Ising model, total positivity, and criticality

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Based on joint works with Pavlo Pylyavskyy, Steven Karp, and Thomas Lam.

Total positivity

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Theorem (Smale (1960), Freedman (1982), Perelman (2003))

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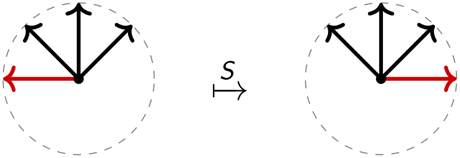

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Ising model

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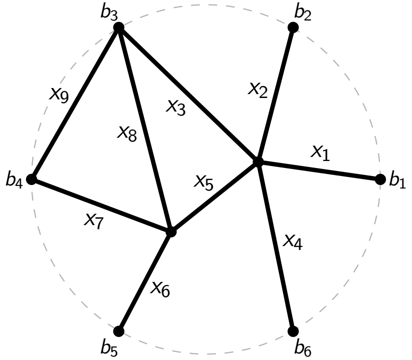

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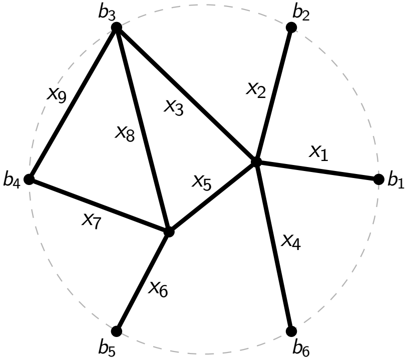

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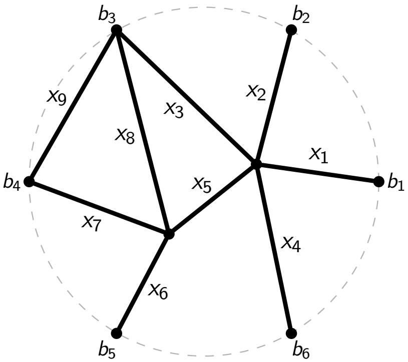

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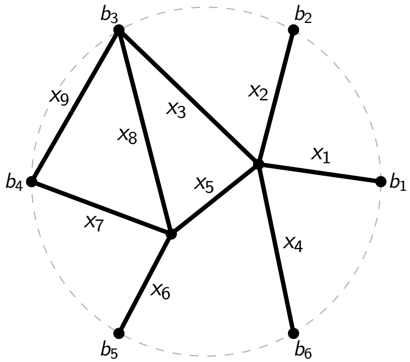

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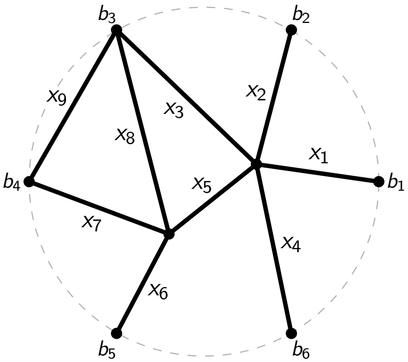

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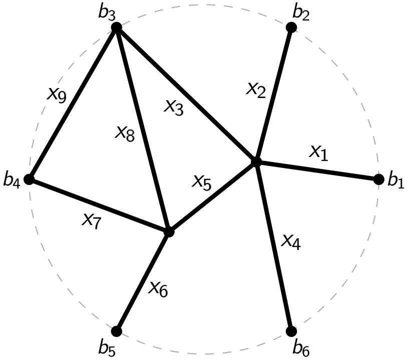

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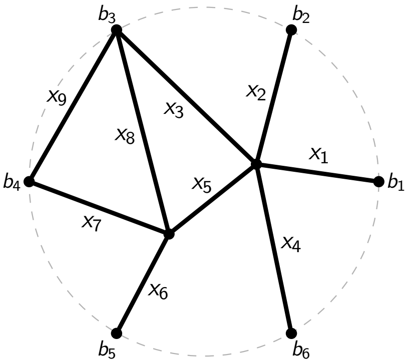

