Ising model, total positivity, and criticality

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Positive Geometries in Scattering Amplitudes and Beyond
Mainz Institute for Theoretical Physics
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[GP20] P. Galashin and P. Pylyavskyy. Ising model and the positive orthogonal Grassmannian. *Duke Math. J.*, 169(10):1877–1942, 2020.

[Gal20] P. Galashin. A formula for boundary correlations of the critical Ising model. `arXiv:2010.13345`.

[Gal21] P. Galashin. Critical varieties in the Grassmannian. `arXiv:2102.13339`.

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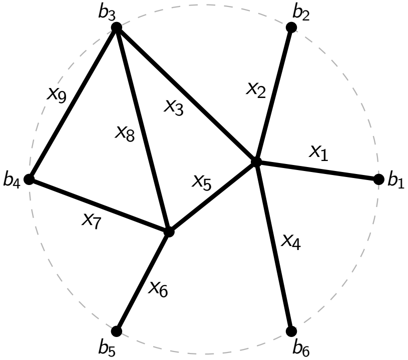

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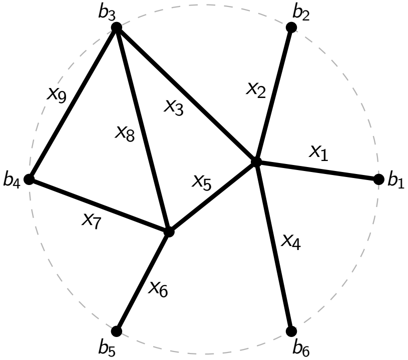

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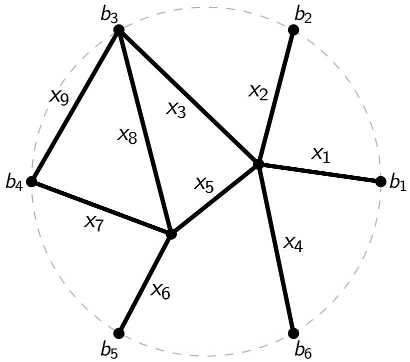

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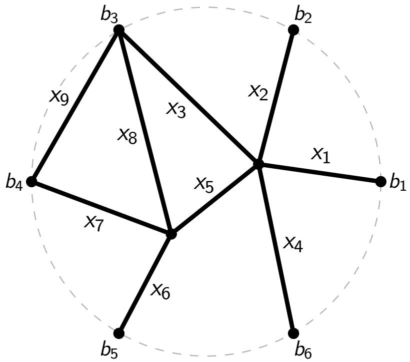

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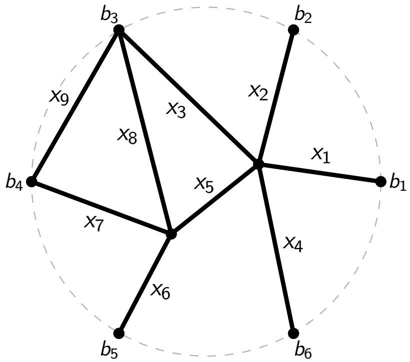

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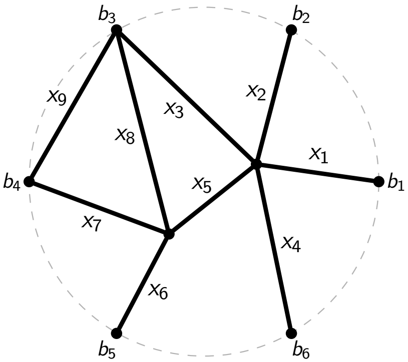

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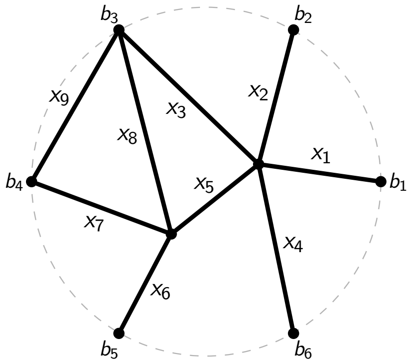

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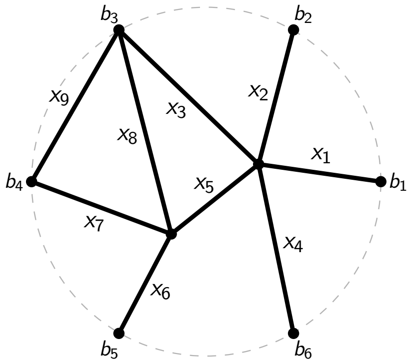

