Shift maps, poset associahedra, and totally nonnegative critical varieties

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Oberwolfach Mini-Workshop: Scattering Amplitudes, Cluster Algebras, and Positive Geometries

December 6, 2021

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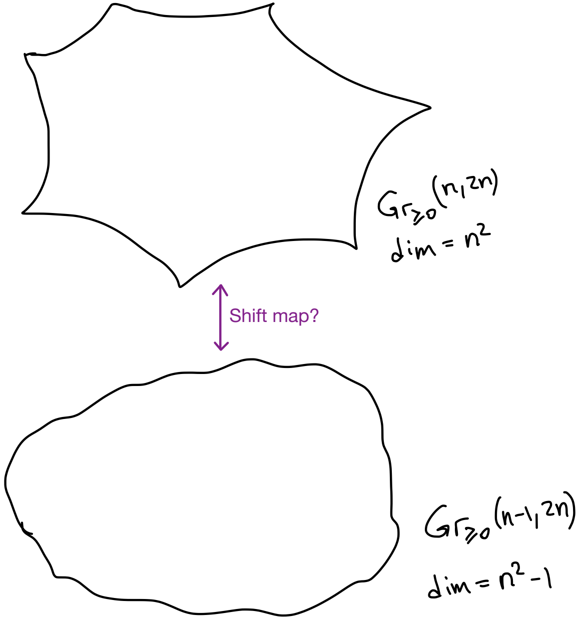
- *Takes the critical part of $\Pi_f^{>0}$ to the critical part of $\Pi_{f^\downarrow}^{>0}$.*


$Gr_{\geq 0}(n,2n)$
$\dim = n^2$
Shift map?
$Gr_{\geq 0}(n-1,2n)$
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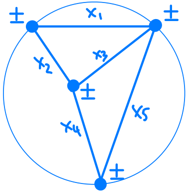
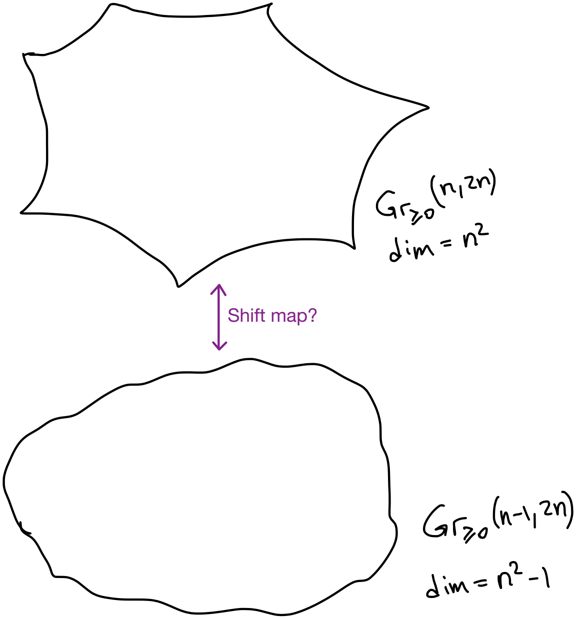
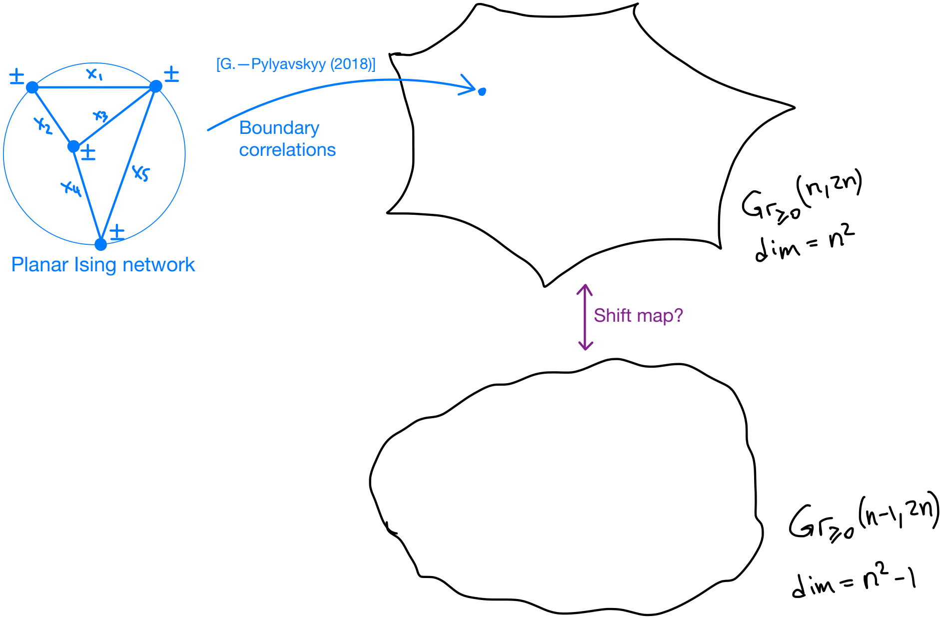


Planar Ising network




[G.—Pylyavskyy (2018)]
Boundary correlations
Planar Ising network
x_1
x_2
x_3
x_4
x_5
$\pm$
$\pm$
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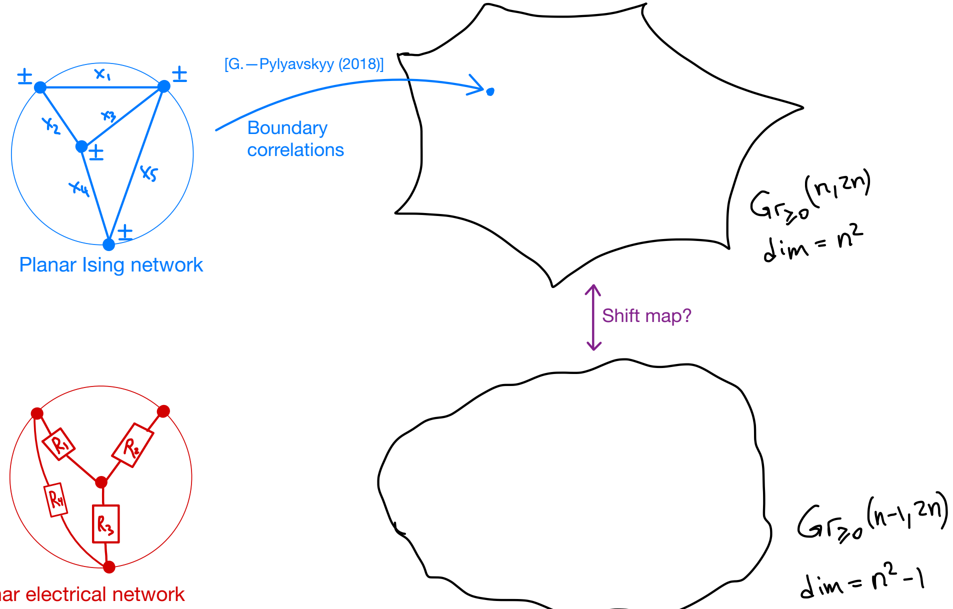
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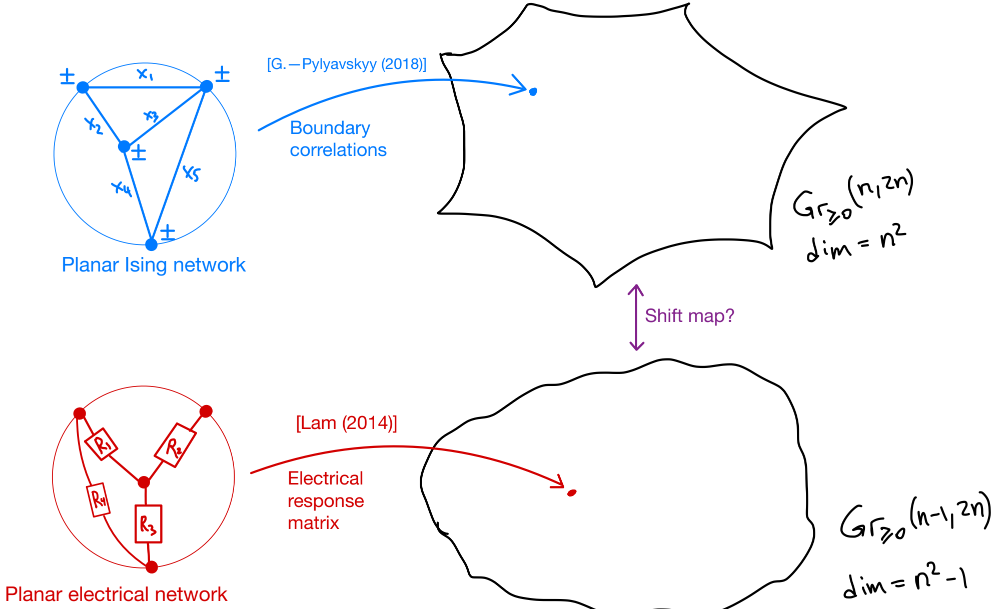
R_1 R_2 R_3 R_4
Planar electrical network



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[Lam (2014)]
Electrical response matrix
R_1
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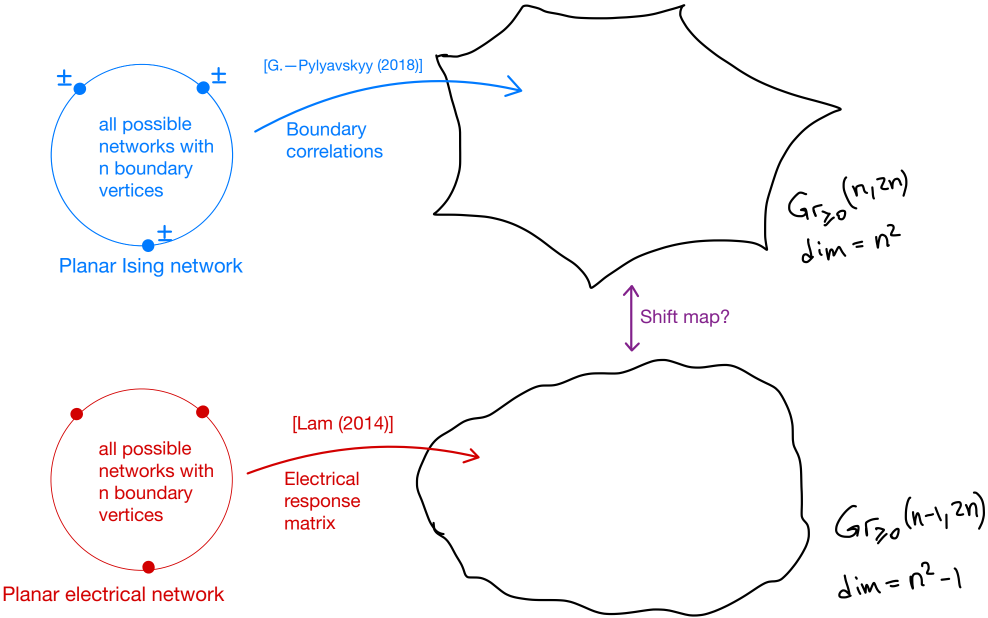
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all possible networks with n boundary vertices
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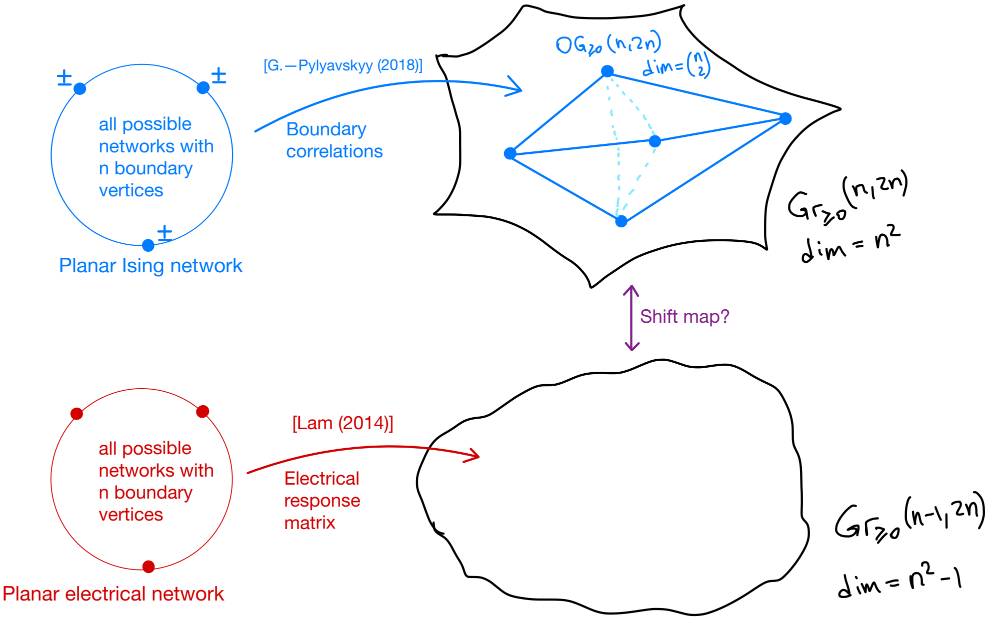

