Zamolodchikov periodicity and integrability

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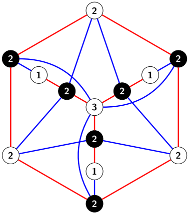
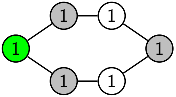
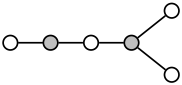
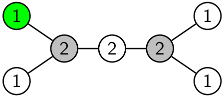
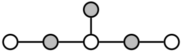
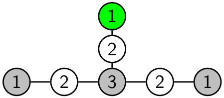
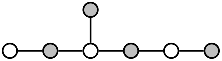
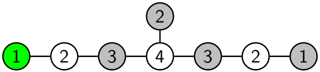
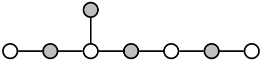
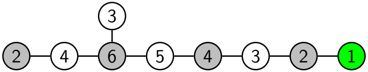
Joint work with Pavlo Pylyavskyy



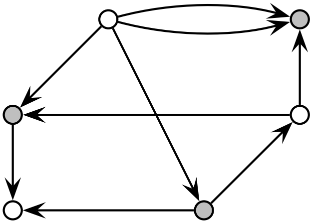
Part 0: Dynkin diagrams

ADE Dynkin diagrams

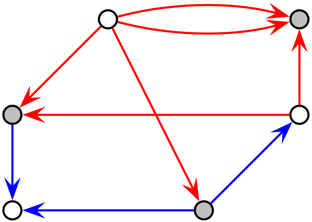
Name	Finite diagram	h	Affine diagram	Name
A_n		$n+1$		$\hat{A}_{n-1}$
D_n		$2n-2$		$\hat{D}_{n-1}$
E_6		12		$\hat{E}_6$
E_7		18		$\hat{E}_7$
E_8		30		$\hat{E}_8$