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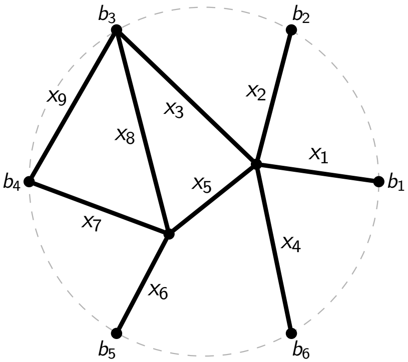

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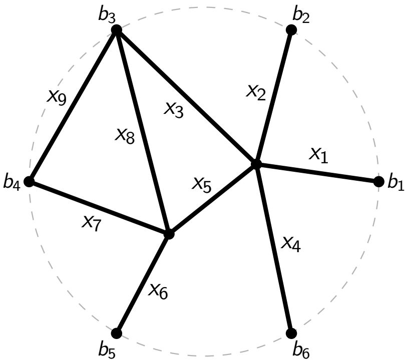
Q1: How to describe correlations by inequalities?

Q2: How to reconstruct the edge weights from boundary correlations?



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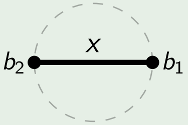

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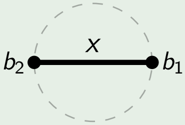

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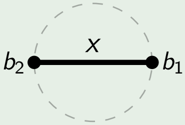

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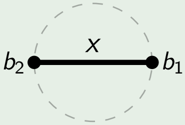

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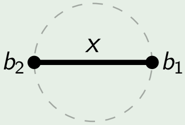

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Definition (Postnikov (2006))

The totally nonnegative Grassmannian is

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Definition (Huang–Wen (2013))

The totally nonnegative orthogonal Grassmannian:

$$\mathrm{OG}_{\geqslant 0}(n,2n) := \{W \in \mathrm{Gr}(n,2n) \mid \Delta_I(W) = \Delta_{[2n]\setminus I}(W) \geqslant 0 \text{ for all } I\}.$$

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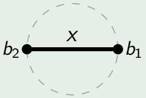

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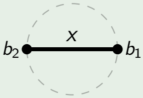

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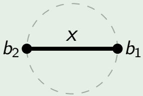

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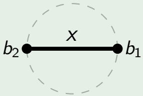

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$$\begin{matrix} \Delta_{12} = 2m & \Delta_{13} = 1 + m^2 & \Delta_{14} = 1 - m^2 \\ \Delta_{34} = 2m & \Delta_{24} = 1 + m^2 & \Delta_{23} = 1 - m^2 \end{matrix}$$

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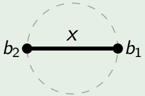

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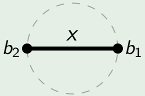

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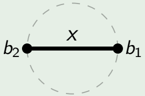

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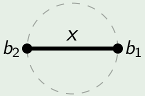

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- *Kramers–Wannier's duality (1941)* $\rightarrow$ *cyclic shift.*

Critical Ising model

Ising model: origin

- Suggested by by W. Lenz to his student E. Ising in 1920.

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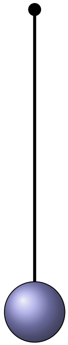

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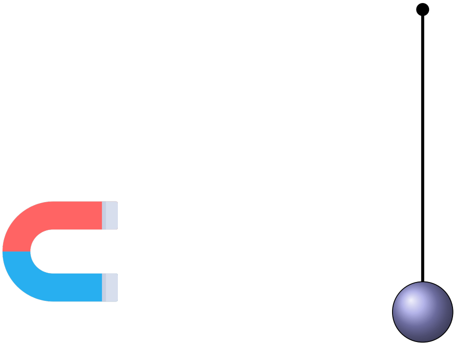

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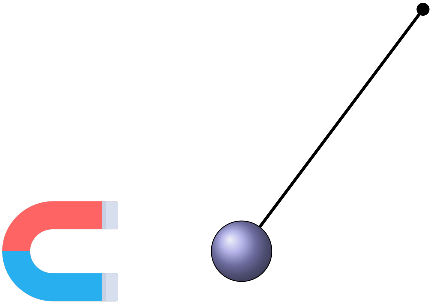

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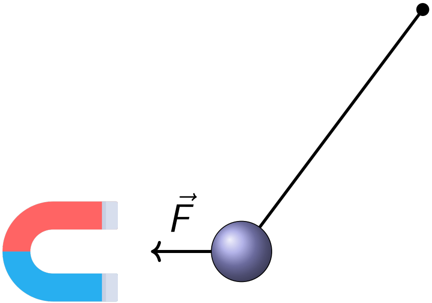

