# Ising model, total positivity, and criticality

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Based on joint works with Pavlo Pylyavskyy, Steven Karp, and Thomas Lam.

# Total positivity vs topology

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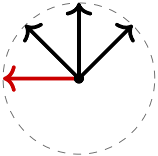

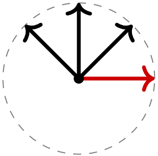

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# Ising model

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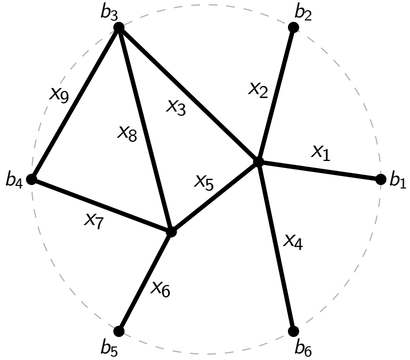

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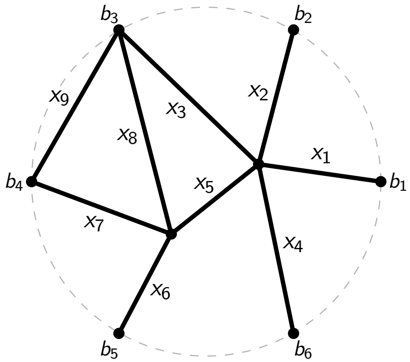

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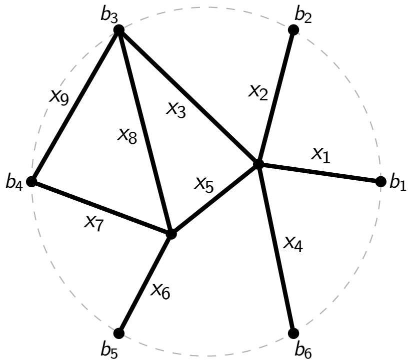

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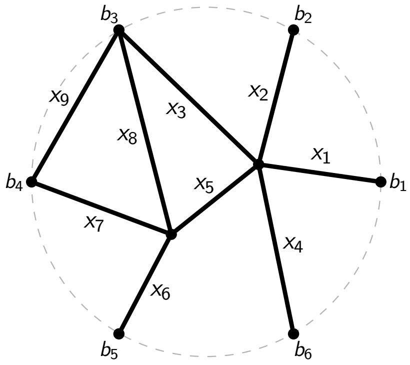

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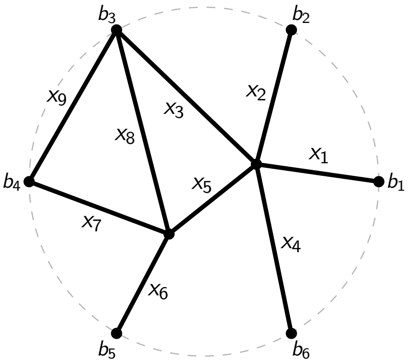

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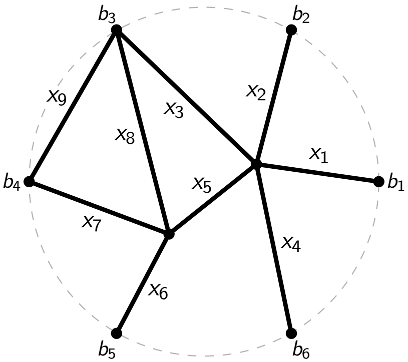

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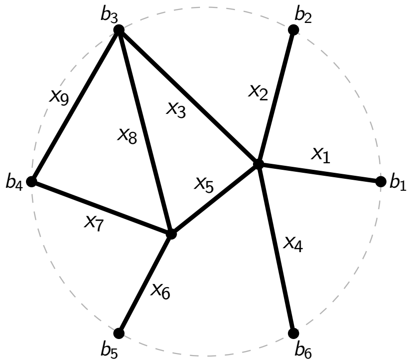