Part 1: T-systems

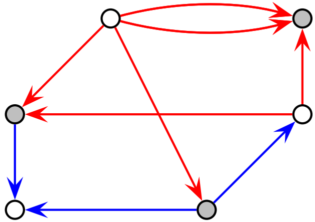
Bipartite quivers

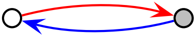


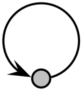
Bipartite quivers



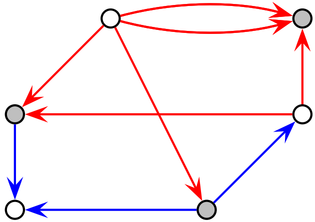
Bipartite quivers

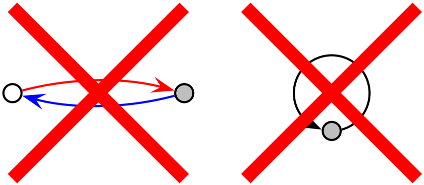




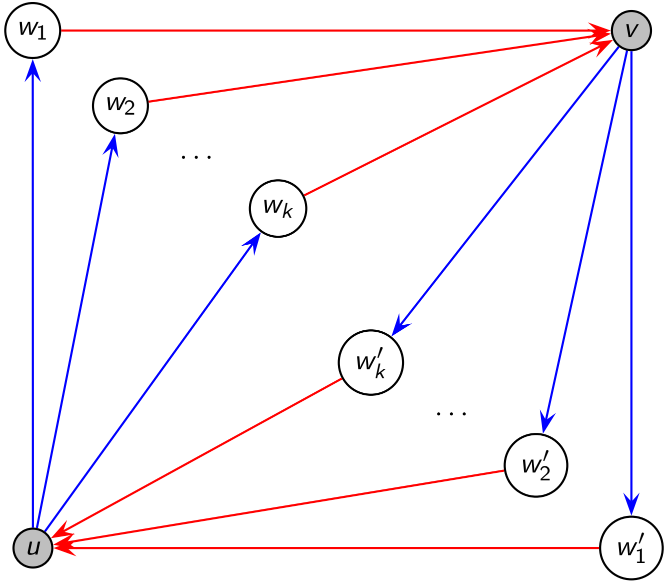


Bipartite quivers

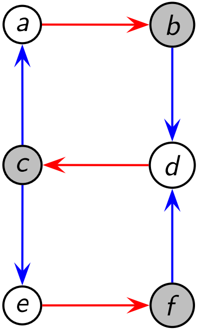




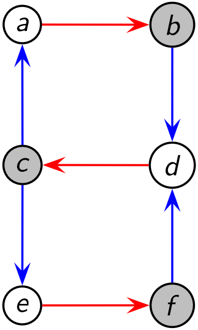
Bipartite **recurrent** quivers



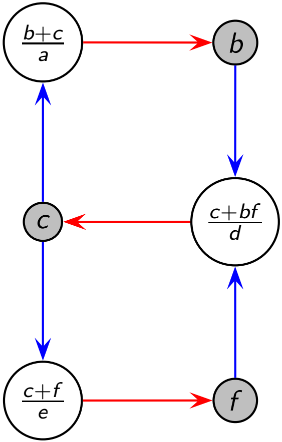
T-system



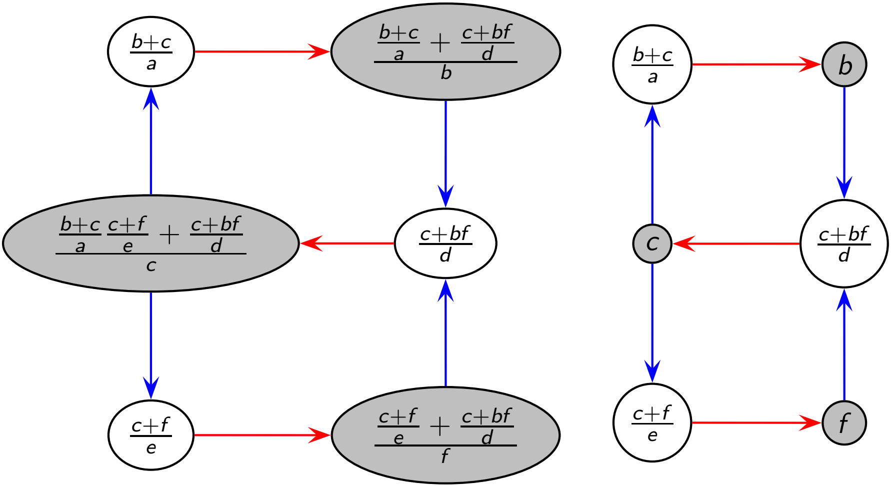
T-system



$\longrightarrow$

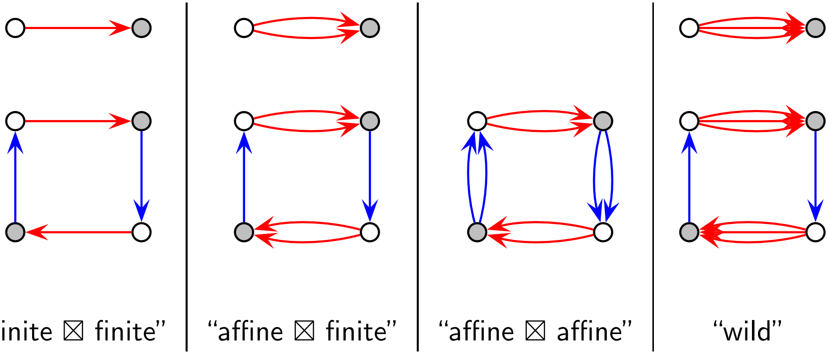


T-system



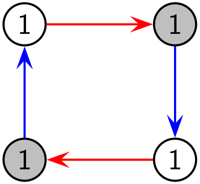
$\frac{b+c}{a}$
$\frac{\frac{b+c}{a}+\frac{c+bf}{d}}{b}$
$\frac{\frac{b+c}{a}\frac{c+f}{e}+\frac{c+bf}{d}}{c}$
$\frac{c+bf}{d}$
$\frac{c+f}{e}$
$\frac{\frac{c+f}{e}+\frac{c+bf}{d}}{f}$
$\frac{b+c}{a}$
b
c
$\frac{c+bf}{d}$
$\frac{c+f}{e}$
f

Four classes of quivers

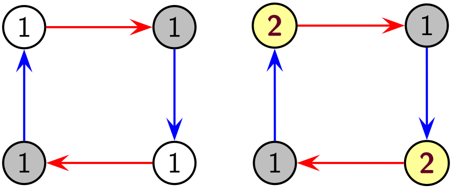


"finite ⊠ finite" | "affine ⊠ finite" | "affine ⊠ affine" | "wild"

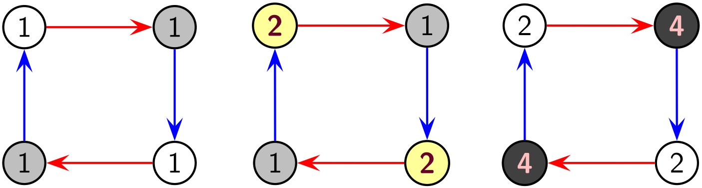
Example: finite $\boxtimes$ finite



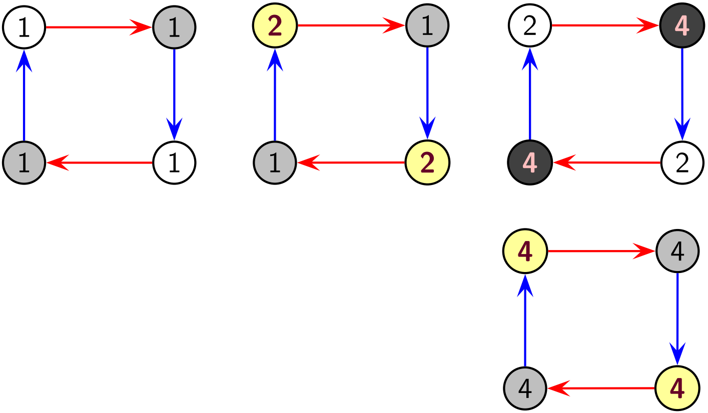
Example: finite $\boxtimes$ finite



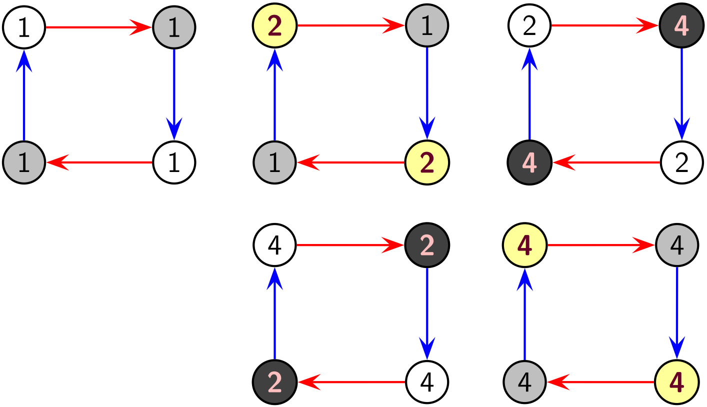
Example: finite ⊠ finite



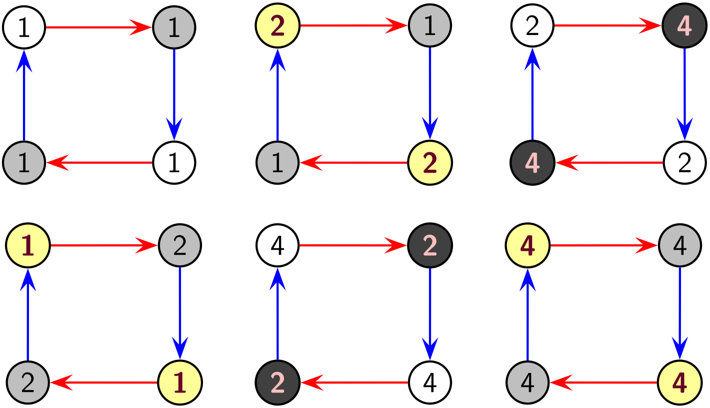
Example: finite ⊠ finite



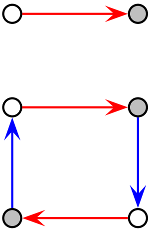
Example: finite ⊠ finite



Example: finite ⊠ finite

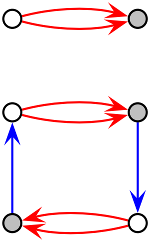


Four classes of quivers

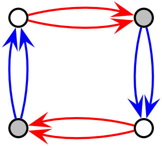


"finite ⊠ finite"

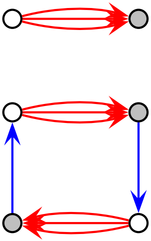
periodic



"affine ⊠ finite"



"affine ⊠ affine"



"wild"

Example: affine ⊠ finite

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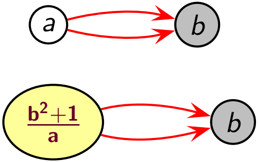

Example: affine ⊠ finite

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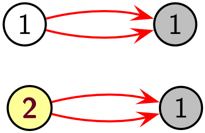
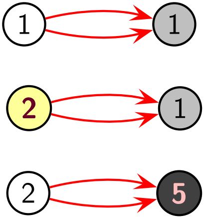
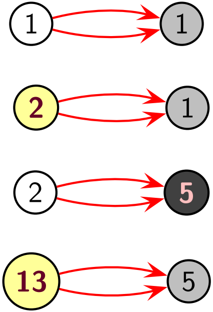