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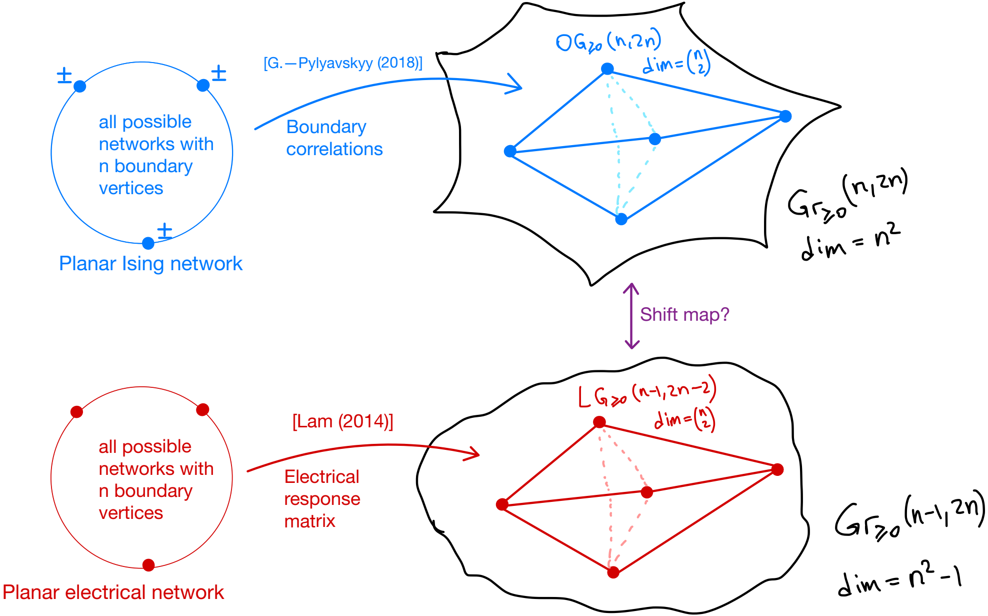
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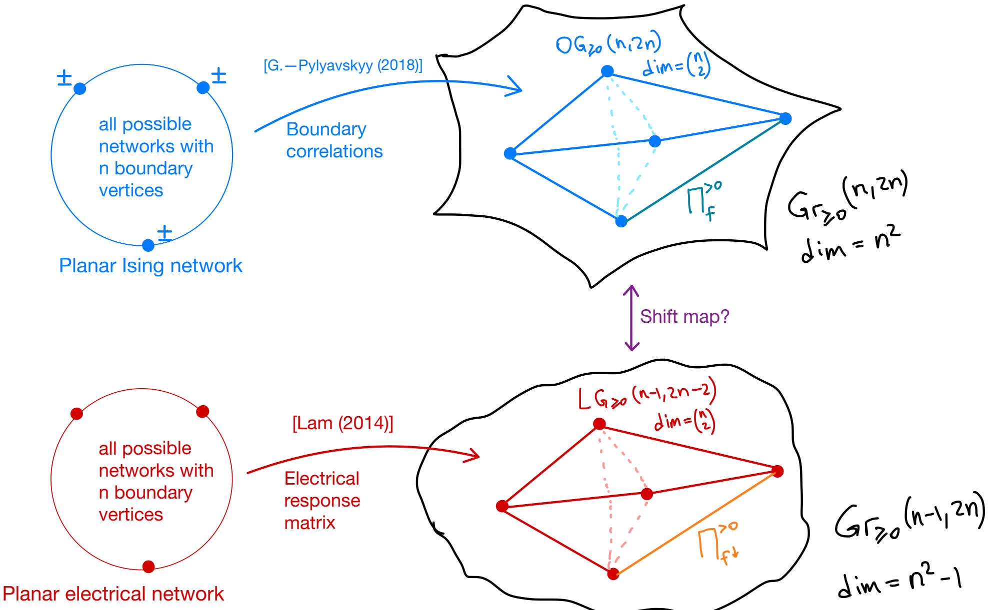

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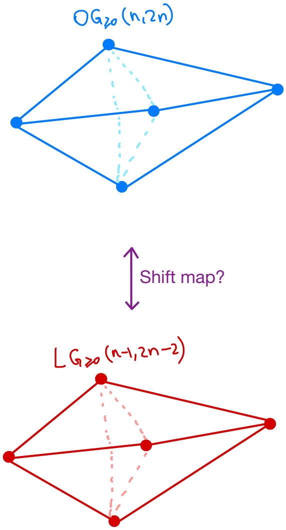
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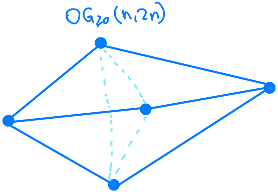
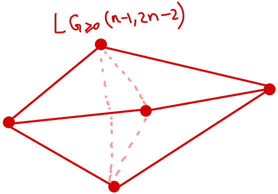
Face posets coincide:

Matchings of 2n elements, ordered by “uncrossing”



$\updownarrow$ Shift map?



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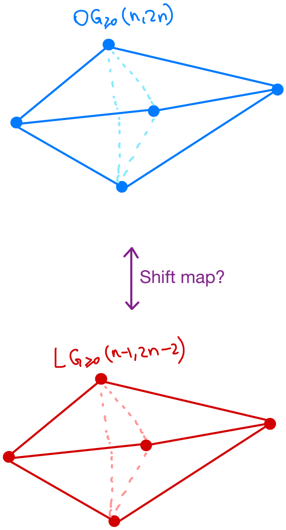

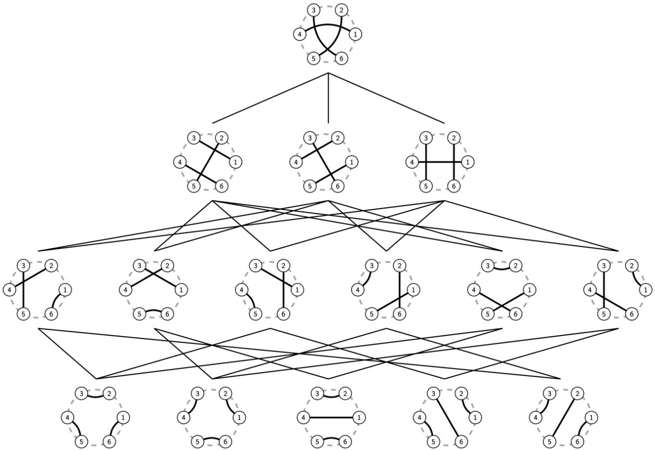

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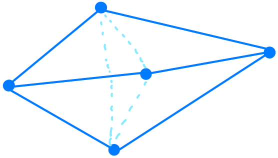

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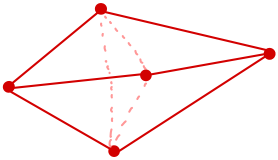

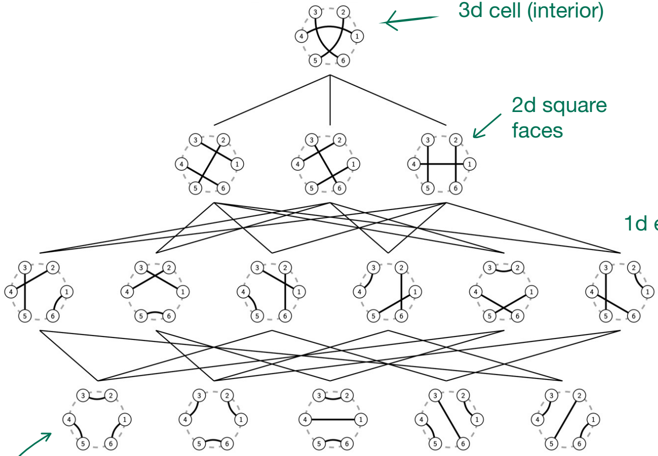

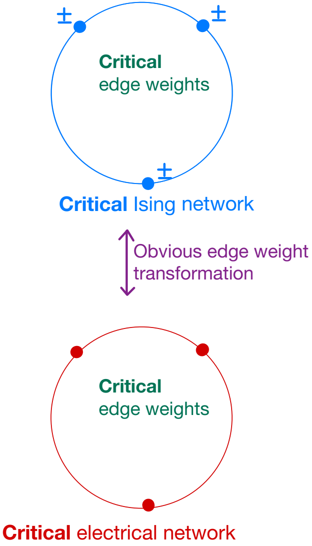
±
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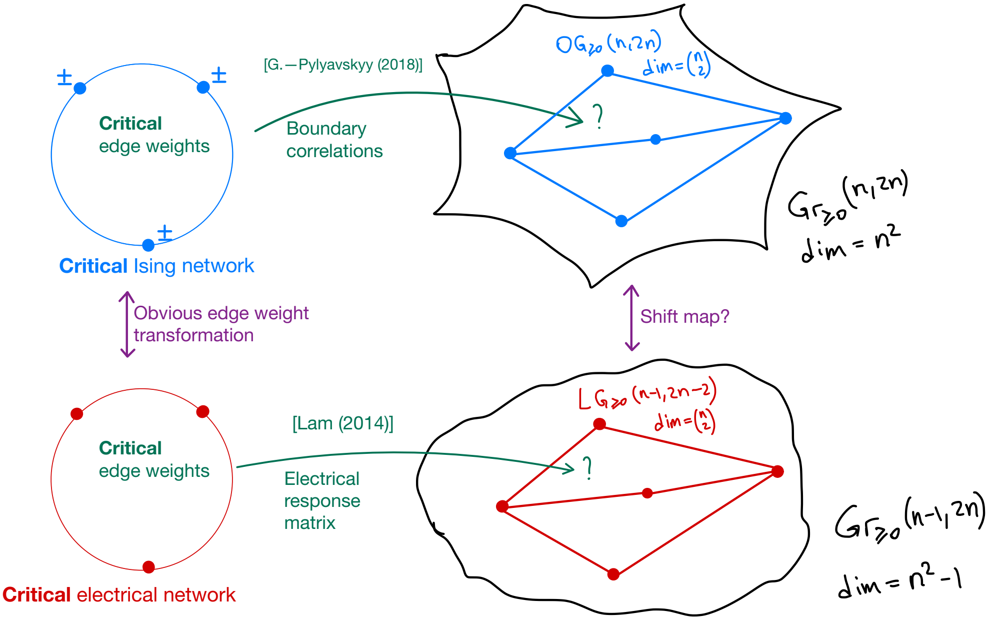
Critical
edge weights
±
Critical Ising network
Obvious edge weight
transformation
Critical
edge weights
Critical electrical network