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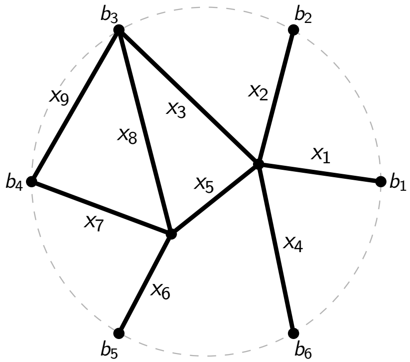

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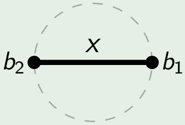

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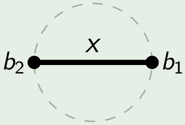

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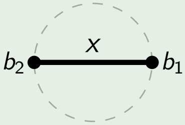

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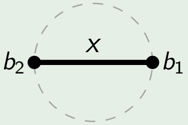

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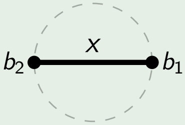

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$$\dim(\mathrm{Gr}_{\geqslant 0}(n,2n)) = n^2$$

The totally nonnegative orthogonal Grassmannian

$\mathrm{Gr}_{\geqslant 0}(k,n) \quad \longleftrightarrow \quad$ amplituhedron $\quad \longleftrightarrow \quad$ $\mathcal{N}=4$ supersymmetric Yang–Mills theory

$\mathrm{OG}_{\geqslant 0}(n,2n) \quad \longleftrightarrow \quad$? $\quad \longleftrightarrow \quad$ $\mathcal{N}=6$ supersymmetric Chern-Simons matter theory

Recall: $\mathrm{Gr}_{\geqslant 0}(k,n) := \{W \in \mathrm{Gr}(k,n) \mid \Delta_I(W) \geqslant 0 \text{ for all } I\}$.

Definition (Huang–Wen (2013))

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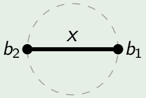

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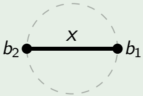

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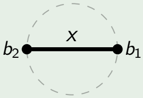

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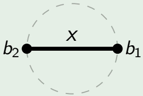

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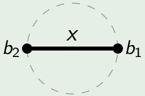

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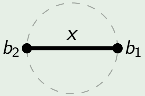

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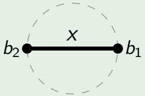

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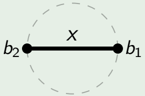

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- *Kramers–Wannier's duality (1941)* $\to$ *cyclic shift.*

Critical Ising model

Phase transition

$$\mathrm{Prob}(\sigma) := \frac{1}{Z} \prod_{\substack{\{u,v\}\in E(G):\\ \sigma_u \neq \sigma_v}} x_{\{u,v\}}.$$

Usually:

- G = large piece of a (e.g. square) lattice;
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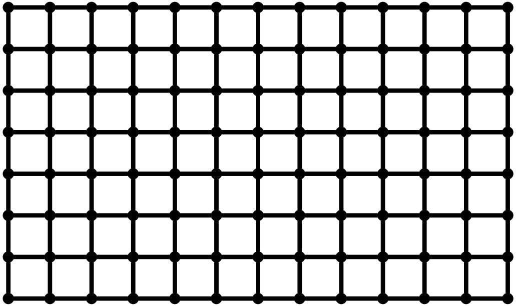

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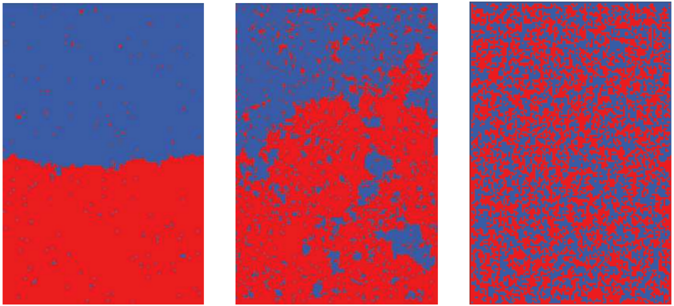