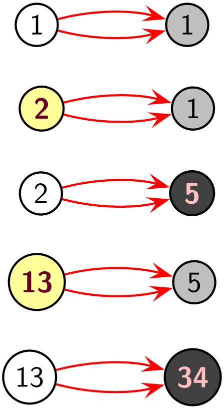
Example: affine ⊠ finite



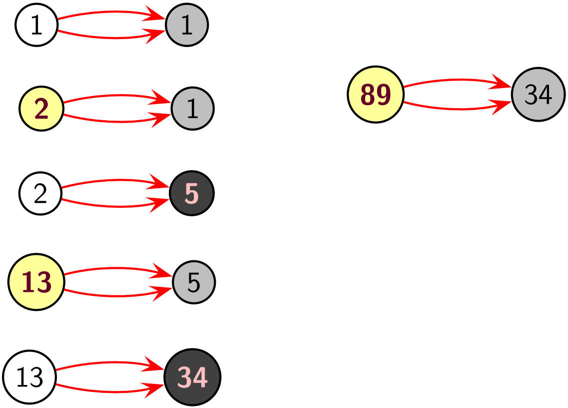
Example: affine ⊠ finite



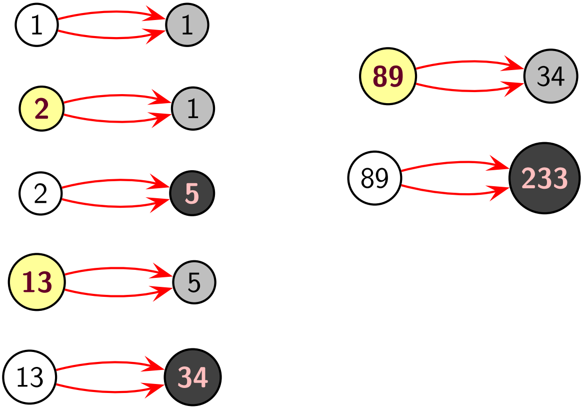
Example: affine ⊠ finite



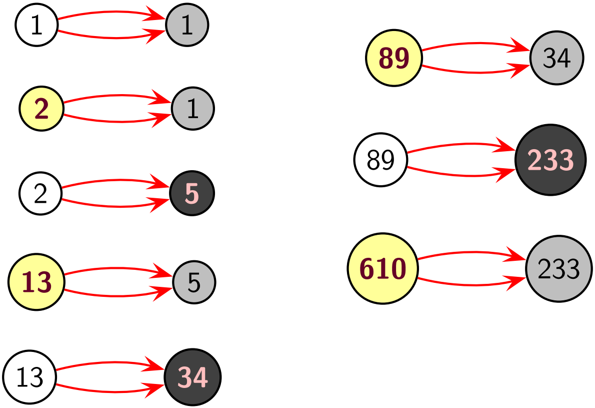
Example: affine ⊠ finite



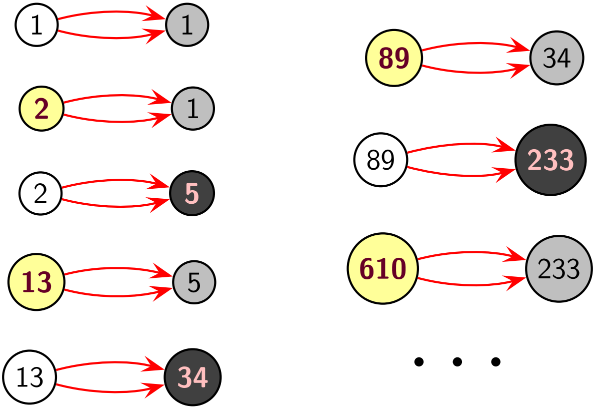
Example: affine ⊠ finite



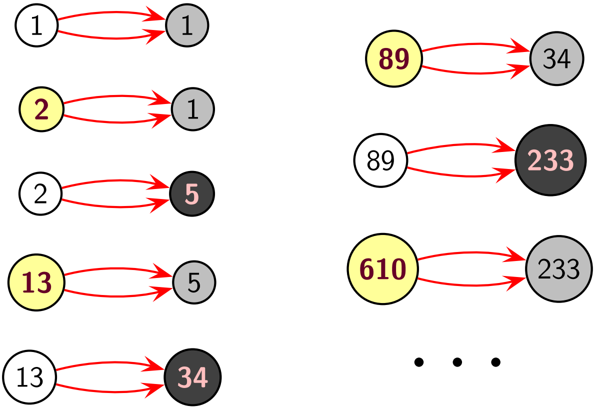
Example: affine ⊠ finite



Example: affine ⊠ finite

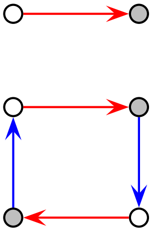


Example: affine ⊠ finite



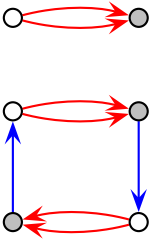
$$x_{n+1} - 3x_n + x_{n-1} = 0$$

Four classes of quivers



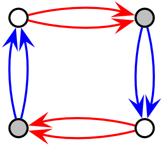
“finite ⊠ finite”

periodic

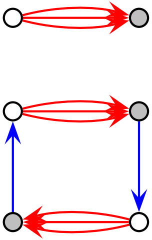


“affine ⊠ finite”

linearizable

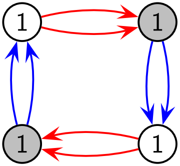


“affine ⊠ affine”

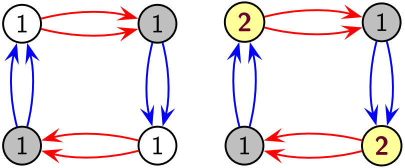


“wild”