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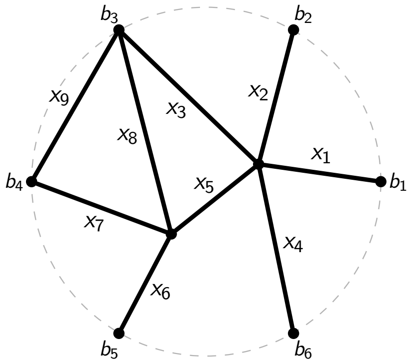

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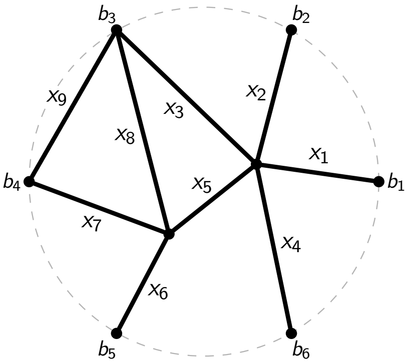
**Q1:** How to describe correlations by inequalities?

**Q2:** How to reconstruct the edge weights from boundary correlations?



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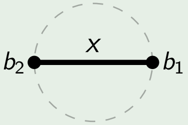

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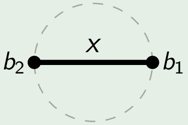


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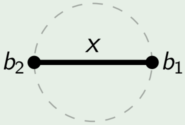


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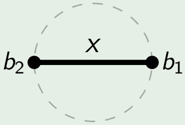


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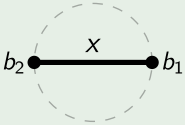


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## Definition (Postnikov (2006))

The totally nonnegative Grassmannian is

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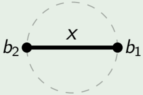

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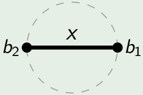


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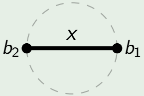


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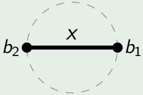


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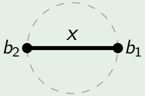


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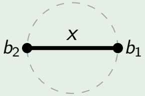


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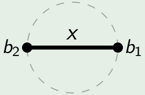


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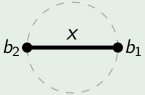


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- *Kramers–Wannier's duality (1941)* $\to$ *cyclic shift.*

# Critical Ising model

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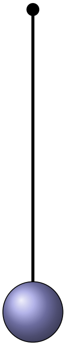

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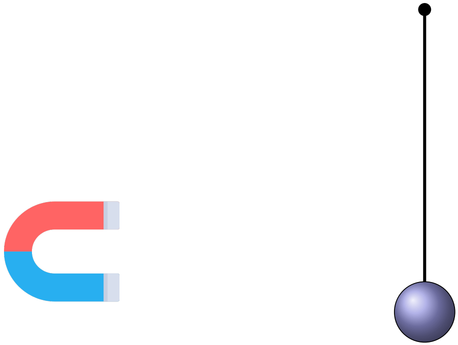

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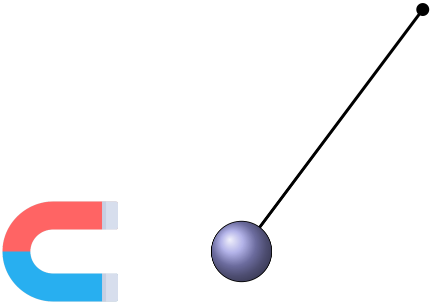

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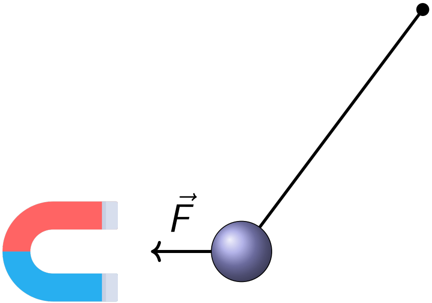