$x < x_{\text{crit}}$ $x = x_{\text{crit}}$ $x > x_{\text{crit}}$

Picture credit: Dmitry Chelkak

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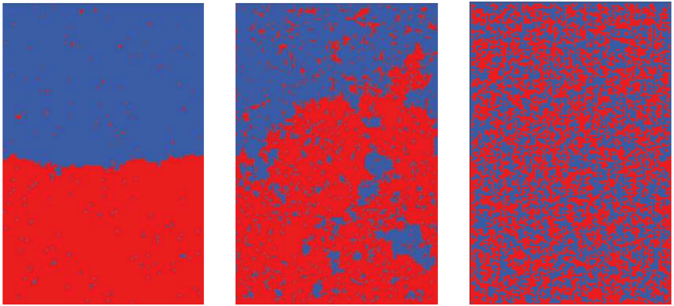

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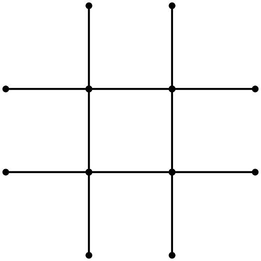

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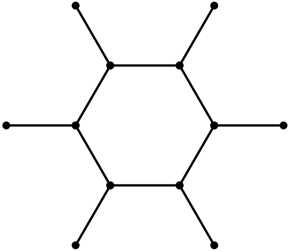
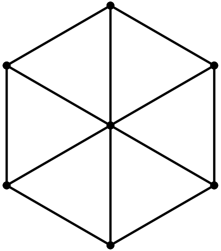


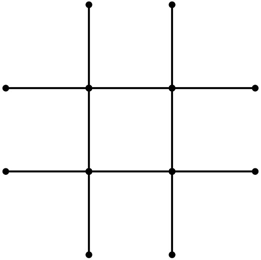
Hexagonal lattice

$x_{\text{crit}} = 2 - \sqrt{3}$



Triangular lattice

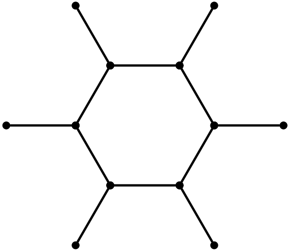
$x_{\text{crit}} = \frac{1}{\sqrt{3}}$



Square lattice

$x_{\text{crit}} = \sqrt{2} - 1$

$x_{\text{crit}} = \tan(\pi/8)$



Hexagonal lattice

$x_{\text{crit}} = 2 - \sqrt{3}$

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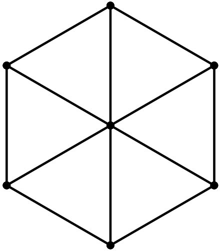

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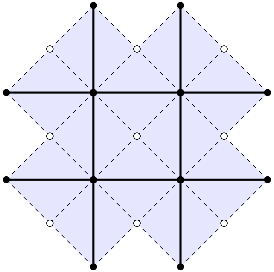
$x_{\text{crit}} = \frac{1}{\sqrt{3}}$

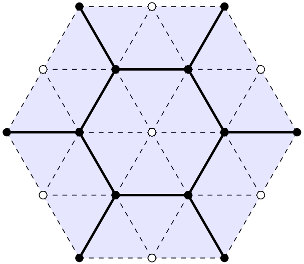
$x_{\text{crit}} = \tan(\pi/6)$



Square lattice

$$x_{\text{crit}} = \sqrt{2} - 1$$

$$x_{\text{crit}} = \tan(\pi/8)$$



Hexagonal lattice

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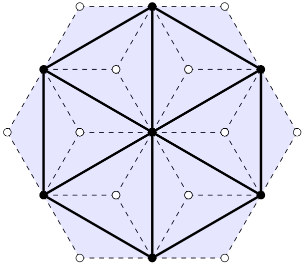
$$x_{\text{crit}} = \tan(\pi/12)$$



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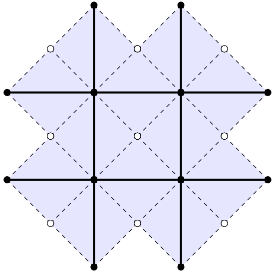

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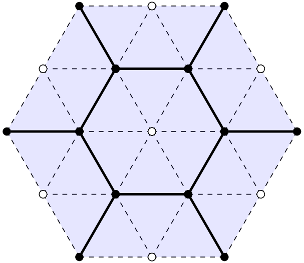