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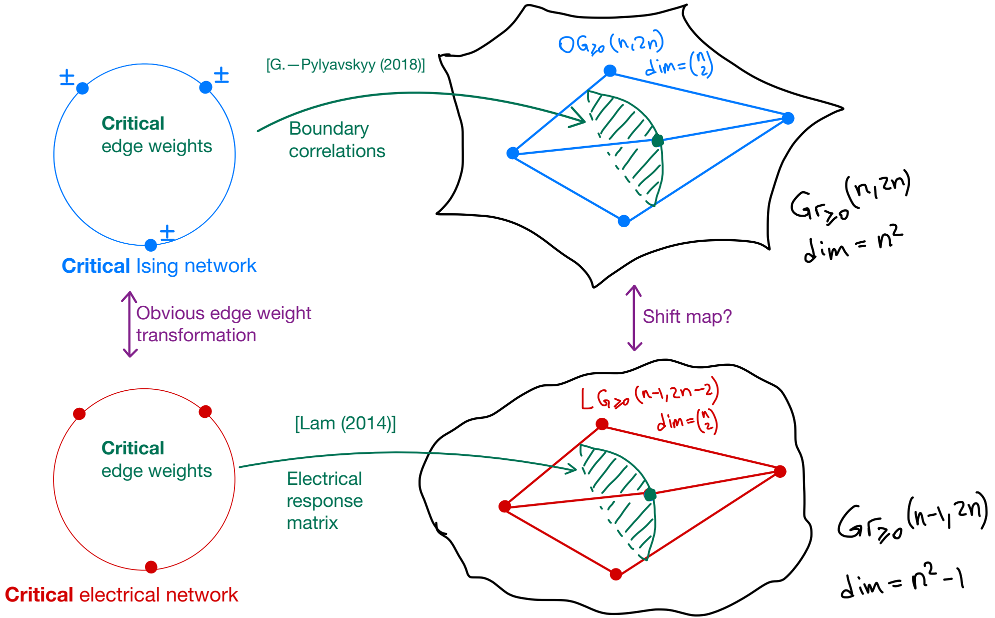
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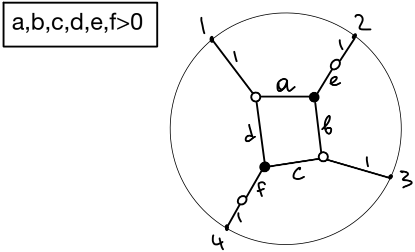
Critical Ising model and
critical electrical networks
are special cases of the
critical dimer model,
introduced by Kenyon in 2002.

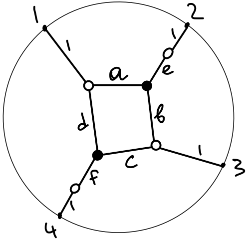
Dimer model



Input: weighted bipartite graph (G,wt) embedded in a disk, with n black degree 1 boundary vertices

Dimer model

$a,b,c,d,e,f>0$



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Dimer model

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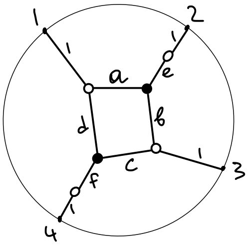
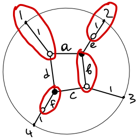
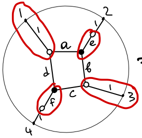
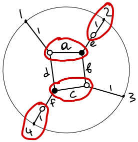
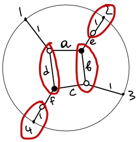


Meas $\longrightarrow$

$\Delta_{12} =$  $= bf$

$\Delta_{13} =$  $= ef$

$\Delta_{24} =$  $+$  $= ac + bd$

... etc ...

Input: weighted bipartite graph (G,wt) embedded in a disk, with n black degree 1 boundary vertices

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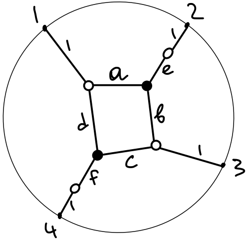
Dimer model

"almost perfect matchings"

a,b,c,d,e,f>0



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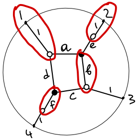
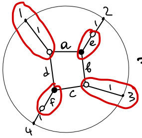
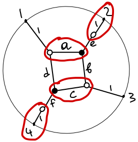
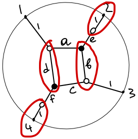
$\Delta_{12} =$  $= bf$

$\Delta_{13} =$  $= ef$

$\Delta_{24} =$  $+$  $= ac + bd$

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Dimer model

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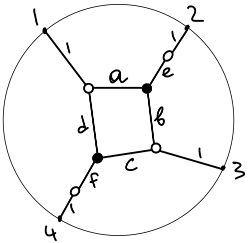
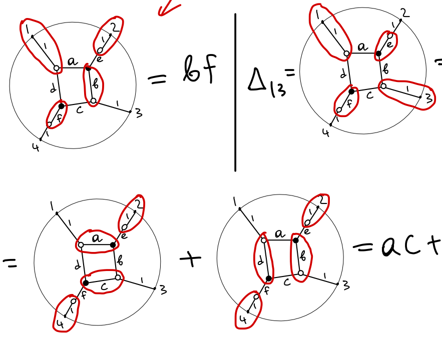


$\xrightarrow{\text{Meas}}$

"almost perfect matchings"



$\Delta_{12} = bf$ | $\Delta_{13} = ef$

$\Delta_{24} = \quad + \quad = ac + bd$

... etc ...

Input: weighted bipartite graph (G,wt) embedded in a disk, with n black degree 1 boundary vertices

Output: $\underbrace{\left(\Delta_I(G, wt)\right)_{I \in \binom{[n]}{k}}}$

(defined up to common rescaling)

Dimer model

"almost perfect matchings"

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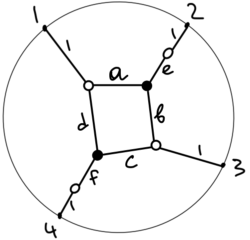
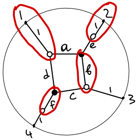
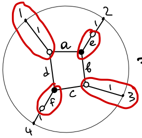
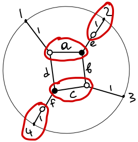
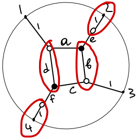


$\xrightarrow{\text{Meas}}$

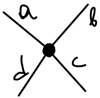
$\Delta_{12} =$  $= bf$ | $\Delta_{13} =$  $= ef$

$\Delta_{24} =$  $+$  $= ac + bd$

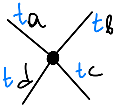
... etc ...

Input: weighted bipartite graph (G,wt) embedded in a disk, with n black degree 1 boundary vertices

Gauge transform:



$\rightarrow$

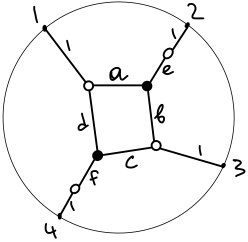


$(t>0)$

Output: $\underbrace{\left(\Delta_I(G,wt)\right)_{I \in \binom{[n]}{k}}}$

(defined up to common rescaling)

Dimer model

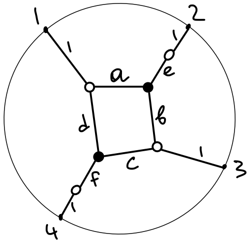


$$\xrightarrow{\text{Meas}}$$

$$\Delta_{12} = bf \qquad \Delta_{13} = ef \qquad \Delta_{14} = ec$$

$$\Delta_{23} = af \qquad \Delta_{24} = ac+bd \qquad \Delta_{34} = de$$

Dimer model



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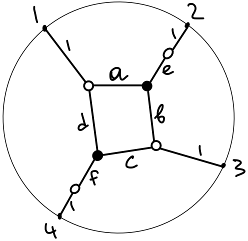
Δ_I = maximal minor w/ column set I

e	a	0	$-de/f$
0	bf/e	f	c

Dimer model



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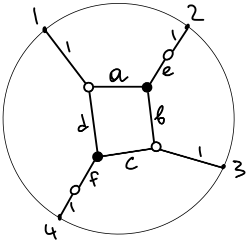
$$\text{RowSpan} \begin{bmatrix} e & a & 0 & -de/f \\ 0 & bf/e & f & c \end{bmatrix} \in Gr_{\geq 0}(2,4)$$

Dimer model



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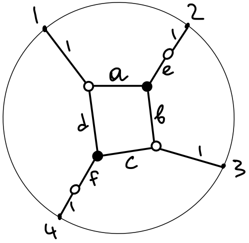
$$\boxed{\text{Recall: } Gr_{\geq 0}(k,n) = \bigsqcup_f \Pi_f^{>0}}$$

Dimer model



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Each G parametrizes a **positroid cell** $\Pi_f^{>0} \subset Gr_{\geq 0}(k,n)$,

where **f** is the **strand permutation** of G.

$$\boxed{\text{Recall: } \quad Gr_{\geq 0}(k,n) = \bigsqcup_f \Pi_f^{>0}}$$

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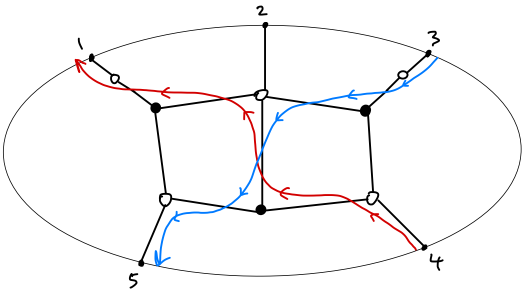

A **strand** in G is a path that
- makes a sharp right turn at each black vertex
- makes a sharp left turn at each white vertex

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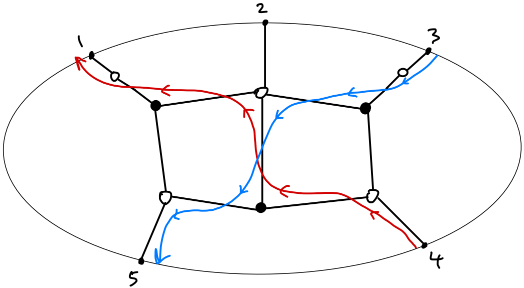

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Strand permutation of G sends 3 to 5, 4 to 1, ...

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$$

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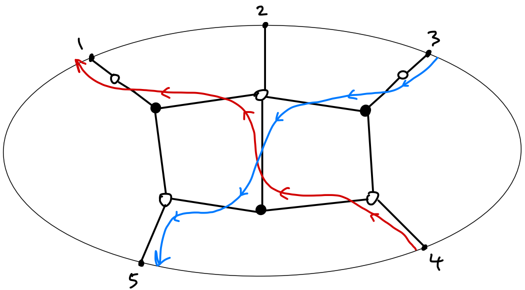

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G is called **reduced** if it has minimal number of faces among all graphs with the same strand permutation.