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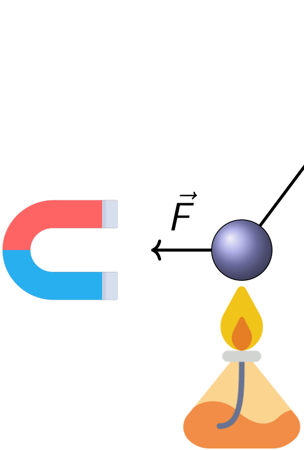

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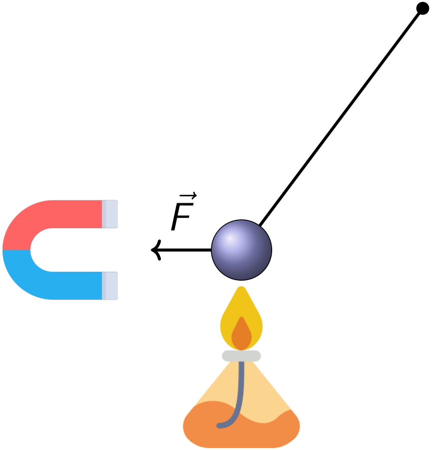

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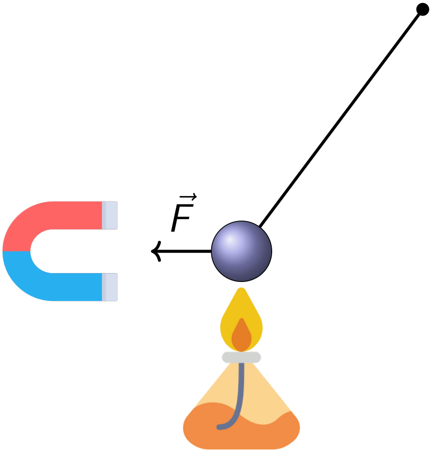

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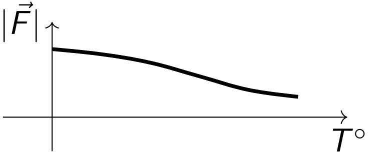

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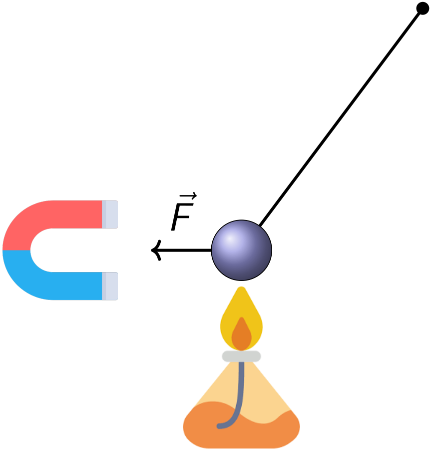

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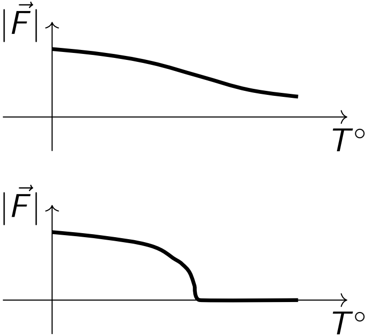

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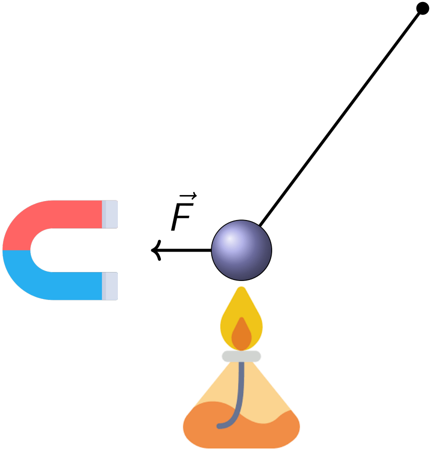