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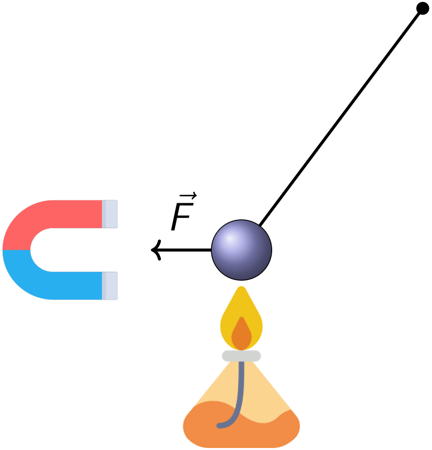

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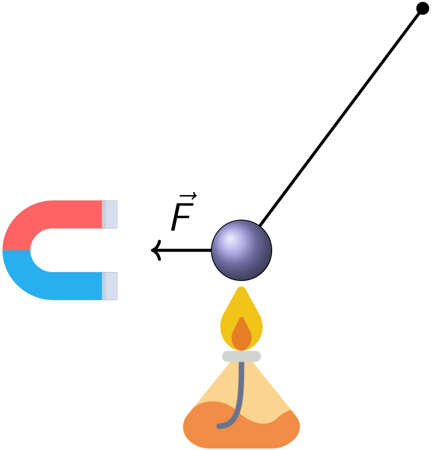

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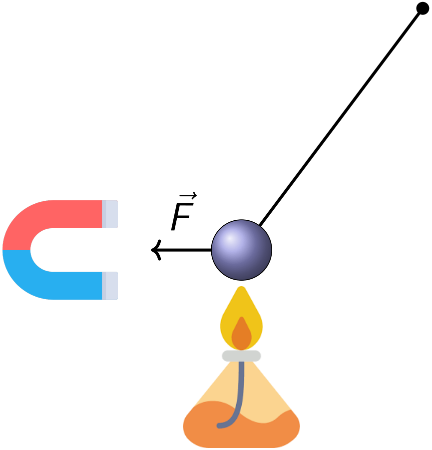


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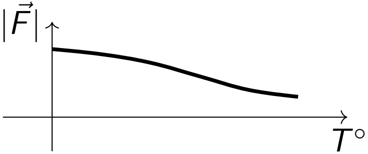

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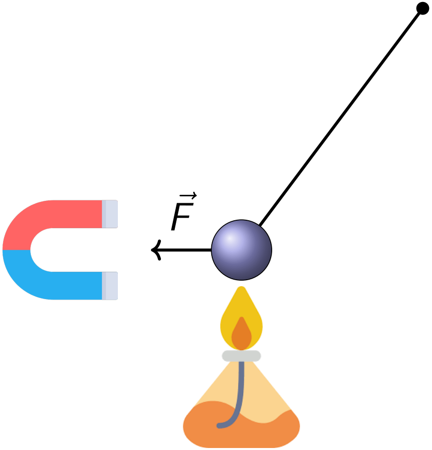


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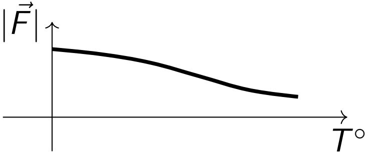


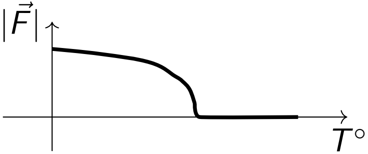

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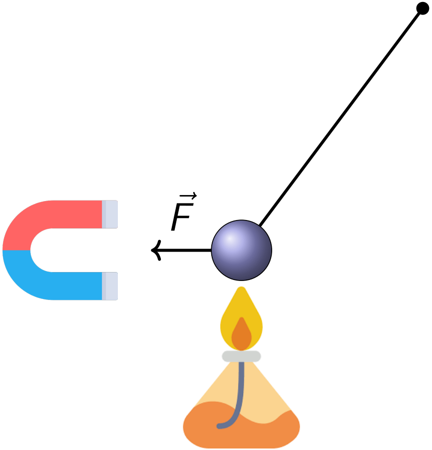