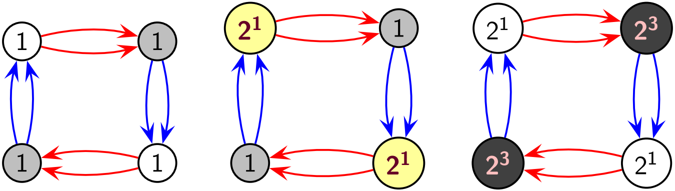
Example: affine $\boxtimes$ affine



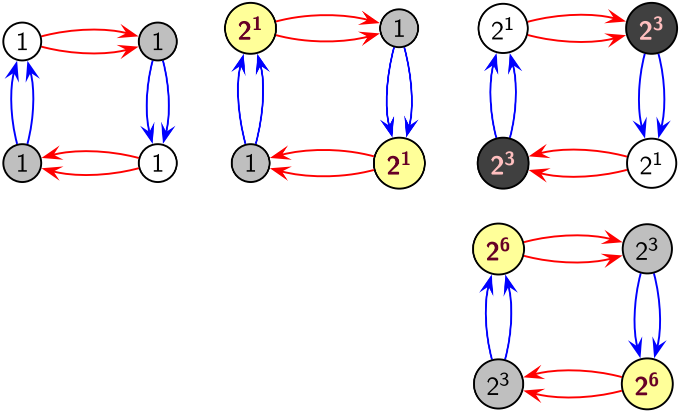
Example: affine ⊠ affine



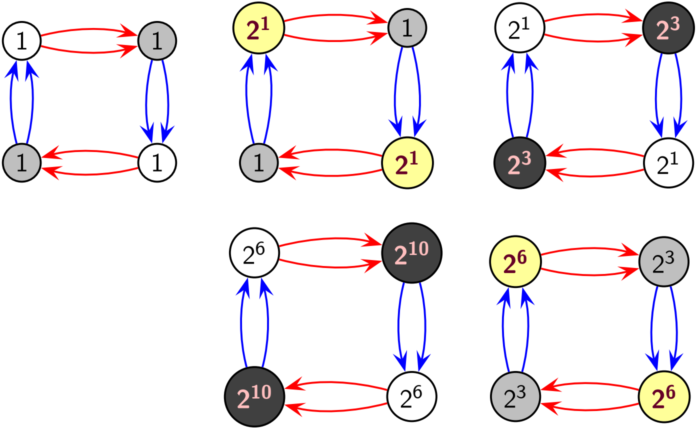
Example: affine ⊠ affine



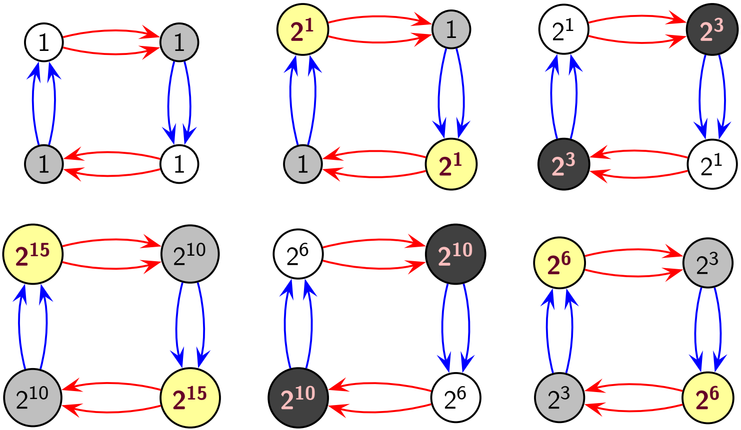
Example: affine ⊠ affine



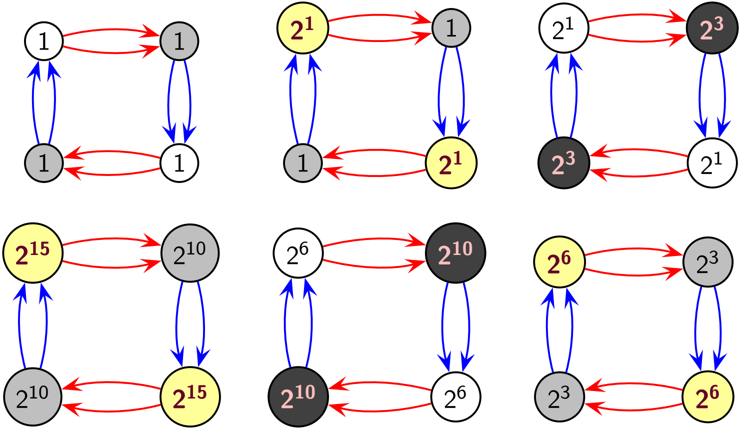
Example: affine ⊠ affine



Example: affine ⊠ affine

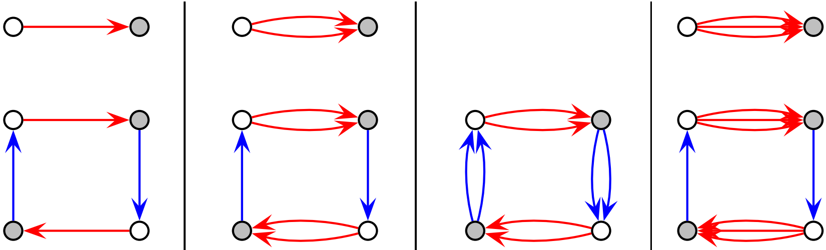


Example: affine ⊠ affine



$$2^{\binom{n}{2}}$$

Four classes of quivers



“finite ⊠ finite” — periodic

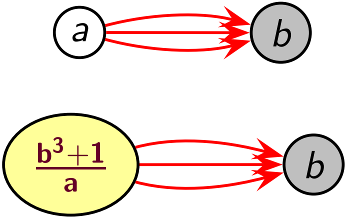
“affine ⊠ finite” — linearizable

“affine ⊠ affine” — **grows as** $\exp(t^2)$

“wild”

Example: wild

Example: wild



Example: wild

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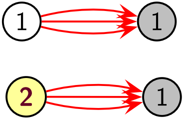
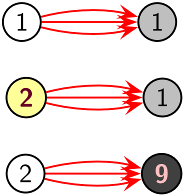
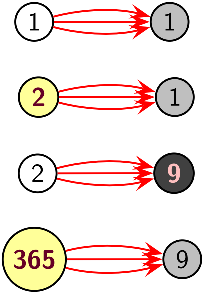