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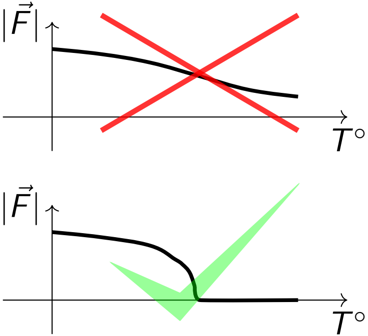

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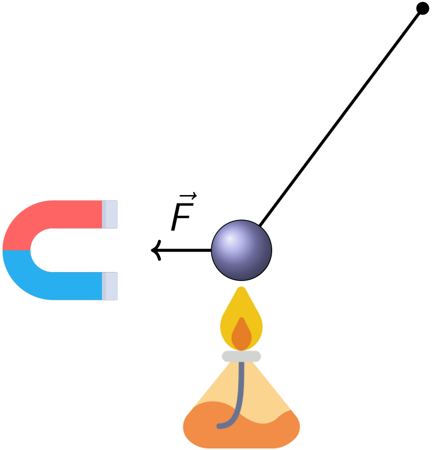

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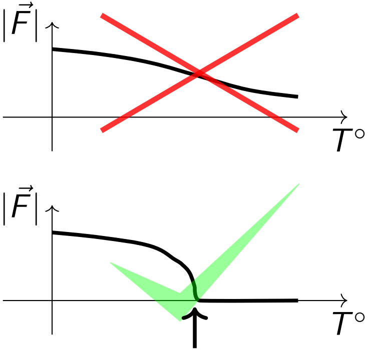

Curie point (P. Curie, 1895)

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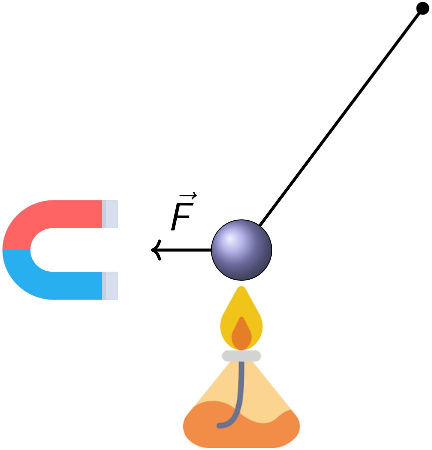

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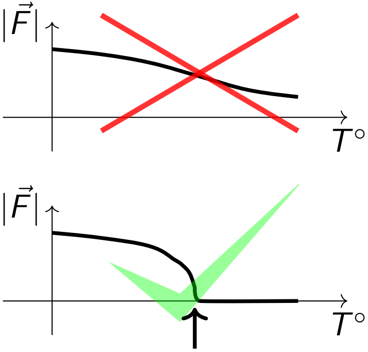

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Phase transition

$$\mathrm{Prob}(\sigma) := \frac{1}{Z} \prod_{\substack{\{u,v\}\in E(G):\\ \sigma_u \neq \sigma_v}} x_{\{u,v\}}.$$

Usually:

- G = large piece of a (e.g. square) lattice;
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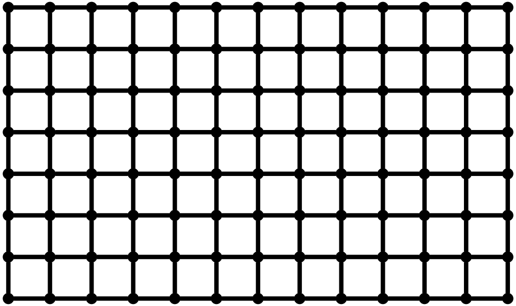

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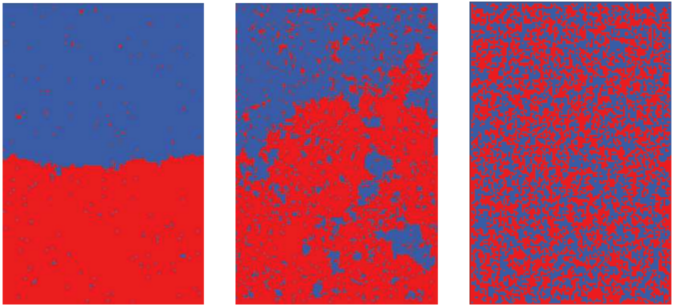

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Picture credit: Dmitry Chelkak