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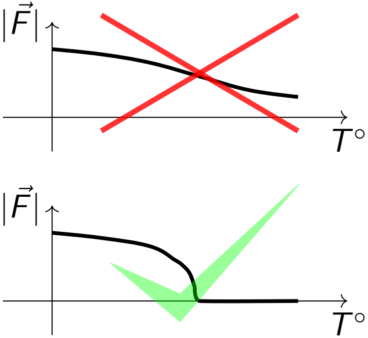

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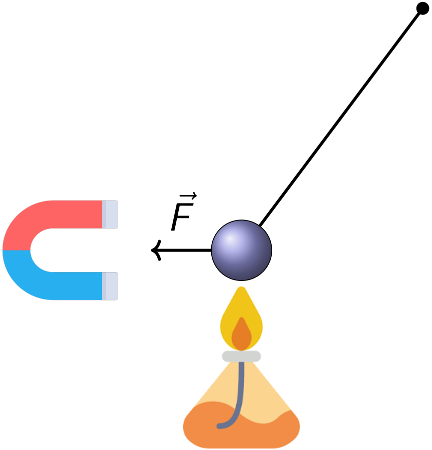


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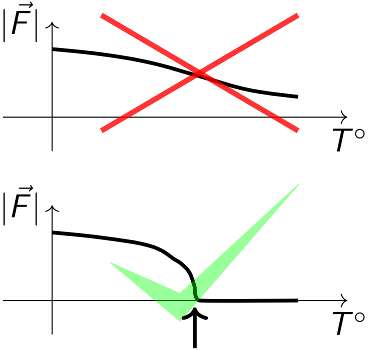


*Curie point* (P. Curie, 1895)

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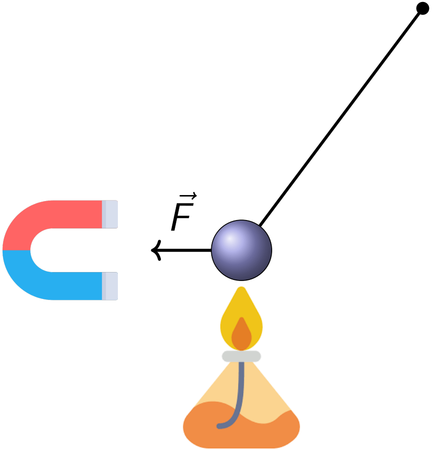


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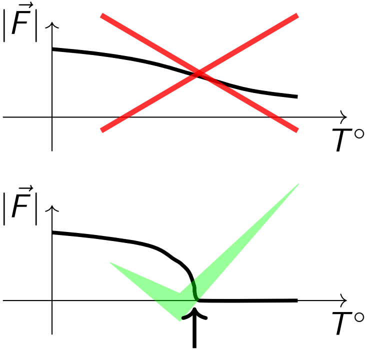


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# Phase transition

$$\text{Prob}(\sigma) := \frac{1}{Z} \prod_{\substack{\{u,v\} \in E(G): \\ \sigma_u \neq \sigma_v}} x_{\{u,v\}}.$$

Usually:

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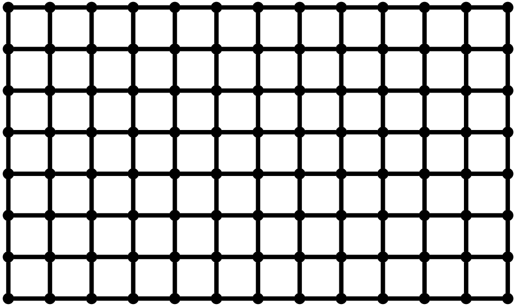