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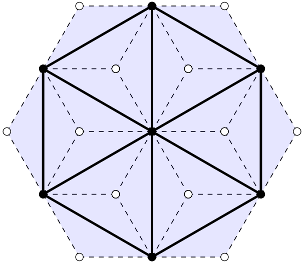

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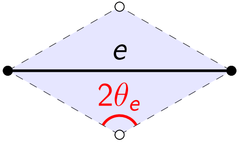

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Critical Z-invariant Ising model

[Bax86] R. J. Baxter. Free-fermion, checkerboard and **Z**-invariant lattice models in statistical mechanics. *Proc. Roy. Soc. London Ser. A*, 404(1826):1–33, 1986.

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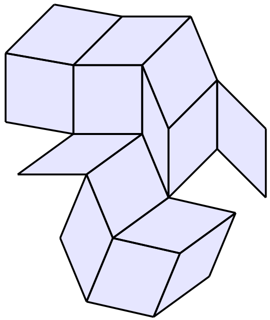

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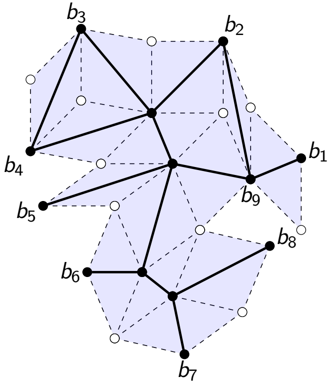

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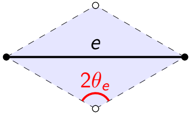

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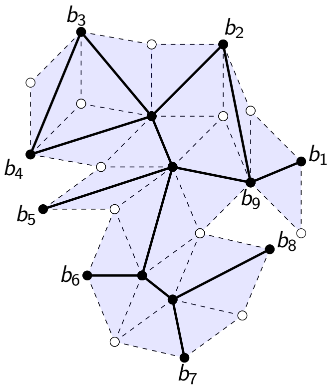

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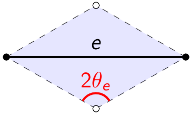

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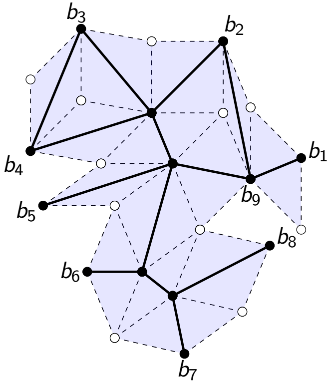

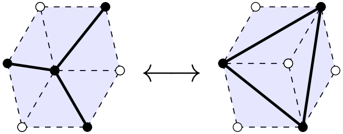

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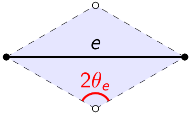

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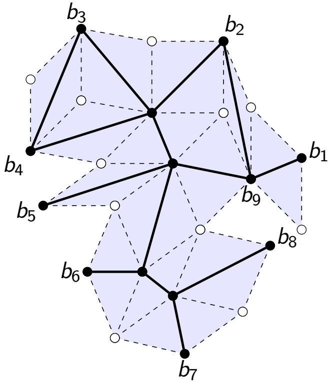

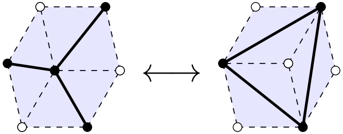

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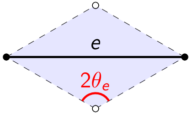

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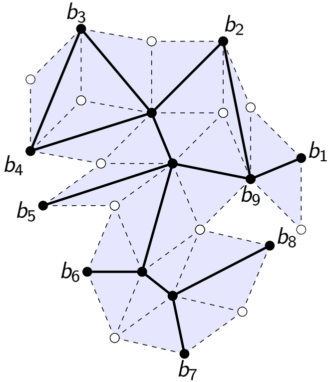
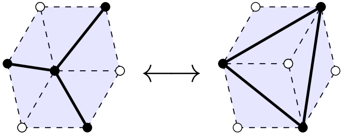
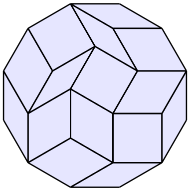
- Formula for $\langle \sigma_i \sigma_j \rangle_R$ in terms of R?