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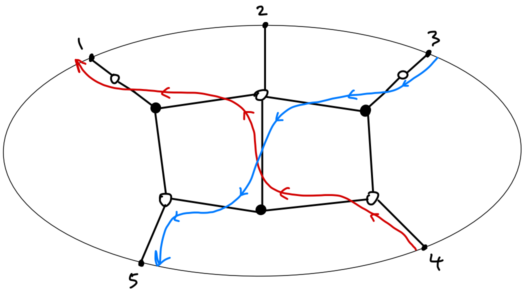

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$\boxed{\begin{matrix} k=2 \\ n=5 \end{matrix}}$

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"Top Cell": $f = f_{k,n}$, sends $i \mapsto i+k \bmod n$ for all i

$\Pi^{>0}_{f_{k,n}} = Gr_{>0}(k,n) = \{V \in Gr(k,n) \mid \Delta_I(V) > 0 \ \forall I\}$

Each G parametrizes a **positroid cell** $\Pi^{>0}_f \subset Gr_{\geq 0}(k,n)$,

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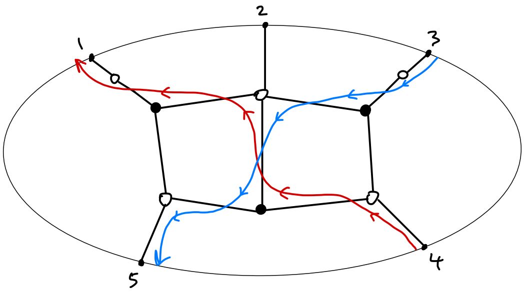

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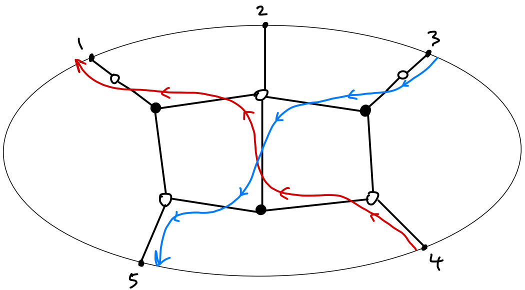

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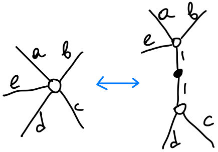
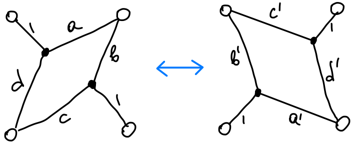
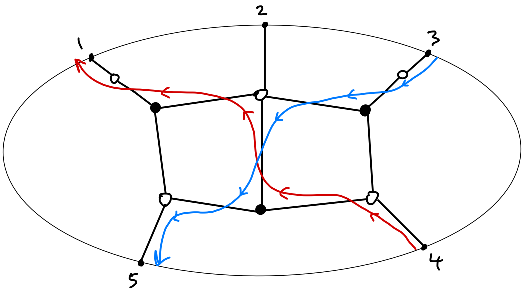
Any two reduced graphs with the same strand permutation are related by these moves:





$$a' = \frac{a}{ac+bd}, \text{ etc}$$



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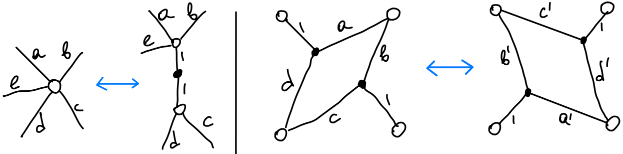
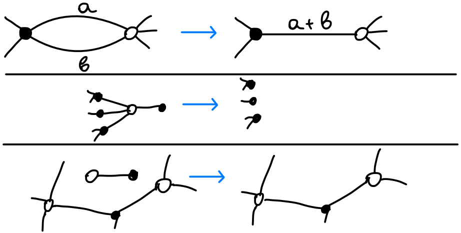


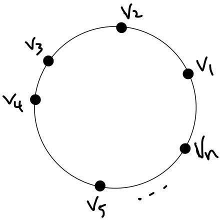
$$a' = \frac{a}{ac+bd}, \text{ etc}$$

Any non-reduced graph can be transformed into a reduced one using these additional moves:



Critical cells

Choose n points on a circle,
labeled counterclockwise:



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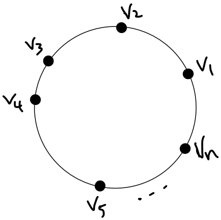
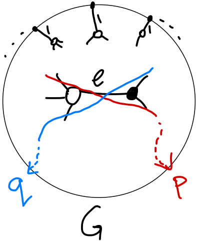


Each edge e of G belongs to exactly two strands, terminating at p and q



Set

$wt(e) := |v_p - v_q|$

$wt\left(\begin{smallmatrix}\text{bdry}\\ \text{edge}\end{smallmatrix}\right) := 1$

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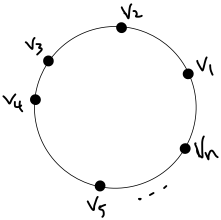

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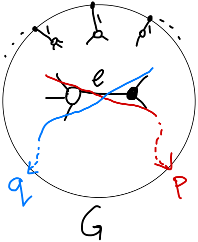

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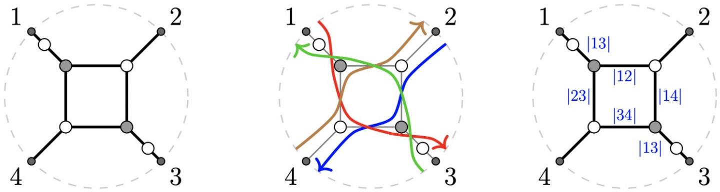

$|pq| = |v_p - v_q|$

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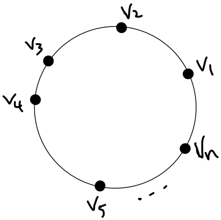

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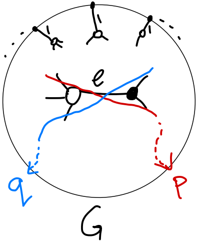

Set

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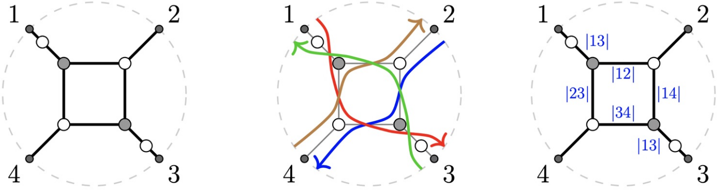
Critical cell $Crit_f^{>0}$ is the subset of the Grassmannian obtained by fixing G and letting v's vary.



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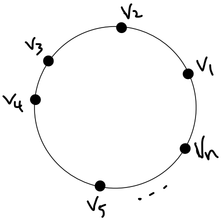

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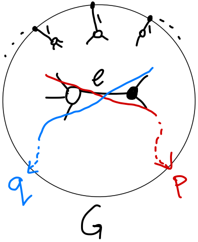

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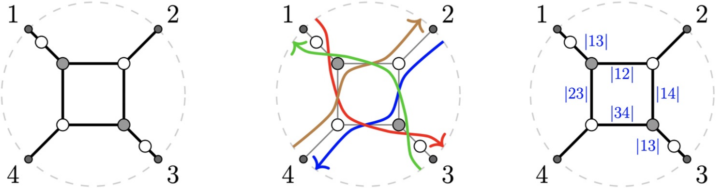
Critical edge weights are invariant under square moves, so the critical cell depends only on f, not on G.



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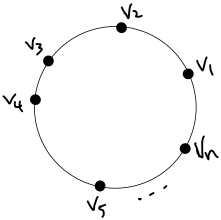

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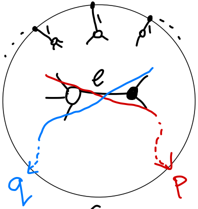

G

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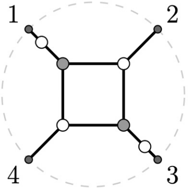
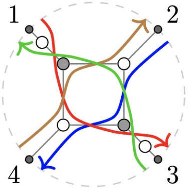
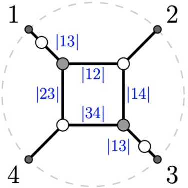
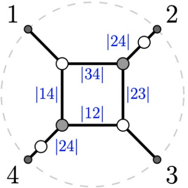


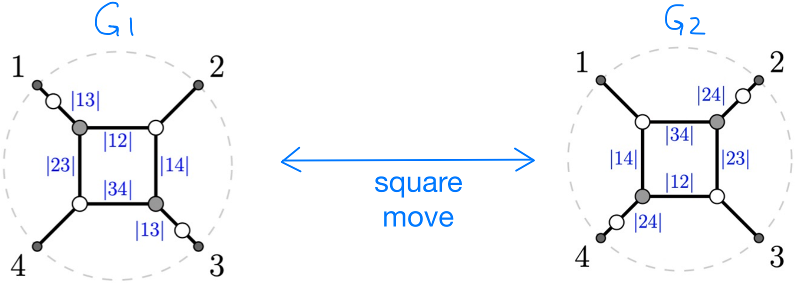




G_1

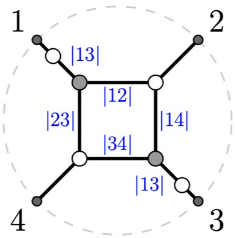


G_2



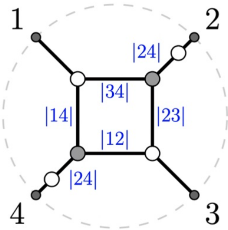
G1
G2
1
2
3
4
|13|
|12|
|23|
|14|
|34|
|13|
square
move
1
2
3
4
|24|
|34|
|14|
|23|
|12|
|24|

G_1

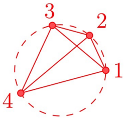


$\longleftrightarrow$ square move

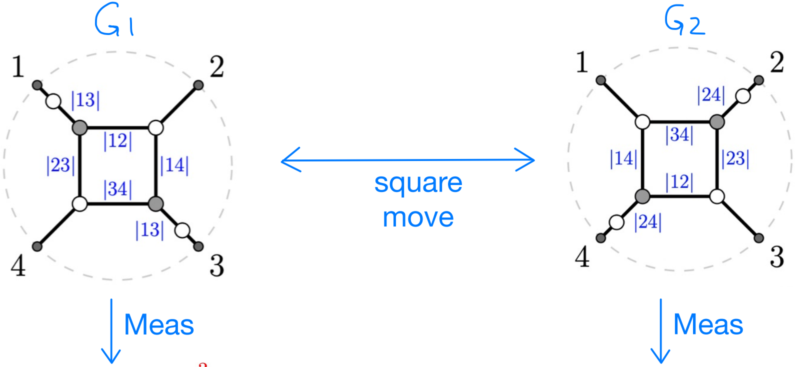
G_2

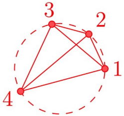


$\downarrow$ Meas



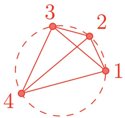
$$\Delta_{12} = |23| \cdot |13|$$
$$\Delta_{23} = |34| \cdot |13|$$
$$\Delta_{34} = |14| \cdot |13|$$
$$\Delta_{14} = |12| \cdot |13|$$
$$\Delta_{13} = |12| \cdot |34| + |14| \cdot |23| \overset{\text{by Ptolemy's theorem}}{=} |24| \cdot |13|$$
$$\Delta_{24} = |13| \cdot |13|$$





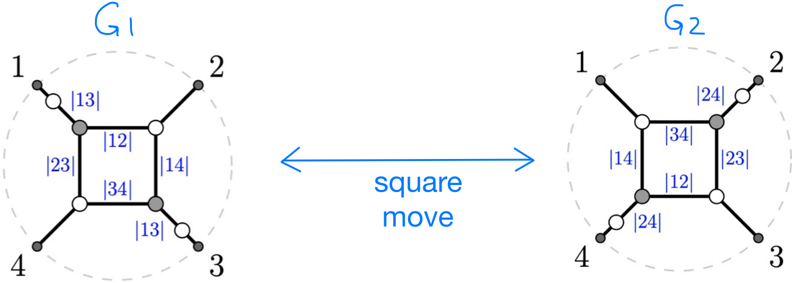
by Ptolemy's theorem

$$
\begin{aligned}
\Delta_{12} &= |23| \cdot |13| \\
\Delta_{23} &= |34| \cdot |13| \\
\Delta_{34} &= |14| \cdot |13| \\
\Delta_{14} &= |12| \cdot |13| \\
\Delta_{13} &= |12| \cdot |34| + |14| \cdot |23| = |24| \cdot |13| \\
\Delta_{24} &= |13| \cdot |13|
\end{aligned}
$$