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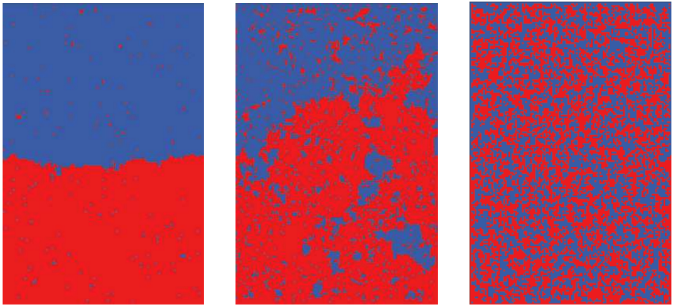

$x < x_{\text{crit}}$ $\quad$ $x = x_{\text{crit}}$ $\quad$ $x > x_{\text{crit}}$

Picture credit: Dmitry Chelkak

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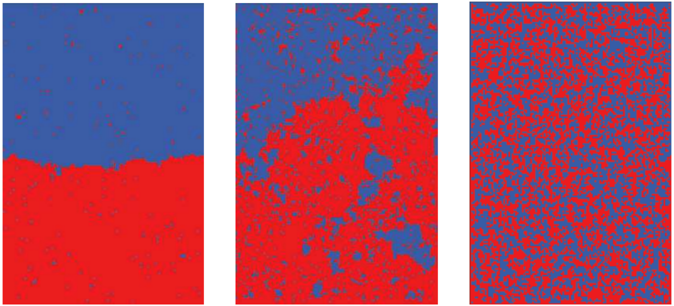

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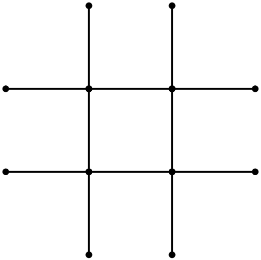
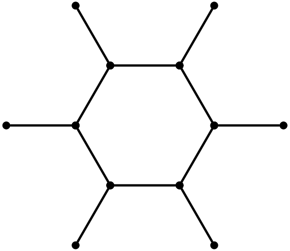
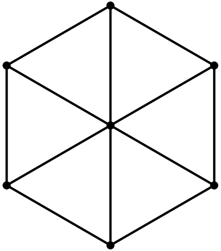


Square lattice
$x_{\text{crit}} = \sqrt{2} - 1$



Hexagonal lattice
$x_{\text{crit}} = 2 - \sqrt{3}$



Triangular lattice
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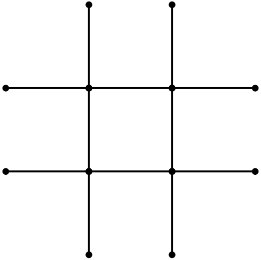

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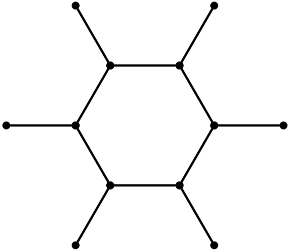

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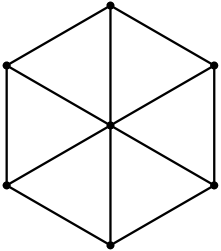

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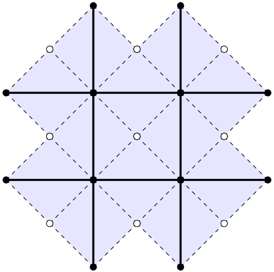

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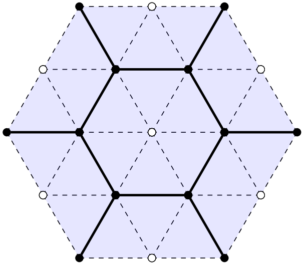

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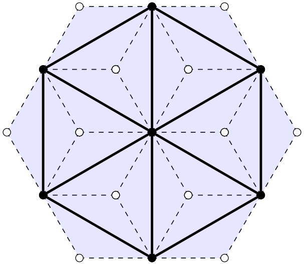