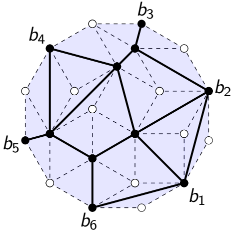




A formula for regular polygons

Let R_N be a regular $2N$-gon and $\langle \sigma_i \sigma_j \rangle_{R_N}$ be the corresponding boundary correlations.





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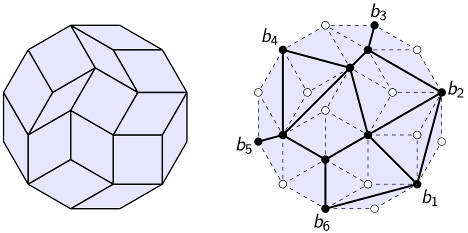

Theorem (G. (2020))

For $1 \leqslant i, j \leqslant N$ and $d := |i - j|$, we have

$$\langle \sigma_i \sigma_j \rangle_{R_N} = \frac{2}{N} \left(\frac{1}{\sin\left((2d-1)\pi/2N\right)} - \frac{1}{\sin\left((2d-3)\pi/2N\right)} + \cdots \pm \frac{1}{\sin\left(\pi/2N\right)} \right) \mp 1.$$

[Gal20] Pavel Galashin. A formula for boundary correlations of the critical Ising model. arXiv:2010.13345, 2020.

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- A: Yes, by the Leibniz formula for π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

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[Hon10] Clement Hongler. *Conformal invariance of Ising model correlations.* PhD thesis, 06/28 2010.

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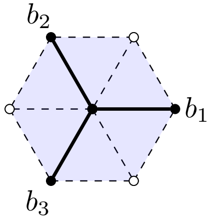

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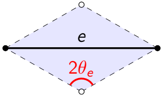

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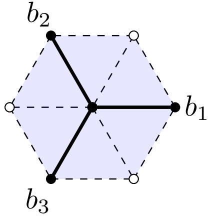

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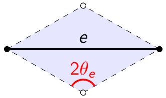

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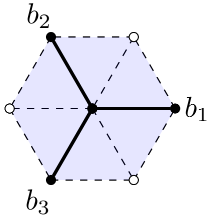

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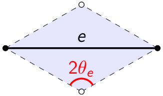

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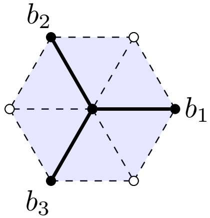

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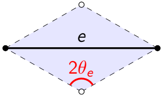

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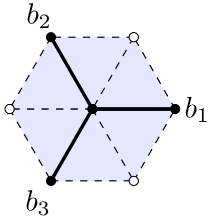

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- Treat each edge of G as a resistor.
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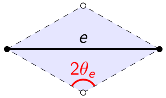


$R_e = \tan(\theta_e)$

- Electrical response matrix $\Lambda : \mathbb{R}^N \to \mathbb{R}^N$, voltages $\mapsto$ currents.
- Λ is invariant under star-triangle moves $\Longrightarrow$ depends only on the region.

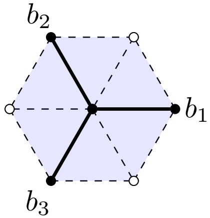
Theorem (G. (2021))

If R is a regular $2N$-gon then for $1 \leqslant i, j \leqslant N$ and $d := |i - j|$, we have

$$\Lambda_{i,j} = \frac{\sin(\pi/N)}{N \cdot \sin((2d-1)\pi/2N) \cdot \sin((2d+1)\pi/2N)}.$$

- Ising model case: $x_e = \tan(\theta_e/2)$ and

$$\langle \sigma_i \sigma_j \rangle_{R_N} = \frac{2}{N}\left(\frac{1}{\sin\left((2d-1)\pi/2N\right)} - \frac{1}{\sin\left((2d-3)\pi/2N\right)} + \cdots \pm \frac{1}{\sin\left(\pi/2N\right)}\right) \mp 1.$$

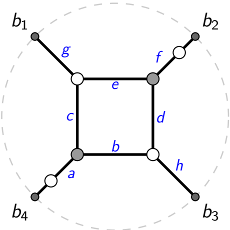


$$R_e = \frac{1}{\sqrt{3}}$$

$$\Lambda = \frac{1}{\sqrt{3}}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

Critical dimer model

- (G, wt) – a weighted planar bipartite graph, with n black boundary vertices $b_1, b_2, \ldots, b_n$ of degree 1.



