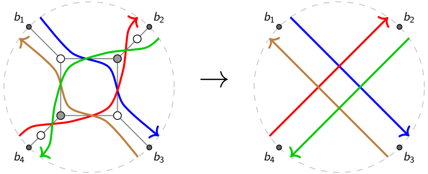

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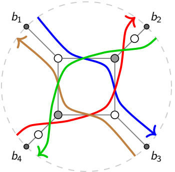

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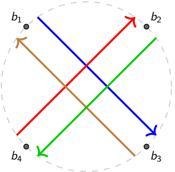

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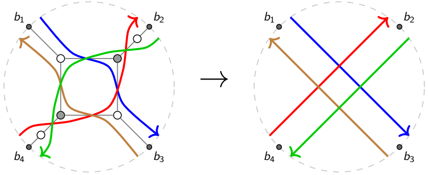

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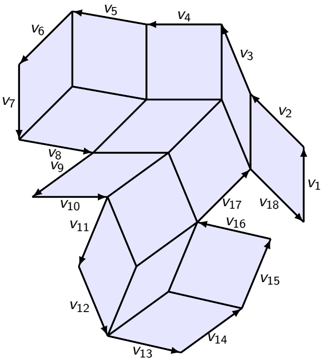
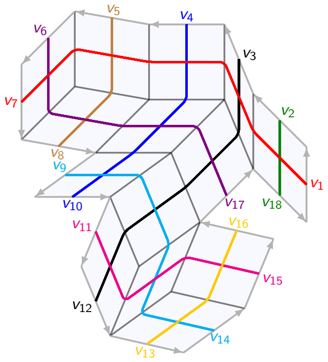


$v_r = \exp(2i\theta_r)$



$\tau(1) = 7$, etc.

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Thanks!