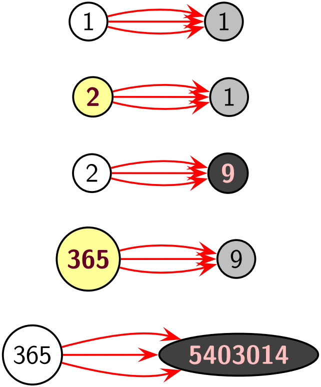
Example: wild



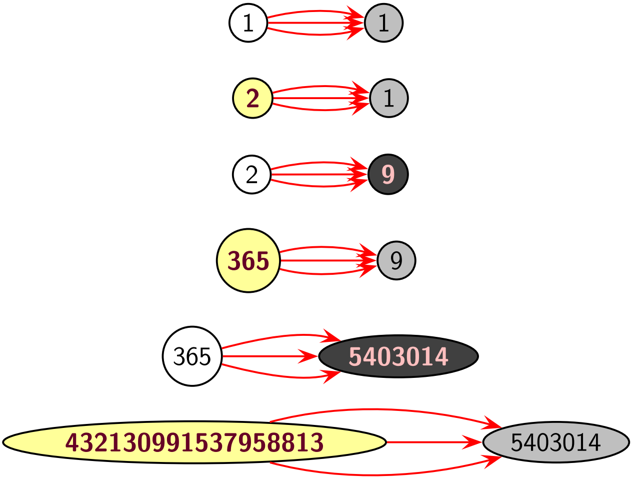
Example: wild



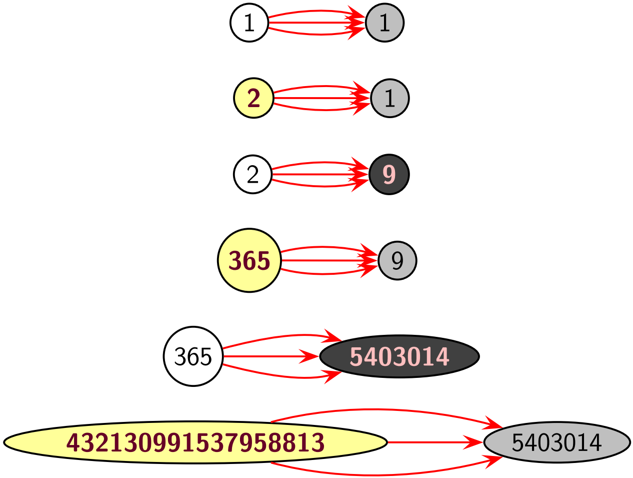
Example: wild



Example: wild

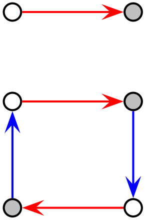


Example: wild



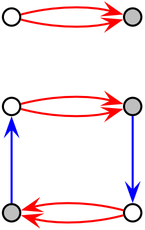
the next number is 14935169284101525874491673463268414536523593057

Four classes of quivers



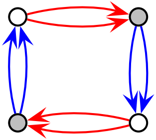
"finite ⊠ finite"

periodic



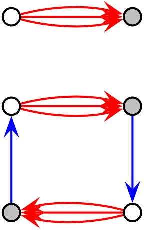
"affine ⊠ finite"

linearizable



"affine ⊠ affine"

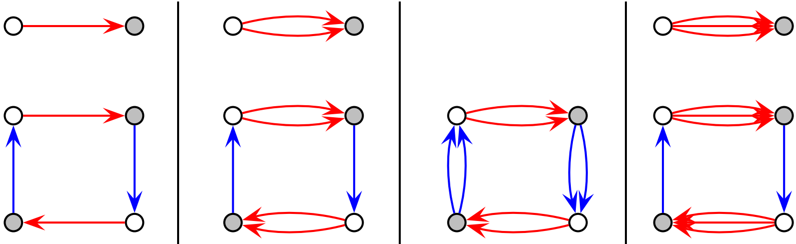
grows as $\exp(t^2)$



"wild"

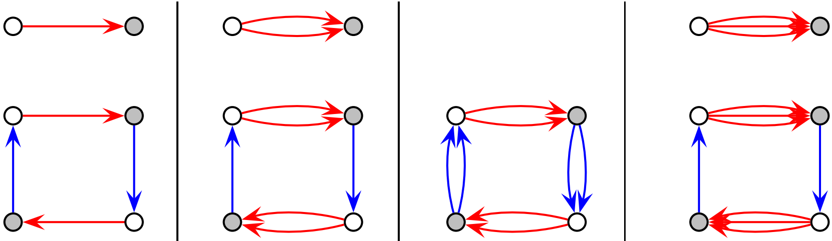
grows as $\exp(\exp(t))$

Four classes of quivers



“finite ⊠ finite”	“affine ⊠ finite”	“affine ⊠ affine”	“wild”
periodic	linearizable	???	???
grows as $\exp(1)$	**grows as** $\exp(t)$	**grows as** $\exp(t^2)$	**grows as** $\exp(\exp(t))$

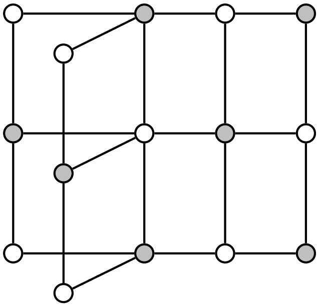
Four classes of quivers



"finite ⊠ finite"	"affine ⊠ finite"	"affine ⊠ affine"	"wild"
periodic	linearizable	**"integrable"?**	**"non-integrable"?**
grows as $\exp(1)$	grows as $\exp(t)$	grows as $\exp(t^2)$	grows as $\exp(\exp(t))$

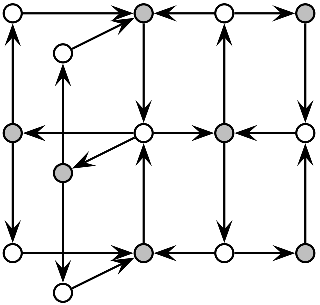
Part 2: Zamolodchikov periodicity

Tensor product



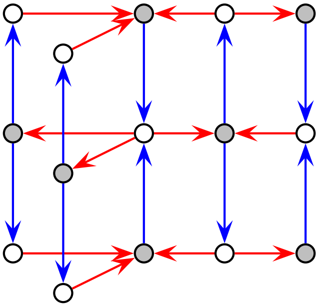
$D_5 \otimes A_3$

Tensor product



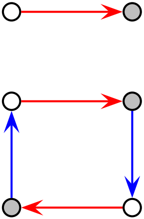
$D_5 \otimes A_3$

Tensor product



$D_5 \otimes A_3$

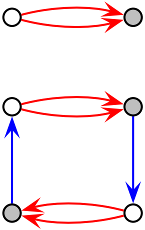
Four classes of quivers



“finite ⊠ finite”

periodic

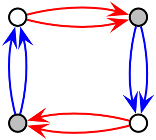
grows as $\exp(1)$



“affine ⊠ finite”

linearizable

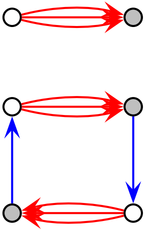
grows as $\exp(t)$



“affine ⊠ affine”

“integrable”?

grows as $\exp(t^2)$



“wild”

“non-integrable”?

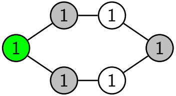
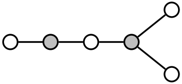
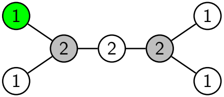
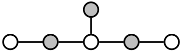
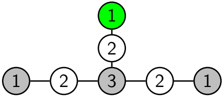
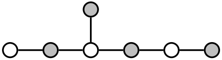
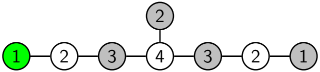
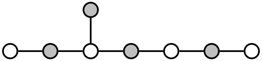
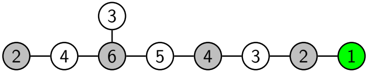
grows as $\exp(\exp(t))$

Zamolodchikov periodicity

Theorem (B. Keller, 2013)

Tensor product of **finite** *Dynkin diagrams* $\Longrightarrow$ *the* T*-system is periodic.*

ADE Dynkin diagrams

Name	Finite diagram	h	Affine diagram	Name
A_n		$n+1$		$\hat{A}_{n-1}$
D_n		$2n-2$		$\hat{D}_{n-1}$
E_6		12		$\hat{E}_6$
E_7		18		$\hat{E}_7$
E_8		30		$\hat{E}_8$