[Pos06] Alexander Postnikov. Total positivity, Grassmannians, and networks. Preprint, arXiv:math/0609764, 2006.

- (G, wt) – a weighted planar bipartite graph, with n black boundary vertices $b_1, b_2, \ldots, b_n$ of degree 1.
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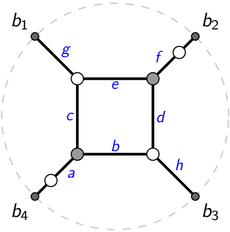

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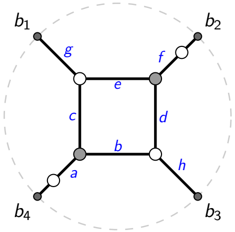
- Boundary measurement map $\mathrm{Meas}_G(\mathrm{wt}) = (\Delta_J(G, \mathrm{wt}))_{J \in \binom{[n]}{k}}$:

$$\Delta_J(G, \mathrm{wt}) := \sum_{\mathcal{A} : \partial(\mathcal{A}) = J} \mathrm{wt}(\mathcal{A}), \quad \text{where} \quad \mathrm{wt}(\mathcal{A}) := \prod_{e \in \mathcal{A}} \mathrm{wt}(e).$$



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- A strand is a path in G that makes a sharp right turn at each black vertex and a sharp left turn at each white vertex.
- Strand permutation: $f_G \in S_n$. (aka "loopless bounded affine permutation")

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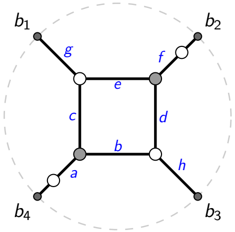

Critical dimer model

[Ken02] R. Kenyon. The Laplacian and Dirac operators on critical planar graphs. *Invent. Math.*, 150(2):409–439, 2002.

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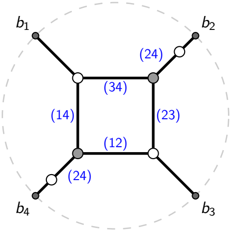

$$(pq) := \sin(\theta_q - \theta_p)$$

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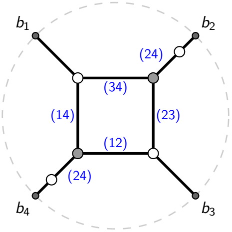

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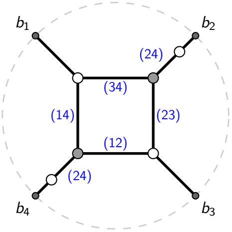
- Each edge e belongs to exactly two strands terminating at b_p and b_q for $1 \leqslant p < q \leqslant n$. Set

$$\mathrm{wt}_{\boldsymbol{\theta}}(e) := \begin{cases} \sin(\theta_q - \theta_p), & \text{if } e \text{ is not a boundary edge,} \\ 1, & \text{otherwise.} \end{cases}$$



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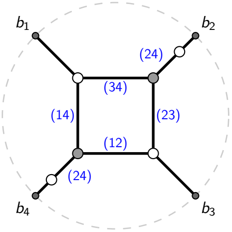

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$$\begin{aligned}
\Delta_{12} &= (23) \cdot (24) \\
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\Delta_{24} &= (14) \cdot (23) + (12) \cdot (34) = (13) \cdot (24)
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by Ptolemy's theorem

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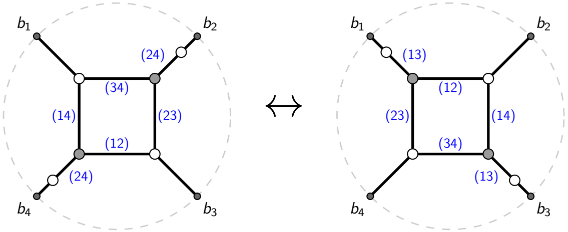