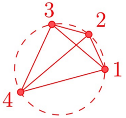


by Ptolemy's theorem

$$
\begin{aligned}
\Delta_{12} &= |23| \cdot |24| \\
\Delta_{23} &= |34| \cdot |24| \\
\Delta_{34} &= |14| \cdot |24| \\
\Delta_{14} &= |12| \cdot |24| \\
\Delta_{13} &= |24| \cdot |24| \\
\Delta_{24} &= |14| \cdot |23| + |12| \cdot |34| = |13| \cdot |24|
\end{aligned}
$$



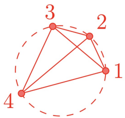
Meas ↓ Meas ↓



$$
\begin{aligned}
\Delta_{12} &= |23| \cdot |13| \\
\Delta_{23} &= |34| \cdot |13| \\
\Delta_{34} &= |14| \cdot |13| \\
\Delta_{14} &= |12| \cdot |13| \\
\Delta_{13} &= |12| \cdot |34| + |14| \cdot |23| = |24| \cdot |13| \\
\Delta_{24} &= |13| \cdot |13|
\end{aligned}
$$

by Ptolemy's theorem

$=$

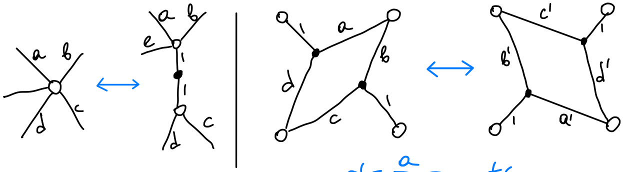


$$
\begin{aligned}
\Delta_{12} &= |23| \cdot |24| \\
\Delta_{23} &= |34| \cdot |24| \\
\Delta_{34} &= |14| \cdot |24| \\
\Delta_{14} &= |12| \cdot |24| \\
\Delta_{13} &= |24| \cdot |24| \\
\Delta_{24} &= |14| \cdot |23| + |12| \cdot |34| = |13| \cdot |24|
\end{aligned}
$$

by Ptolemy's theorem

Equal modulo rescaling

Any two reduced graphs with the same strand permutation are related by these moves:



$$a' = \frac{a}{ac+bd}, \text{ etc}$$

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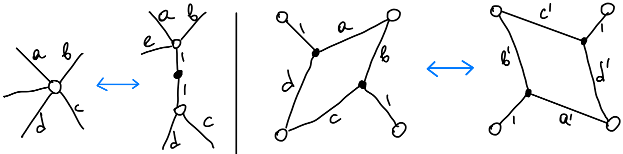
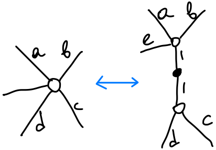


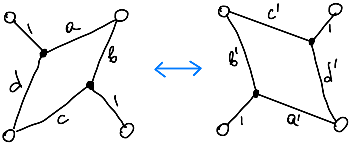
$$a' = \frac{a}{ac+bd}, \text{ etc}$$

For critical weights, Meas is invariant under these moves.

Thus Meas depends only on f and on $v_1, v_2, \dots, v_n$.

Any two reduced graphs with the same strand permutation are related by these moves:





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Formula?

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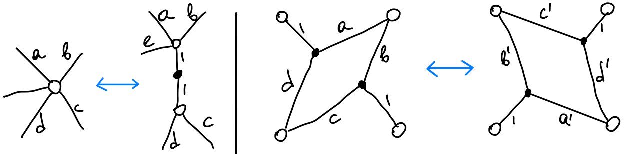

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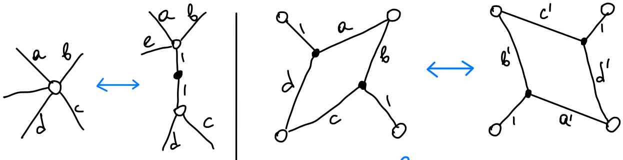
Formula?

linear span of all points on a curve

critical edge weights

<u>Theorem</u>. (G. '21) $\mathrm{Meas}(G, wt) = \mathrm{Span}(\gamma(t))$ inside $Gr(k,n)$

Any two reduced graphs with the same strand permutation are related by these moves:



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Formula?

critical edge weights

linear span of all points on a curve

Theorem. (G. '21) $\text{Meas}(G, wt) = \text{Span}(\gamma(t))$ inside $Gr(k,n)$, where $\gamma(t)$ is an explicit curve in $\mathbb{R}^n$ which depends only on f and $v_1, \ldots, v_n$ but not on G.

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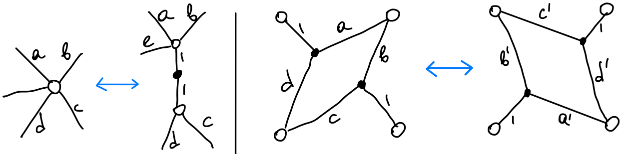


$a' = \frac{a}{ac+bd}$, etc

For critical weights, Meas is invariant under these moves.

Thus Meas depends only on f and on $v_1, v_2, \dots, v_n$.

Formula?

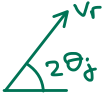
---

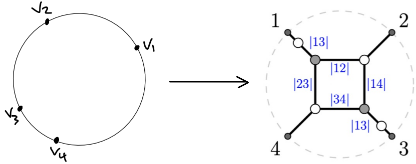
critical edge weights

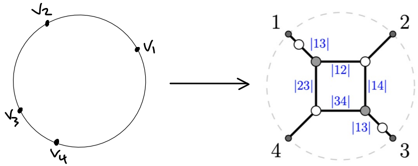
linear span of all points on a curve

<u>Theorem</u>. (G. '21) $\mathrm{Meas}(G, wt) = \mathrm{Span}(\gamma(t))$ inside $Gr(k,n)$, where $\gamma(t)$ is an explicit curve in $\mathbb{R}^n$ which depends only on f and $v_1, \dots, v_n$ but not on G.

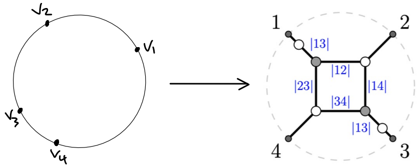
$$\gamma_r(t) = \pm \prod_{j \in I_r(f) \setminus r} \sin(t - \theta_j), \quad \text{where}$$



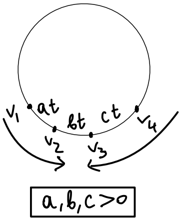

v2
v1
v3
v4
1
2
3
4
|13|
|12|
|23|
|14|
|34|
|13|

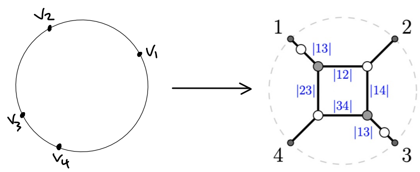


Q: what happens when points start to collide?



Q: what happens when points start to collide?





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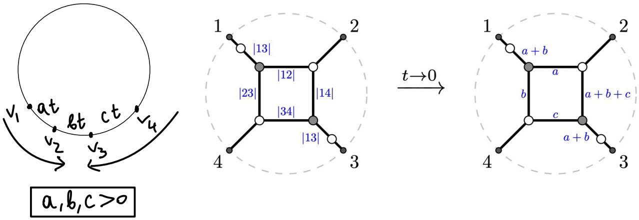

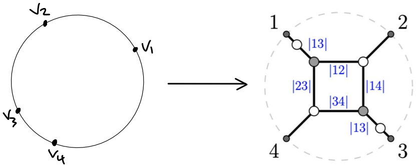

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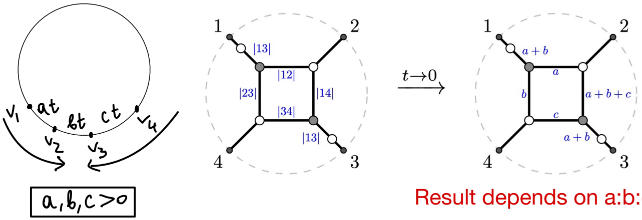
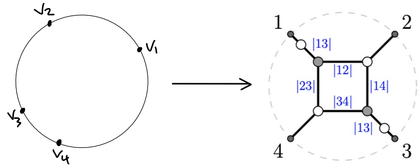
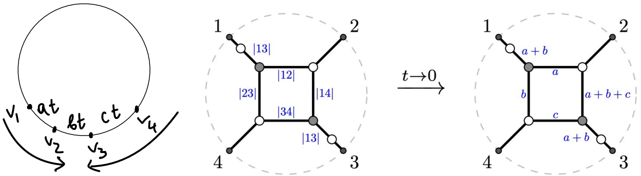


Result depends on a:b:c

dim=2



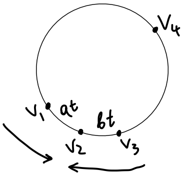
Q: what happens when points start to collide?



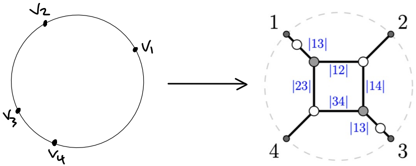
$\boxed{a, b, c > 0}$

Result depends on a:b:c

dim=2



$\boxed{a, b > 0}$



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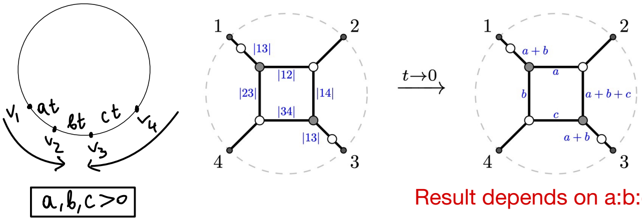
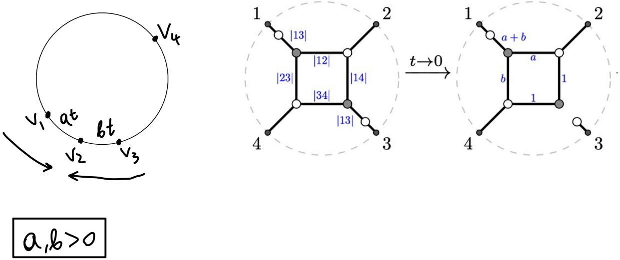
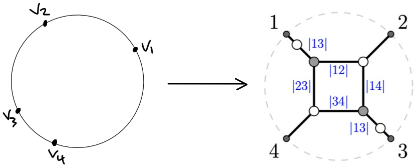


Result depends on a:b:c

dim=2





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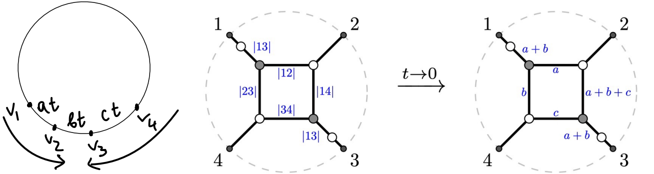
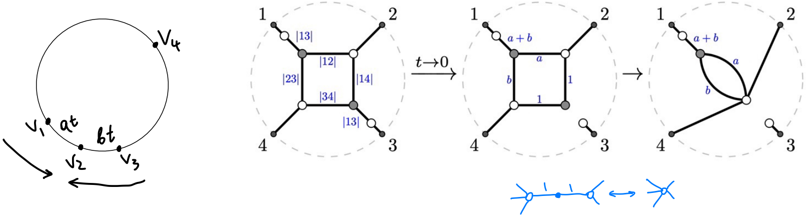