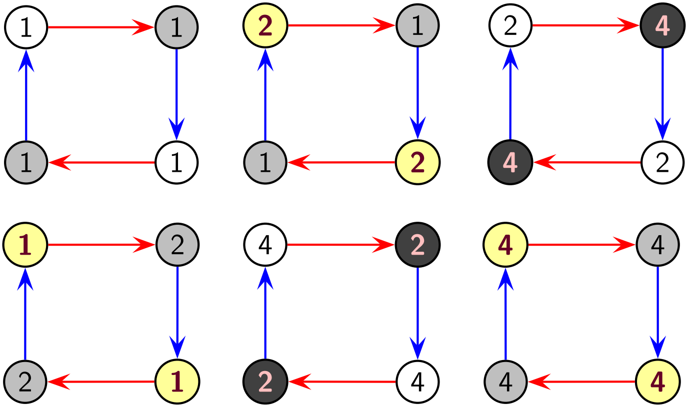
Zamolodchikov periodicity

Theorem (B. Keller, 2013)

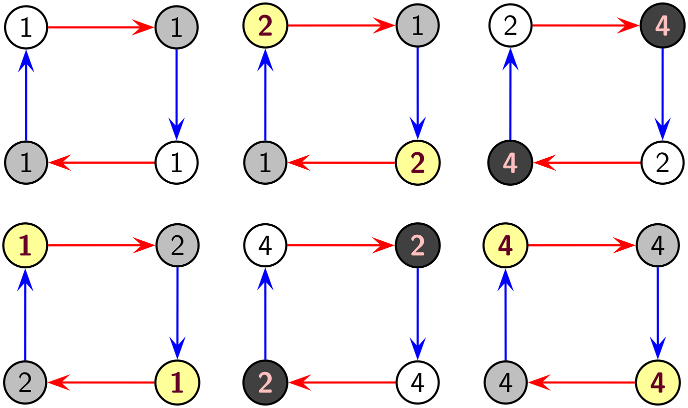
Tensor product of **finite** *Dynkin diagrams* $\Longrightarrow$ *the T-system is periodic* **with period dividing**

$$2(h+h').$$

Example: $A_2 \otimes A_2$

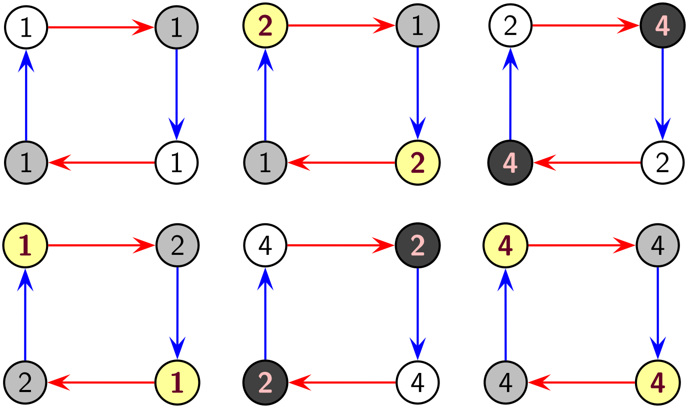


Example: $A_2 \otimes A_2$



6 steps!

Example: $A_2 \otimes A_2$



12

~~6~~ **steps!**

Zamolodchikov periodicity

Tensor product of **finite** Dynkin diagrams $\Longrightarrow$ the T-system is periodic

$\Longleftarrow$ the T-system is periodic

Zamolodchikov periodicity

Tensor product of **finite** Dynkin diagrams $\Longrightarrow$ the T-system is periodic

?

$\Longleftarrow$ the T-system is periodic

The result

Theorem (G.-Pylyavskyy, 2016)

Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1. *The T-system associated with Q is periodic.*
2.

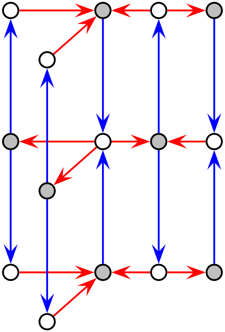
The result

Theorem (G.-Pylyavskyy, 2016)

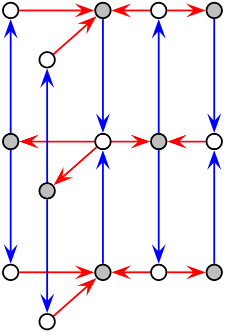
Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1. *The T-system associated with Q is periodic.*
2. Q **is a finite $\boxtimes$ finite quiver.**

Finite ⊠ finite quivers

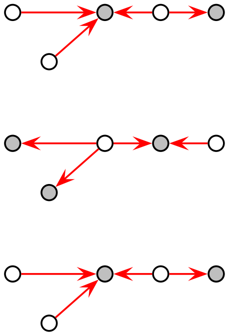


Finite ⊠ finite quivers



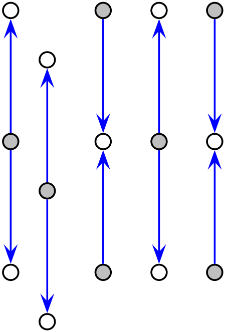
- Bipartite recurrent quiver

Finite ⊠ finite quivers



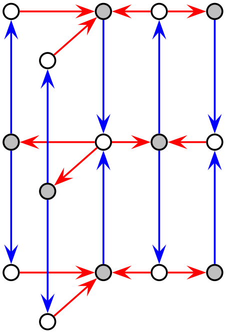
- Bipartite recurrent quiver
- All red components are **finite** Dynkin diagrams

Finite ⊠ finite quivers



- Bipartite recurrent quiver
- All red components are **finite** Dynkin diagrams
- All blue components are **finite** Dynkin diagrams

Finite ⊠ finite quivers



- Bipartite recurrent quiver
- All red components are **finite** Dynkin diagrams
- All blue components are **finite** Dynkin diagrams

↑

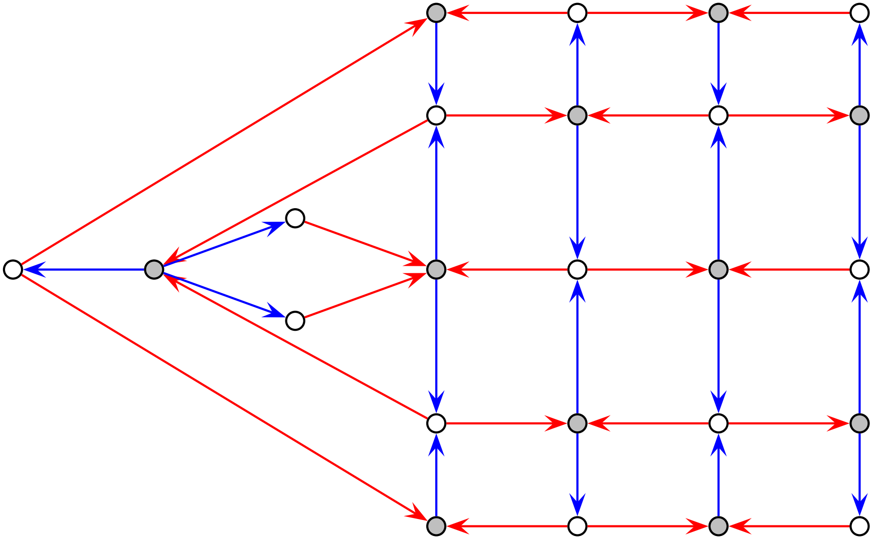
"Finite ⊠ finite quiver"