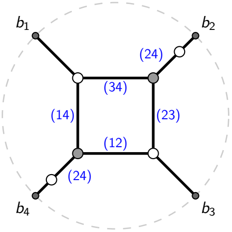

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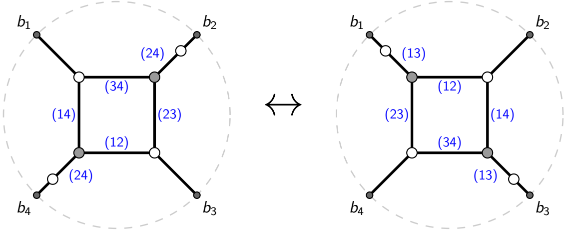
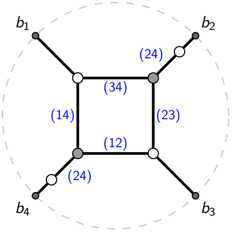


- A graph G is reduced if it has the minimal number of faces among all graphs with the same strand permutation.



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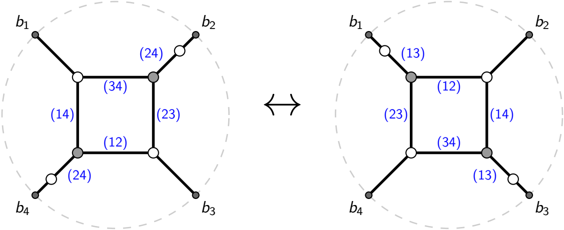

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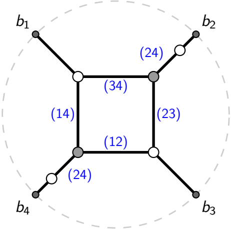

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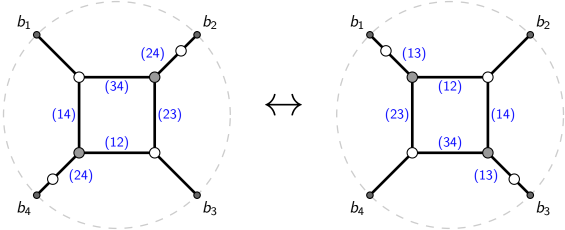

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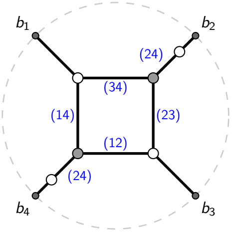

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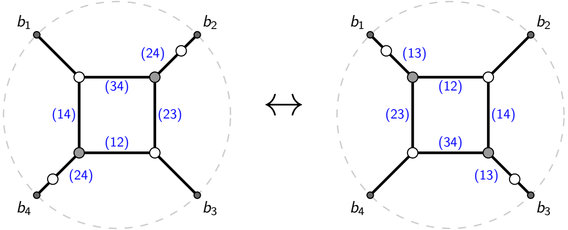

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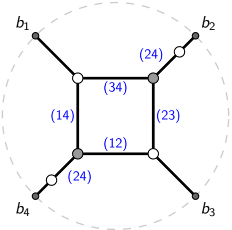
- Formula for $\mathrm{Meas}_f(\boldsymbol{\theta})$ in terms of f and $\boldsymbol{\theta}$?



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- Strand: turn right/left at each black/white vertex.
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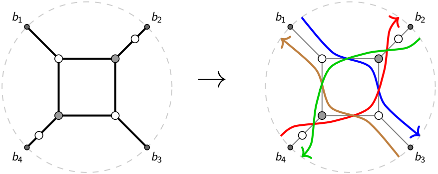

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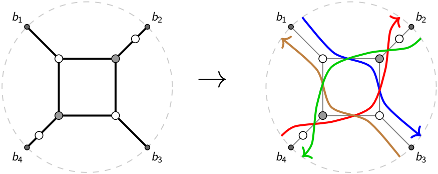

$J_1 = \{2\}$ $J_2 = \{3\}$ $J_3 = \{4\}$ $J_4 = \{1\}$

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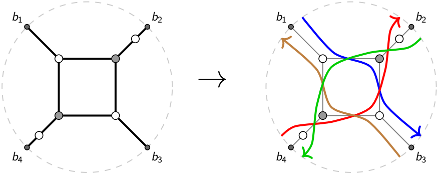
- Consider a curve $\boldsymbol{\gamma}_{f,\boldsymbol{\theta}}(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))$:

$$\gamma_r(t) := \epsilon_r \prod_{p \in J_r} \sin(t - \theta_p) \qquad \text{for } r \in [n],$$

where $\epsilon_r := (-1)^{\#\{p \in [n] \mid f(p) \leqslant p < r\}}$.