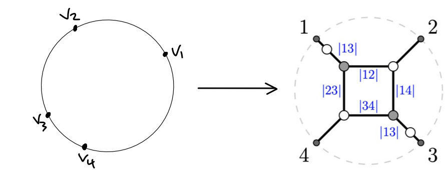
$\boxed{a,b,c>0}$

Result depends on a:b:c

dim=2



$\boxed{a,b>0}$



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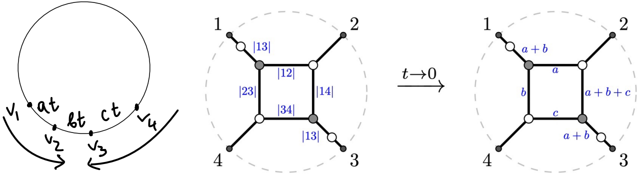

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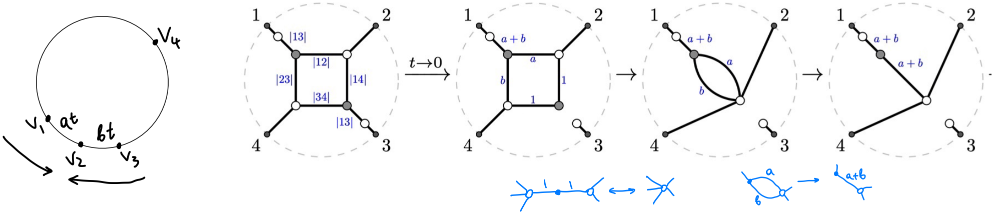
dim=2



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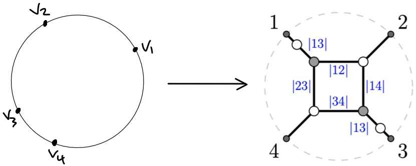

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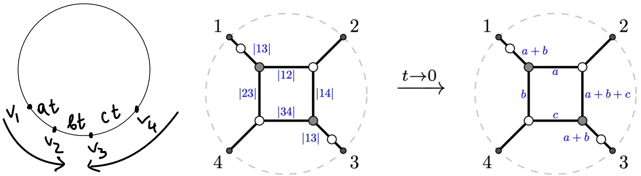

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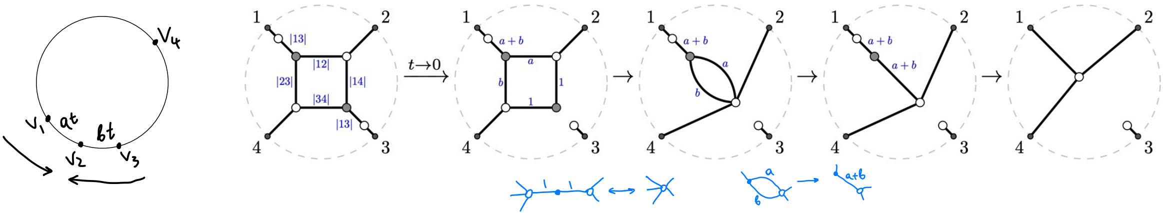

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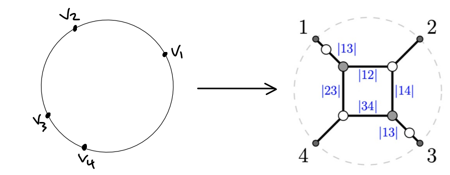

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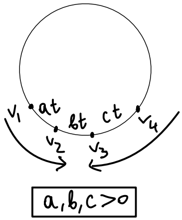
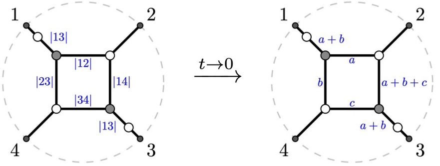
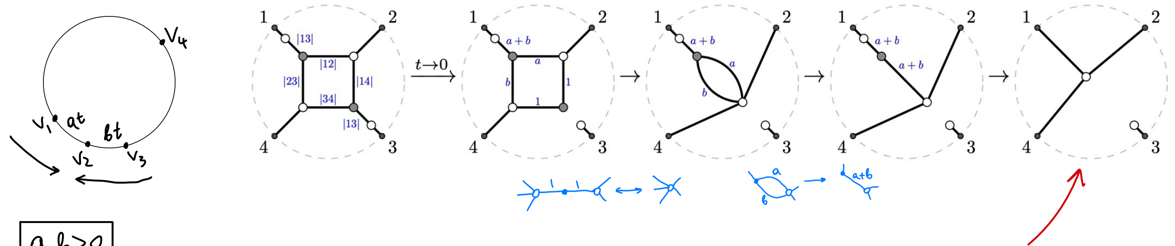




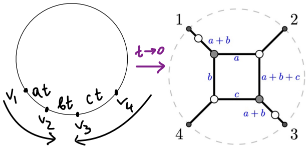
Result depends on a:b:c

dim=2



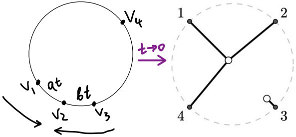
Result does not depend on a:b!

dim=0



Result depends on a:b:c

dim=2



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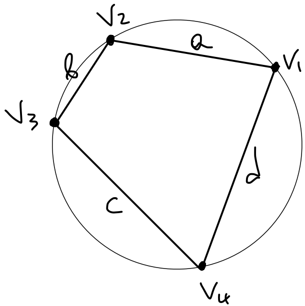
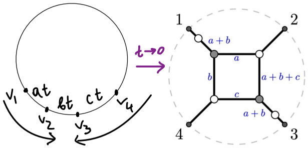
dim=0

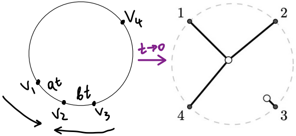


Space of (possibly degenerate) inscribed n-gons



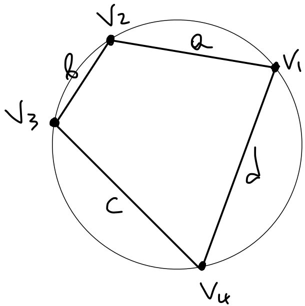
Result depends on a:b:c

dim=2



Result does not depend on a:b!

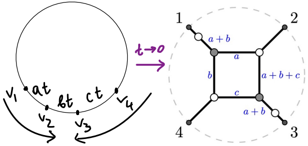
dim=0



homeomorphism

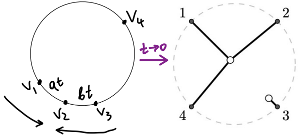
$$(a,b,c,d) \longmapsto (a:b:c:d)$$

Space of (possibly degenerate) inscribed n-gons



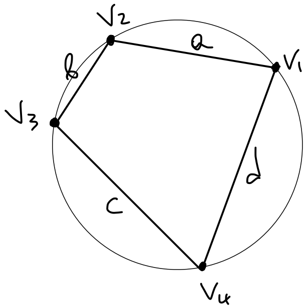
Result depends on a:b:c

dim=2



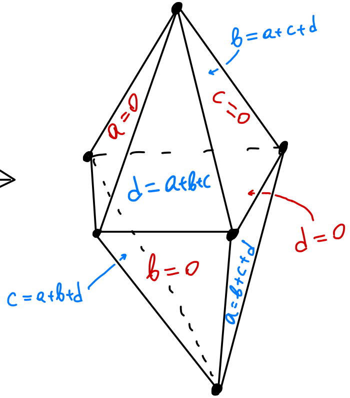
Result does not depend on a:b!

dim=0

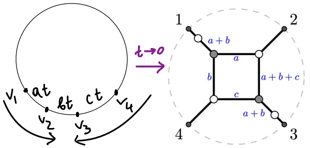


homeomorphism

$$(a,b,c,d) \mapsto (a:b:c:d)$$

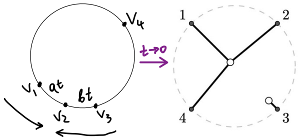


Space of (possibly degenerate) inscribed n-gons



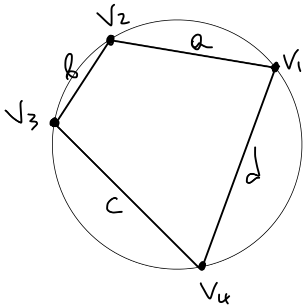
Result depends on a:b:c

dim=2

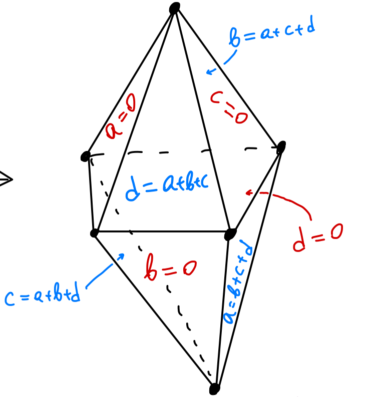


Result does not depend on a:b!

dim=0

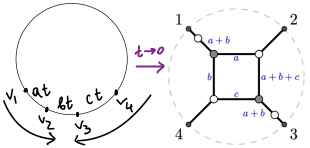


$$\xrightarrow[(a,b,c,d)\mapsto(a:b:c:d)]{\text{homeomorphism}}$$



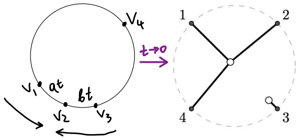
Space of (possibly degenerate) inscribed n-gons

Second hypersimplex $\Delta_{2,n}$



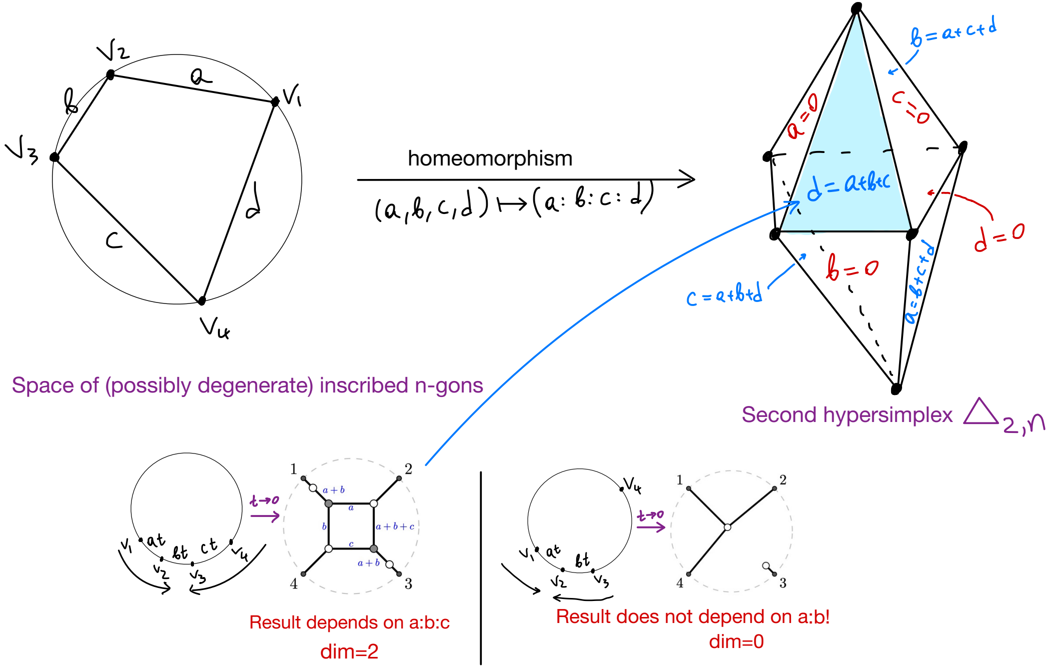
Result depends on a:b:c

dim=2

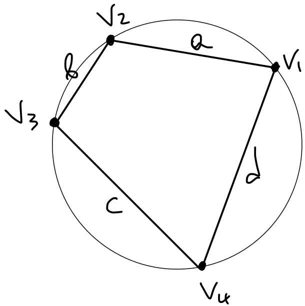


Result does not depend on a:b!