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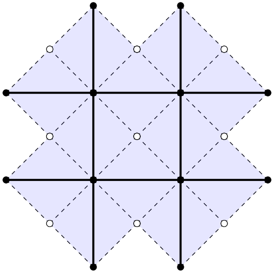

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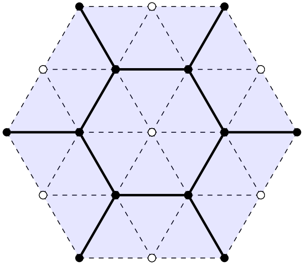

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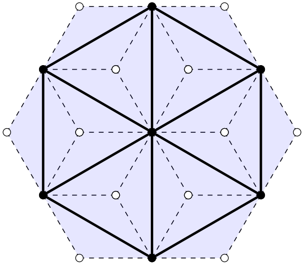

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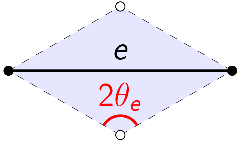


$x_e = \tan(\theta_e/2)$

# Critical $Z$-invariant Ising model

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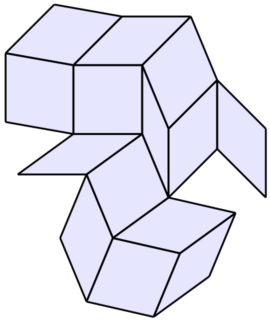

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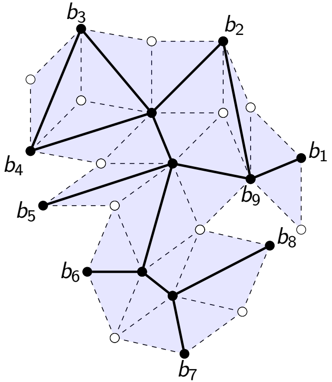

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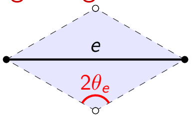
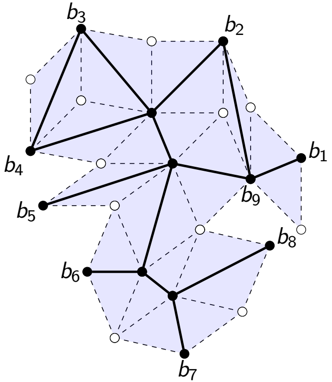



$$\longrightarrow \quad x_e = \tan(\theta_e/2)$$



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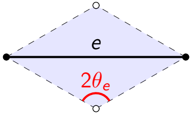


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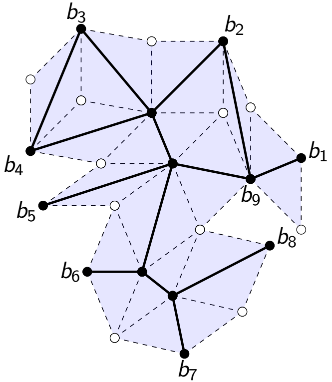


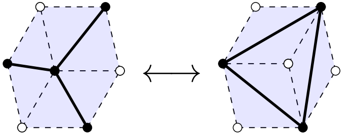

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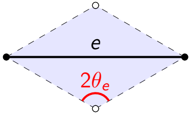


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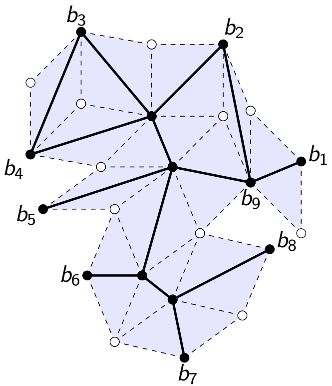


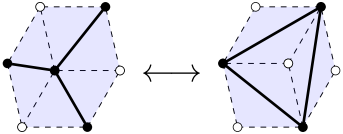

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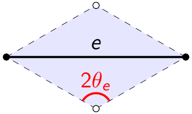