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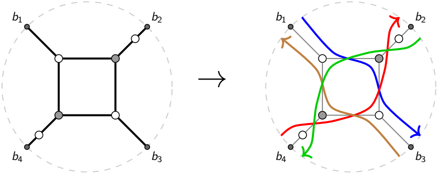
Theorem (G. (2021))

$$\mathrm{Meas}_f(\boldsymbol{\theta}) = \mathrm{Span}(\boldsymbol{\gamma}_{f,\boldsymbol{\theta}}) \quad \textit{inside } \mathrm{Gr}(k,n).$$



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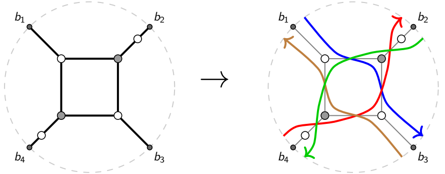

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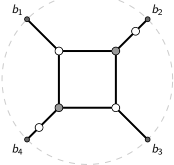
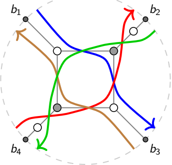


$\to$



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Theorem (G. (2021))

$$\mathrm{Meas}_f(\boldsymbol{\theta}) = \mathrm{Span}(\boldsymbol{\gamma}_{f,\boldsymbol{\theta}}) \quad \textit{inside } \mathrm{Gr}(k,n).$$

$\Delta_{12} = (23) \cdot (24)$ $\qquad$ $(pq) := \sin(\theta_q - \theta_p)$

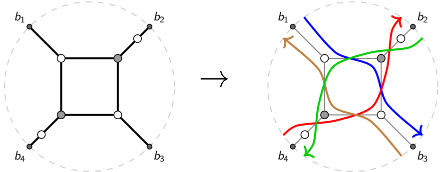
$\Delta_{23} = (34) \cdot (24)$

$\Delta_{34} = (14) \cdot (24)$ $\qquad$ by Ptolemy's theorem

$\Delta_{14} = (12) \cdot (24)$

$\Delta_{13} = (24) \cdot (24)$

$\Delta_{24} = (14) \cdot (23) + (12) \cdot (34) = (13) \cdot (24)$



$J_1 = \{2\} \quad J_2 = \{3\} \quad J_3 = \{4\} \quad J_4 = \{1\}$

$$\boldsymbol{\gamma}_{f,\boldsymbol{\theta}}(t) = (\sin(t - \theta_2), \sin(t - \theta_3), \sin(t - \theta_4), -\sin(t - \theta_1))$$

$\mathrm{Span}(\boldsymbol{\gamma}_{f,\boldsymbol{\theta}})$ is the row span of

$$\begin{pmatrix} -\sin(\theta_2) & -\sin(\theta_3) & -\sin(\theta_4) & \sin(\theta_1) \\ \cos(\theta_2) & \cos(\theta_3) & \cos(\theta_4) & -\cos(\theta_1) \end{pmatrix}$$

$\Delta_{12} = \sin(\theta_3 - \theta_2) \quad \Delta_{23} = \sin(\theta_4 - \theta_3)$

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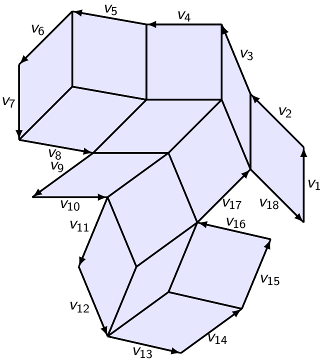
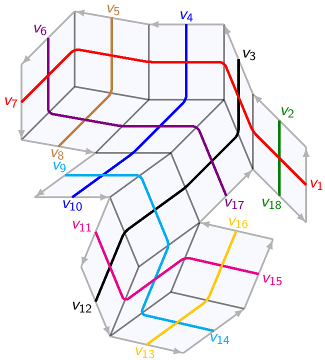


$v_r = \exp(2i\theta_r)$



$\tau(1) = 7$, etc.

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- If $\theta_r = r\pi/n$ for all $1 \leqslant r \leqslant n$, we get $\mathrm{Meas}_{f_{k,n}}(\boldsymbol{\theta}) = X_0^{(k,n)}$.
- This yields the above formulas for regular polygons in the Ising and electrical cases.

Thanks!