dim=0

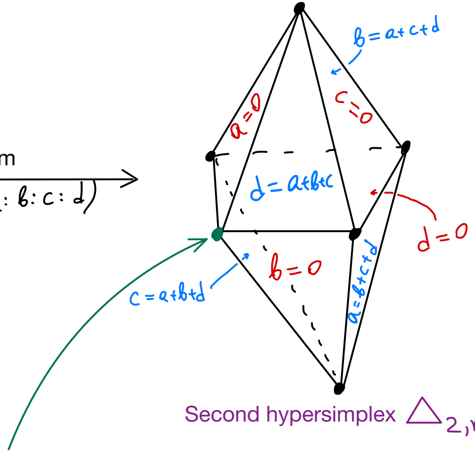


V2
a
V1
b
V3
c
d
V4
homeomorphism
(a, b, c, d) ↦ (a : b : c : d)
b = a + c + d
a = 0
c = 0
d = a + b + c
d = 0
b = 0
c = a + b + d
a = b + c + d
Space of (possibly degenerate) inscribed n-gons
Second hypersimplex △2,n
t→0
V1
at
bt
ct
V4
V2
V3
1
2
3
4
a + b
a
b
c
a + b + c
a + b
Result depends on a:b:c
dim=2
V4
t→0
V1
at
bt
V2
V3
1
2
3
4
Result does not depend on a:b!
dim=0



homeomorphism

$$(a,b,c,d) \mapsto (a:b:c:d)$$



Space of (possibly degenerate) inscribed n-gons

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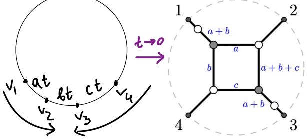
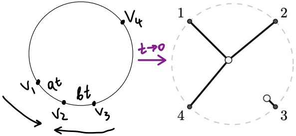


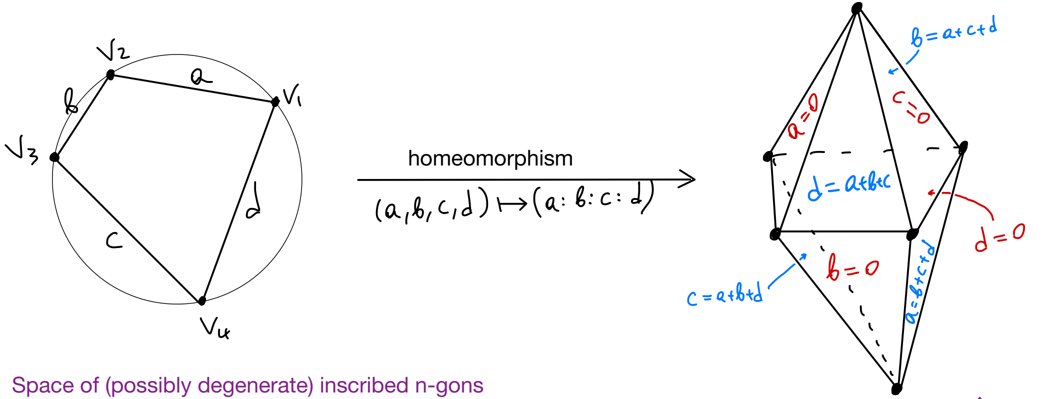
Result depends on a:b:c

dim=2



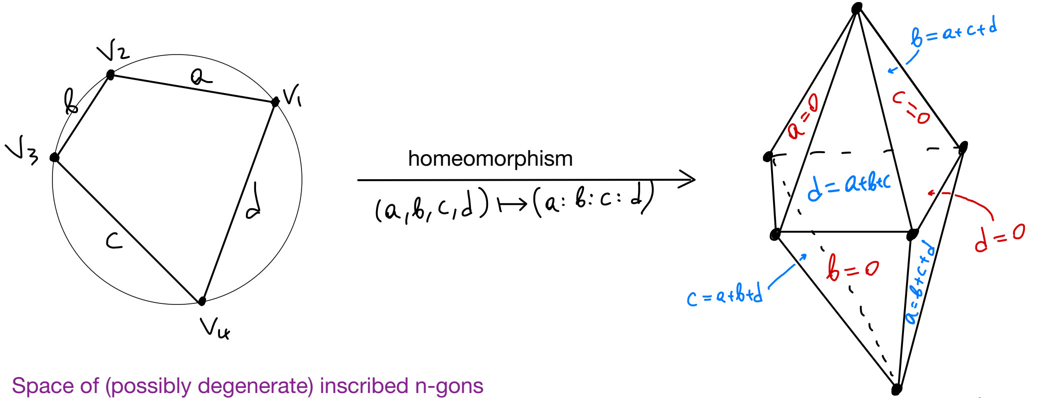
Result does not depend on a:b!

dim=0



Space of (possibly degenerate) inscribed n-gons

Second hypersimplex $\triangle_{2,n}$



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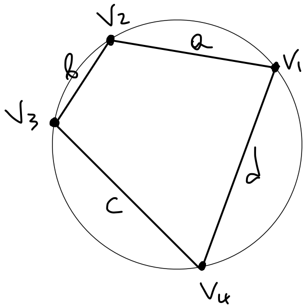
<u>Theorem</u> (G. '21) For top cell $f = f_{k,n}$, $\boxed{2 \le k \le n-1}$

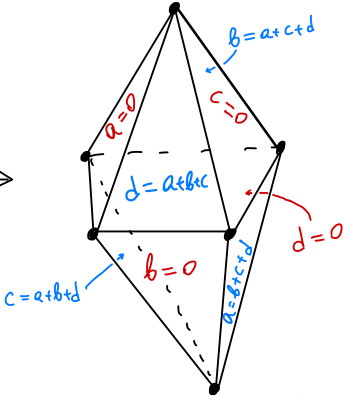
$$\mathrm{Crit}^{\geq 0}_{f_{k,n}} \cong \Delta_{2,n}$$

closure of $\mathrm{Crit}^{>0}_{f_{k,n}}$ inside $\mathrm{Gr}_{\geq 0}(k,n)$



Space of (possibly degenerate) inscribed n-gons

$$\xrightarrow[(a,b,c,d)\mapsto(a:b:c:d)]{\text{homeomorphism}}$$



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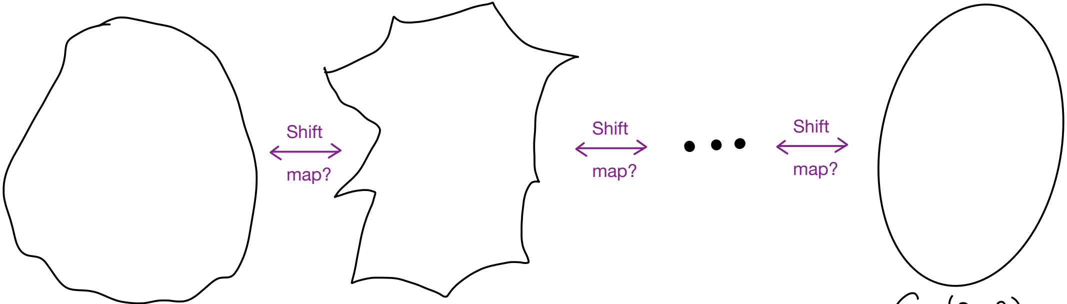

$Gr_{\geq 0}(n-1,n)$ $\dim = n-1$ $\quad$ $Gr_{\geq 0}(n-2,n)$ $\dim = 2(n-2)$ $\quad \ldots \quad$ $Gr_{\geq 0}(k,n)$ $\dim = k(n-k)$ $\quad \ldots \quad$ $Gr_{\geq 0}(2,n)$ $\dim = 2(n-2)$

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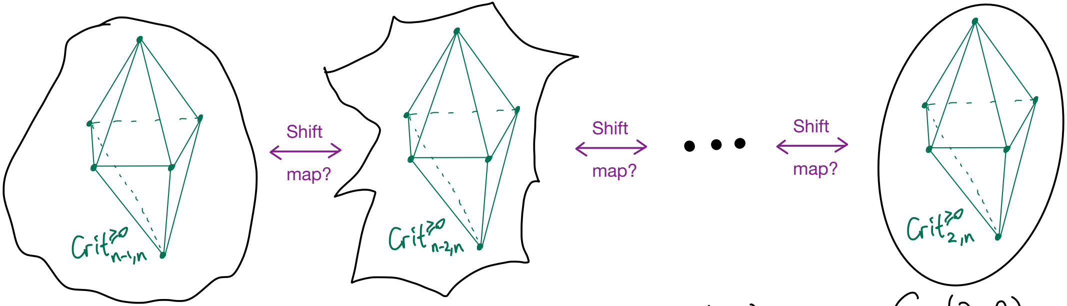

$Gr_{\geq 0}(n-1,n)$ $\dim = n-1$ | $Gr_{\geq 0}(n-2,n)$ $\dim = 2(n-2)$ | $\cdots$ | $Gr_{\geq 0}(k,n)$ $\dim = k(n-k)$ | $\cdots$ | $Gr_{\geq 0}(2,n)$ $\dim = 2(n-2)$

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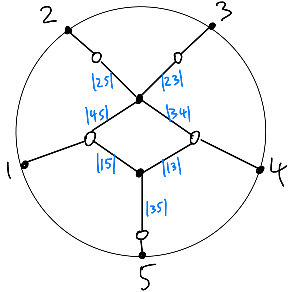

V_1 can pass through V_2

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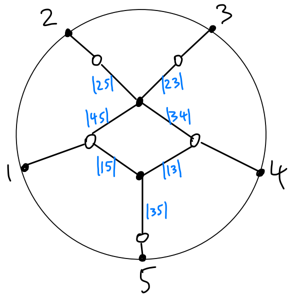
Problem #1: have to allow some v's to pass through each other.