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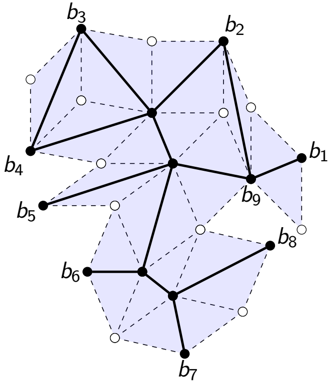


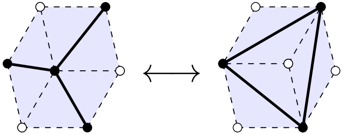

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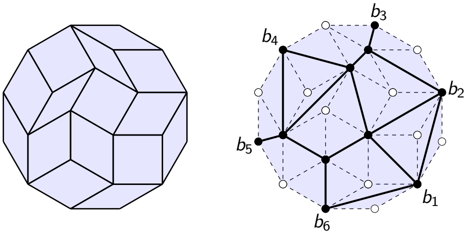

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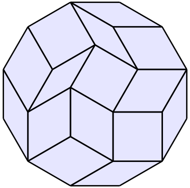

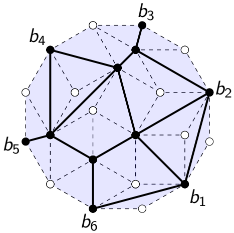


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[Gal20] Pavel Galashin. A formula for boundary correlations of the critical Ising model. arXiv:2010.13345, 2020.

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- A: Yes, by the Leibniz formula for $\pi$:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

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[Hon10] Clement Hongler. *Conformal invariance of Ising model correlations.* PhD thesis, 06/28 2010.

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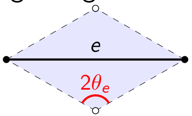


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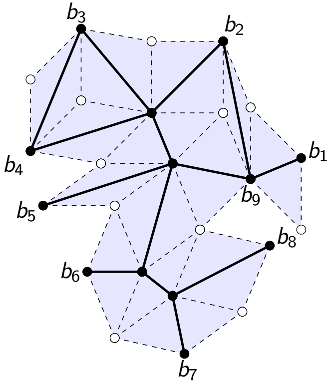


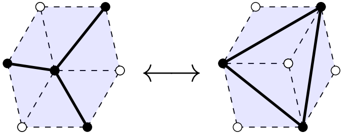

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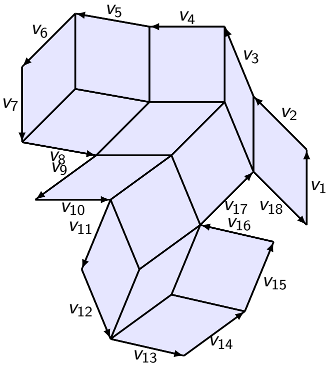


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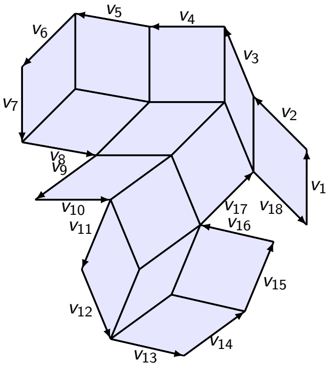


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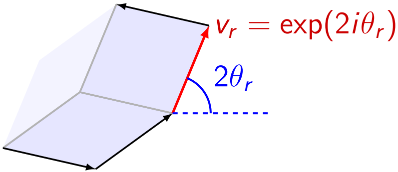

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Let $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_{2n})$ be given by $v_r = \exp(2i\theta_r)$.
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For non-convex $R$, the $r$-th coordinate is $\pm \prod_{j \in J_r} \sin(t - \theta_j)$, where $|J_r| = n - 1$.

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# Thank you!