The classification of Zamolodchikov periodic quivers

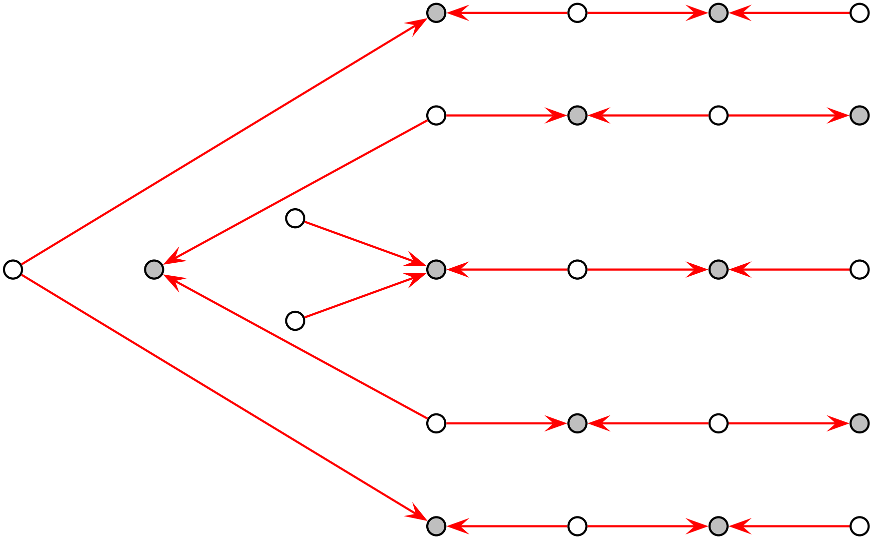
Theorem (G.-Pylyavskyy, 2016)

Let Q be a bipartite recurrent quiver. Then the following are equivalent.

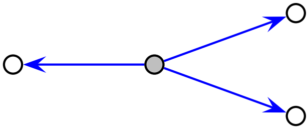
1. *The T-system associated with Q is periodic.*
2. *Q is a* **finite $\boxtimes$ finite quiver**.