Problem #2: the limit usually depends on "infinitesimal ratios"



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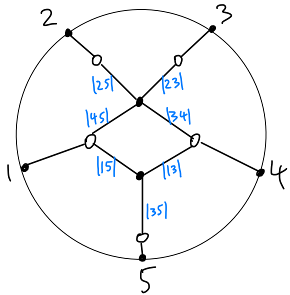

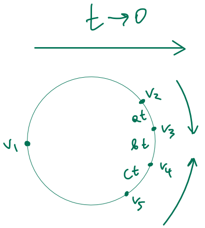

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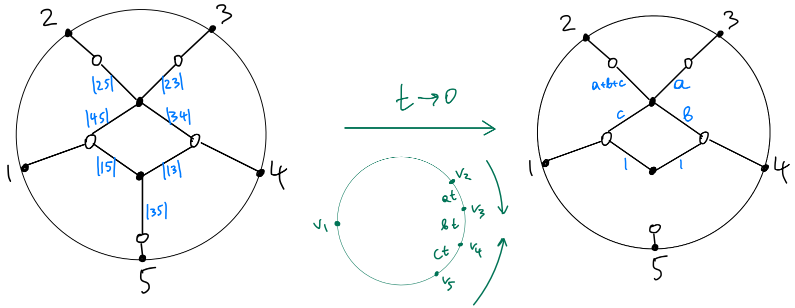

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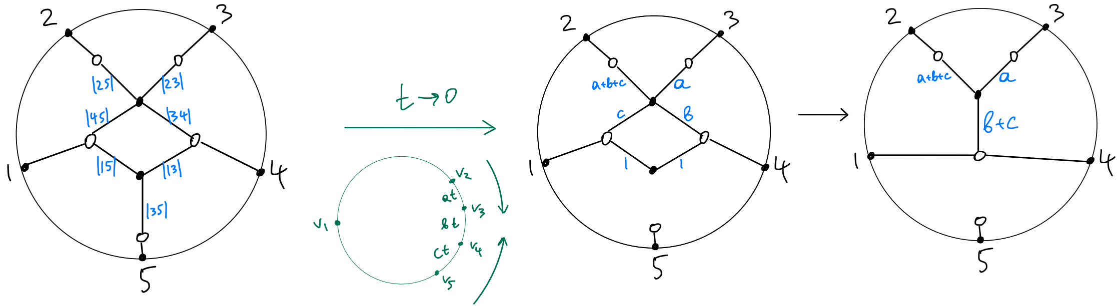

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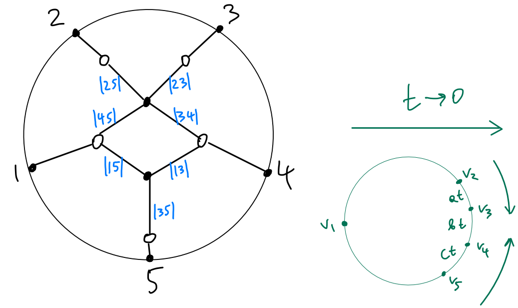

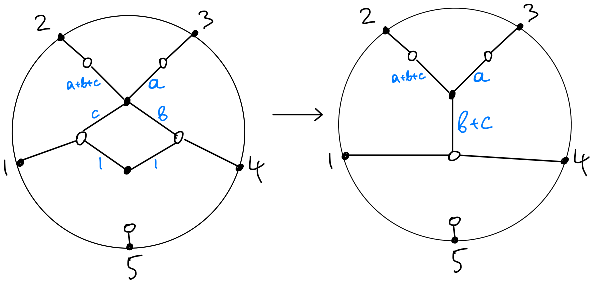

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- Top cell $f = f_{k,n} \Rightarrow \mathrm{Cyc}_f$ = usual cyclohedron

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Poset associahedra

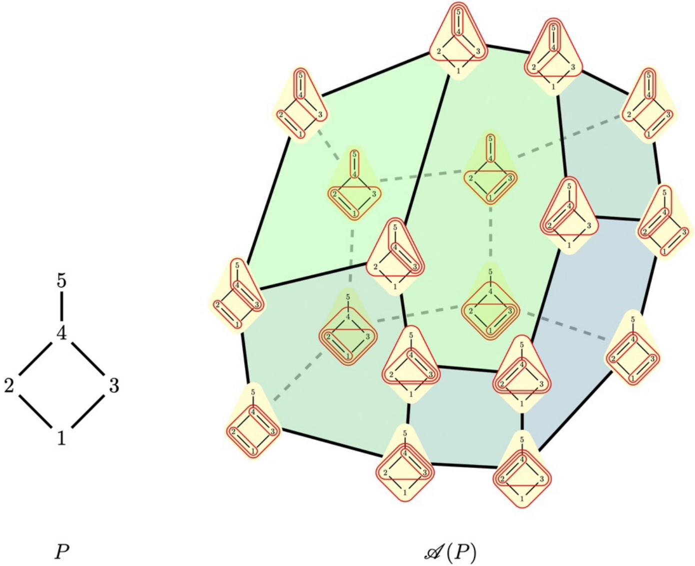
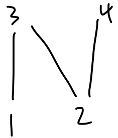


FIGURE 1. A poset associahedron.

Suppose P is a connected
poset (=partially ordered set)

$P =$ 

$1<3, \ 2<3, \ 2<4$

Suppose P is a connected poset (=partially ordered set)

$P =$ 

$1 < 3, \; 2 < 3, \; 2 < 4$

P-configuration space:

$$\mathcal{O}^{\circ}(P) := \left\{ x \in \mathbb{R}^{P} \;\middle|\; x_i < x_j \text{ if } i <_P j \right\} \Big/ \text{shifts, rescaling}$$

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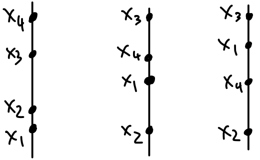

etc

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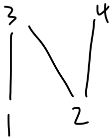

$1<3,\ 2<3,\ 2<4$

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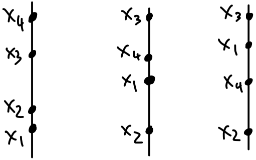
$$\mathcal{O}^0(P) := \{x \in \mathbb{R}^P \mid x_i < x_j \text{ if } i <_P j\} \big/ \text{shifts, rescaling}$$



etc

Q: how to compactify the P-configuration space?

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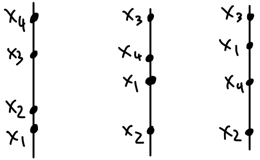

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Option #1: order polytope $\mathcal{O}(P)$

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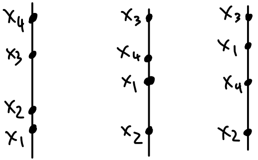

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Option #2: poset associahedron $\mathcal{A}(P)$

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τ is **convex** if

$i \leq_P j \leq_P k, \quad i, k \in \tau \Rightarrow j \in \tau.$

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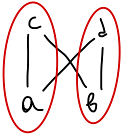
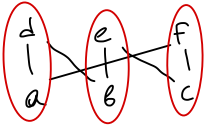


not acyclic!

a=c, b=d, but
a<d, b<c!



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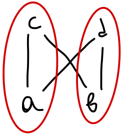

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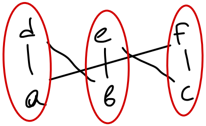

not acyclic!

Proposition. The faces of the order polytope $\mathcal{O}(P)$ are in bijection with acyclic collections of disjoint P-tubes whose union is P.

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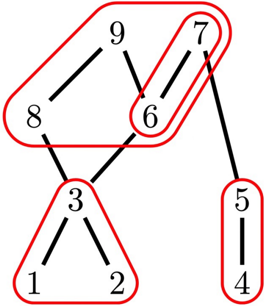

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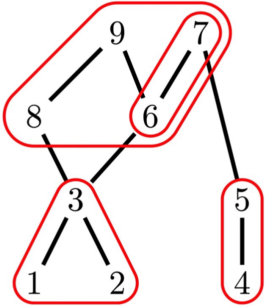

Theorem. (G. '21) There exists a convex polytope $\mathcal{A}(P)$ ("poset associahedron") whose faces correspond to P-tubings, ordered by reverse inclusion.

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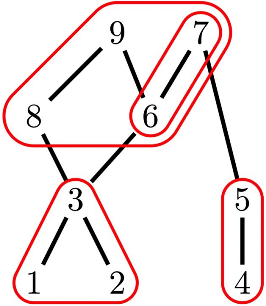

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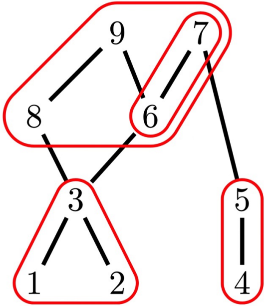

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- dimension = |P|-2
- vertices = maximal P-tubings

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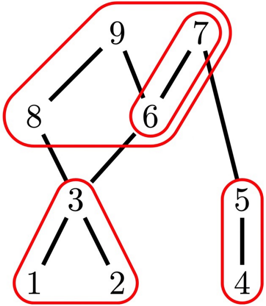

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- facets = P-tubes

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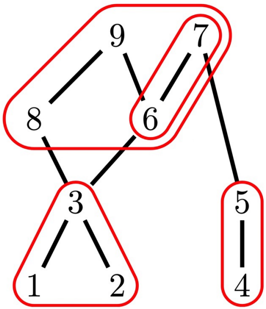

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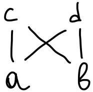
Ex. $P =$ 

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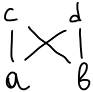
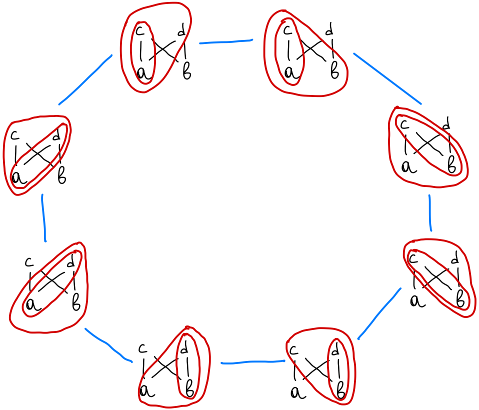


$\mathcal{A}(P) =$



<u>Definition.</u> A **P-tubing** is an acyclic collection of P-tubes, such that any two tubes are either nested or disjoint.

<u>Theorem. (G. '21)</u> There exists a convex polytope $\mathcal{A}(P)$ ("poset associahedron") whose faces correspond to P-tubings, ordered by reverse inclusion.

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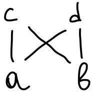
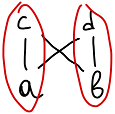
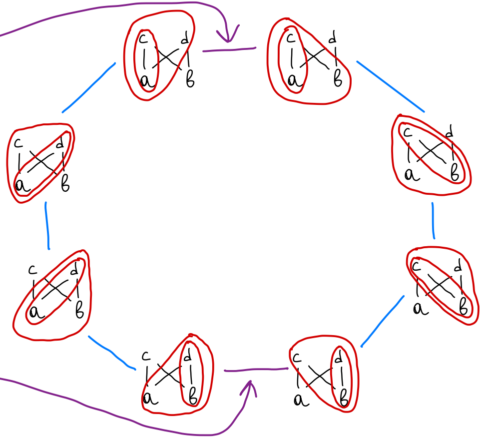
Ex. $P =$ 

Note: the **acyclic** condition is important!



bad!

$\mathcal{A}(P) =$ 

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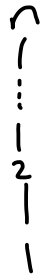
<u>Ex.</u> P = chain = 

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Ex. P = chain = 

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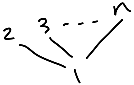
Ex. P = claw = 

$\Rightarrow \mathcal{A}(P)$ = permutohedron

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Similarly, one can define **affine posets** and **affine poset cyclohedra**.

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Similarly, one can define **affine posets** and **affine poset cyclohedra**.

- dimension = |P|-1
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Similarly, one can define **affine posets** and **affine poset cyclohedra**.

- dimension = |P|-1
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- P = affine chain => Cyc(P) = usual cyclohedron
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- poset associahedron = compactification of the space of points on a line

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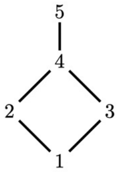
Similarly, one can define **affine posets** and **affine poset cyclohedra**.

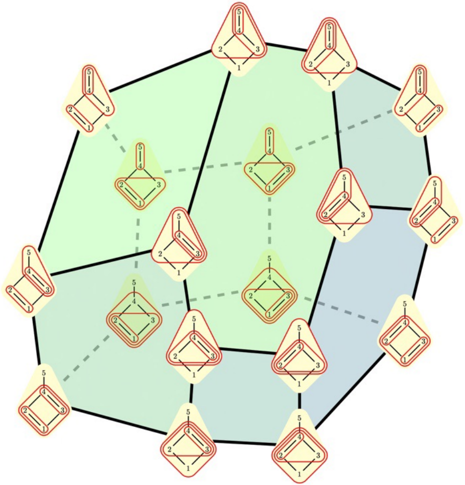
- dimension = |P|-1
- faces = products of smaller poset associahedra and affine poset cyclohedra
- P = affine chain => Cyc(P) = usual cyclohedron
- P = affine claw => Cyc(P) = type B permutohedron

- poset associahedron = compactification of the space of points on a line
- affine poset cyclohedron = compactifiaction of the space of points on a circle

Thanks!



P



$\mathscr{A}(P)$

FIGURE 1. A poset associahedron.