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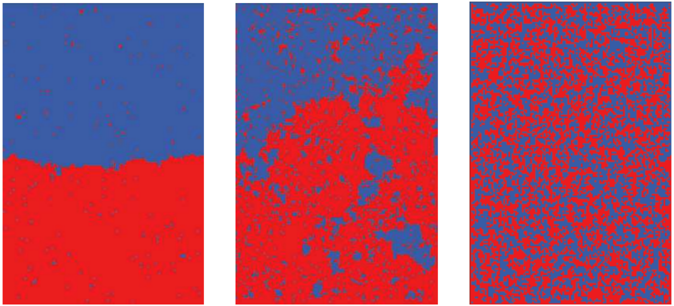
- Kramers–Wannier (1941): Square lattice: $x_{\mathrm{crit}} = \sqrt{2} - 1$.



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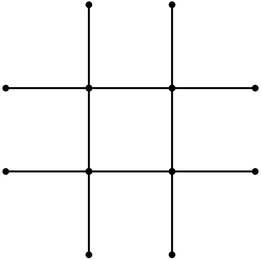

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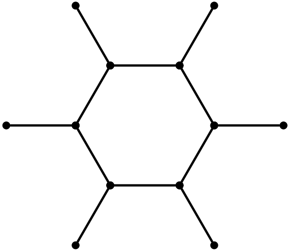
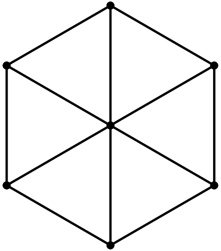


Hexagonal lattice

$x_{\text{crit}} = 2 - \sqrt{3}$



Triangular lattice

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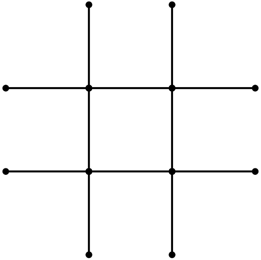

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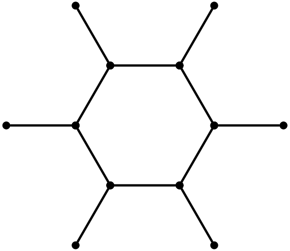

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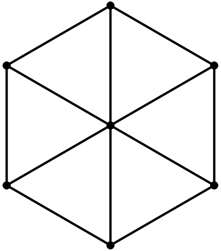

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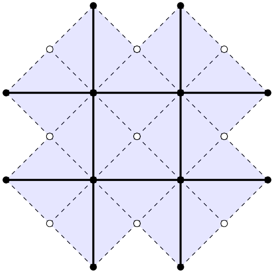

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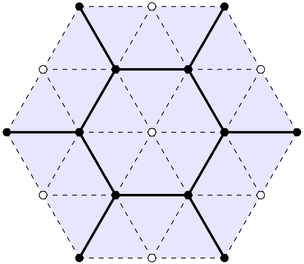

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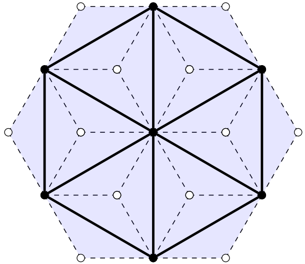

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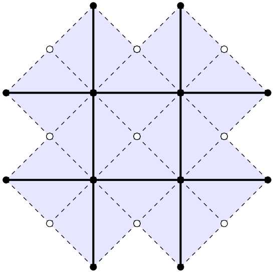

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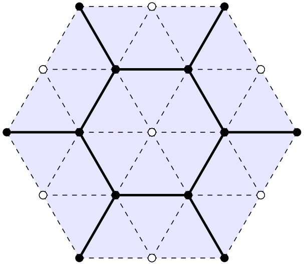

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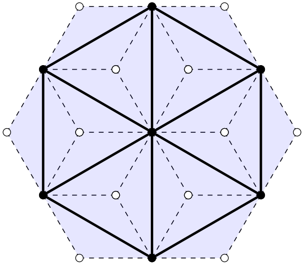

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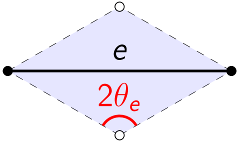

$x_e = \tan(\theta_e/2)$

Critical Z-invariant Ising model

[Bax86] R. J. Baxter. Free-fermion, checkerboard and **Z**-invariant lattice models in statistical mechanics. *Proc. Roy. Soc. London Ser. A*, 404(1826):1–33, 1986.

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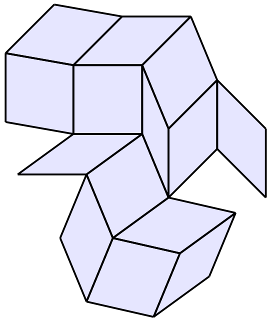

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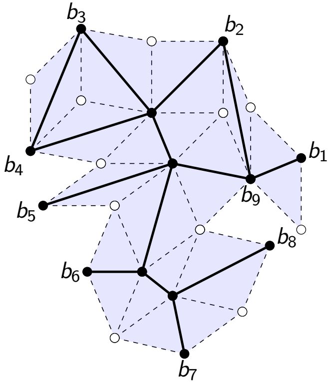

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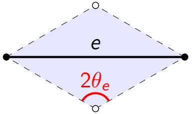
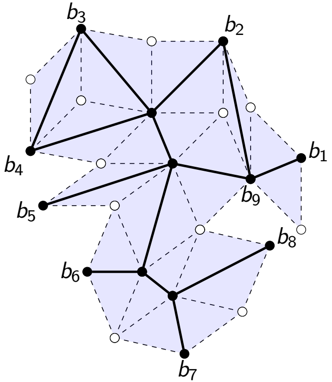


$$\longrightarrow \quad x_e = \tan(\theta_e/2)$$



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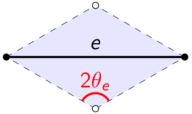

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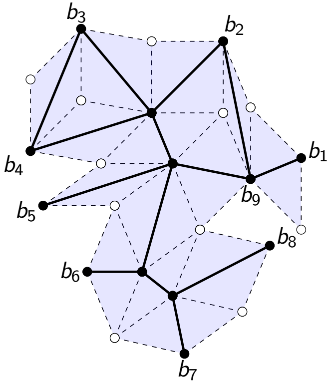

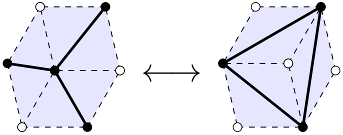

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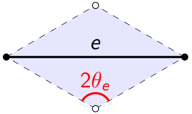

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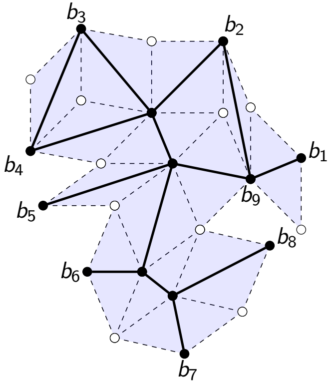
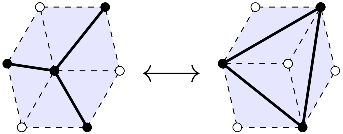
- Conclusion: $M_R := (\langle \sigma_i \sigma_j \rangle_R)_{i,j=1}^n$ depends only on the polygonal region R.





Critical Z-invariant Ising model

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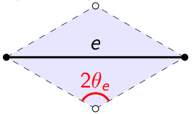

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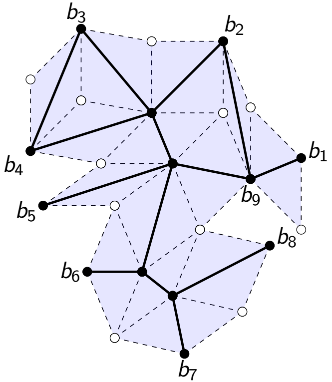
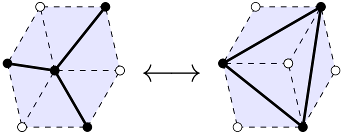
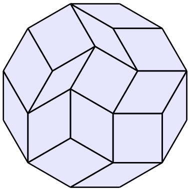
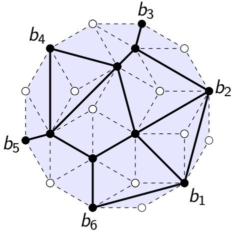




A formula for regular polygons

Let R_n be a regular $2n$-gon and $\langle \sigma_i \sigma_j \rangle_{R_n}$ be the corresponding boundary correlations.





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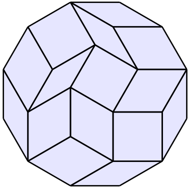

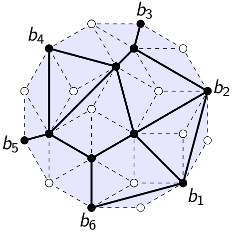

Theorem (G. (2020))

For $1 \leqslant i, j \leqslant n$ and $d := |i - j|$, we have

$$\langle \sigma_i \sigma_j \rangle_{R_n} = \frac{2}{n} \left(\frac{1}{\sin\left((2d-1)\pi/2n\right)} - \frac{1}{\sin\left((2d-3)\pi/2n\right)} + \cdots \pm \frac{1}{\sin\left(\pi/2n\right)} \right) \mp 1.$$

[Gal20] Pavel Galashin. A formula for boundary correlations of the critical Ising model. arXiv:2010.13345, 2020.

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- A: Yes, by the Leibniz formula for π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

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[Hon10] Clement Hongler. *Conformal invariance of Ising model correlations.* PhD thesis, 06/28 2010.

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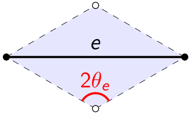

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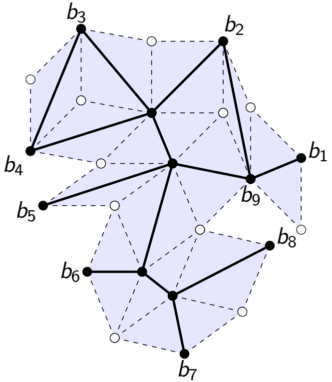
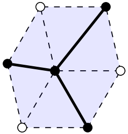
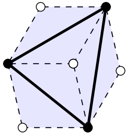




$\longleftrightarrow$



Shape of a polygonal region R

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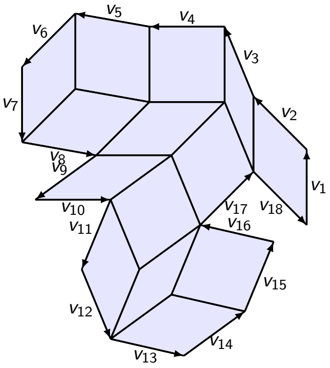

R

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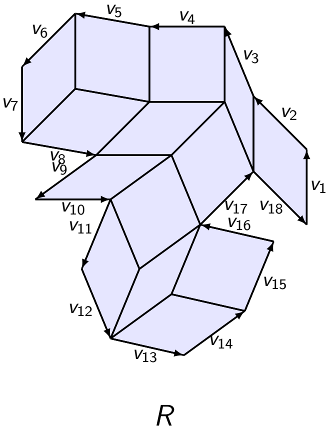

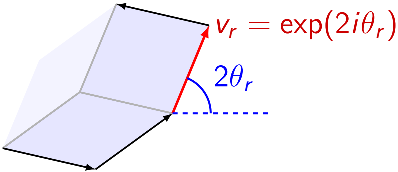

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- The inverse of ϕ is easily computed using linear algebra.

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Let $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_{2n})$ be given by $v_r = \exp(2i\theta_r)$.
Given this data, we define a certain curve $\gamma_{\boldsymbol{\theta}} : \mathbb{R} \to \mathbb{R}^{2n}$.

For non-convex R, the r-th coordinate is $\pm \prod_{j \in J_r} \sin(t - \theta_j)$, where $|J_r| = n - 1$.

Theorem (G. (2021))

$$\phi(M_R) = \mathrm{Span}(\boldsymbol{\gamma_\theta}) \quad \textit{inside } \mathrm{OG}_{\geqslant 0}(n, 2n).$$

Here $\phi : \overline{\mathcal{X}}_n \xrightarrow{\sim} \mathrm{OG}_{\geqslant 0}(n, 2n)$ is the doubling map, $M_R := (\langle \sigma_i \sigma_j \rangle_R)_{i,j=1}^n$ is the critical boundary correlation matrix, and $\mathrm{Span}(\boldsymbol{\gamma_\theta})$ is the linear span of all points on $\boldsymbol{\gamma_\theta}$.

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Thank you!