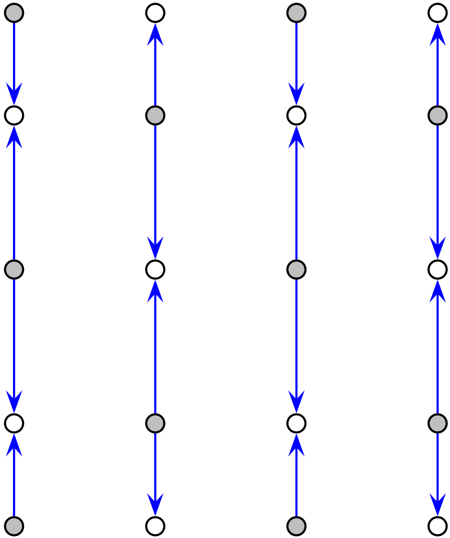


Finite ⊠ finite quivers

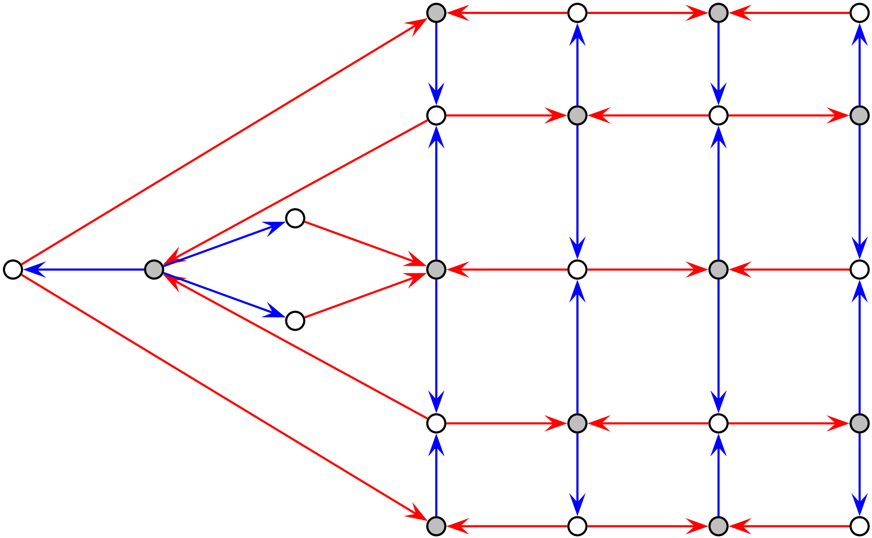


Finite ⊠ finite quivers

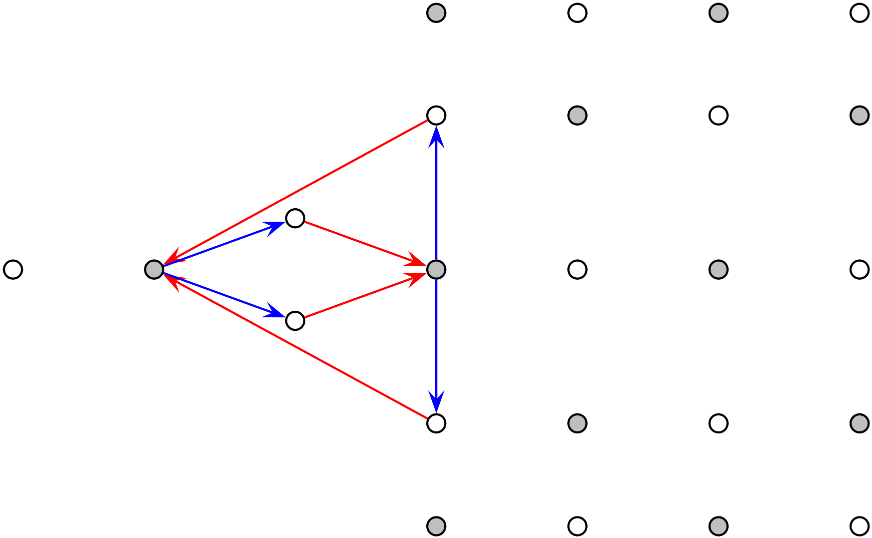




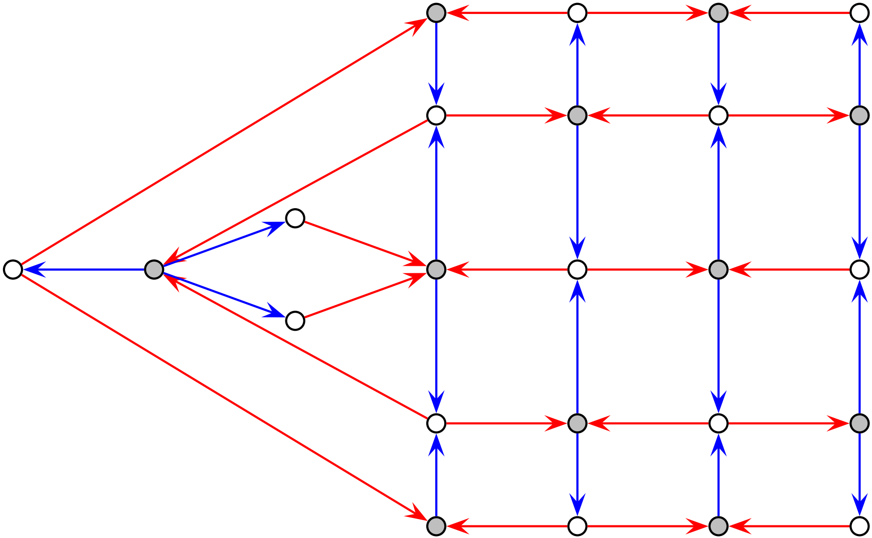
Finite ⊠ finite quivers



Finite ⊠ finite quivers



Finite ⊠ finite quivers



The classification of Zamolodchikov periodic quivers

Theorem (G.-Pylyavskyy, 2016)

Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1. *The T-system associated with Q is periodic.*
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The classification of Zamolodchikov periodic quivers

Theorem (G.-Pylyavskyy, 2016)

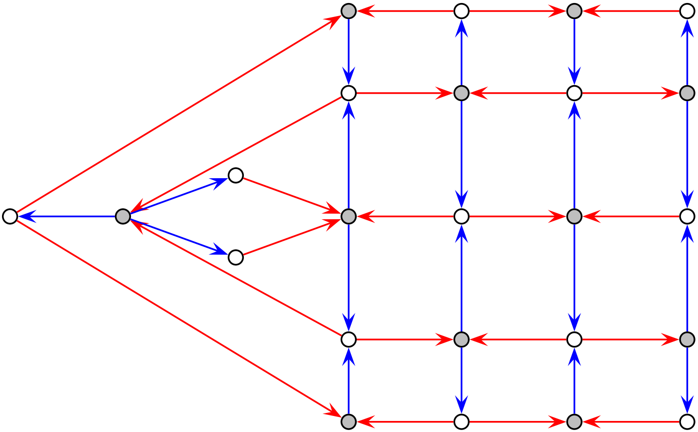
Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1. *The T-system associated with Q is periodic.*
2. *Q is a finite $\boxtimes$ finite quiver.*

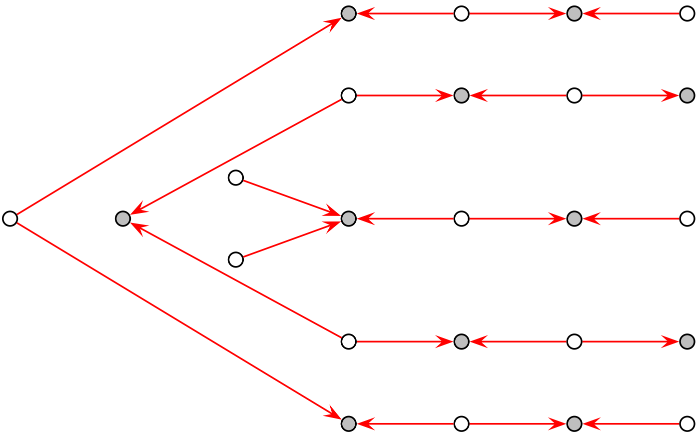
In addition, the T-system has period dividing

$$2(h + h').$$

Finite ⊠ finite quivers

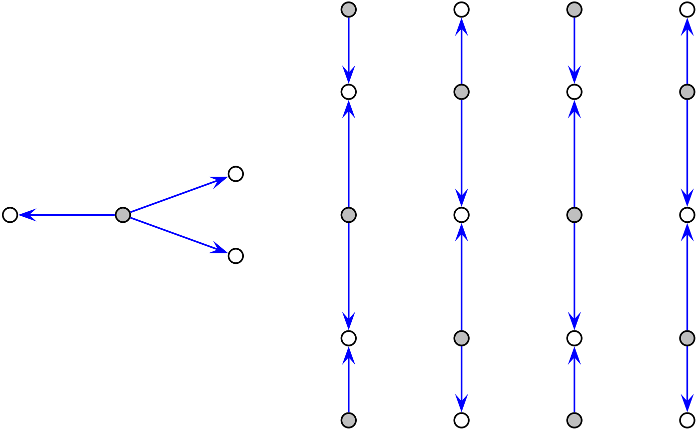


Finite ⊠ finite quivers



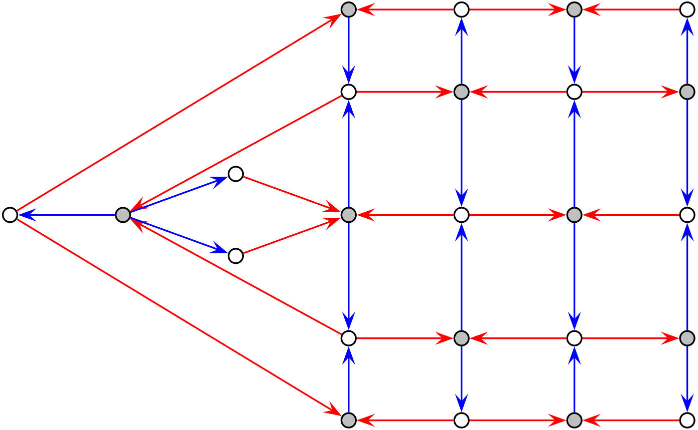
$\mathbf{h = 9 + 1 = 12 - 2 = 10}$;

Finite ⊠ finite quivers



$h = 9 + 1 = 12 - 2 = 10;\quad \mathbf{h' = 5 + 1 = 8 - 2 = 6};$

Finite ⊠ finite quivers



$h = 9 + 1 = 12 - 2 = 10$; $h' = 5 + 1 = 8 - 2 = 6$; Period $=$ **32**

The classification of Zamolodchikov periodic quivers

Theorem (G.-Pylyavskyy, 2016)

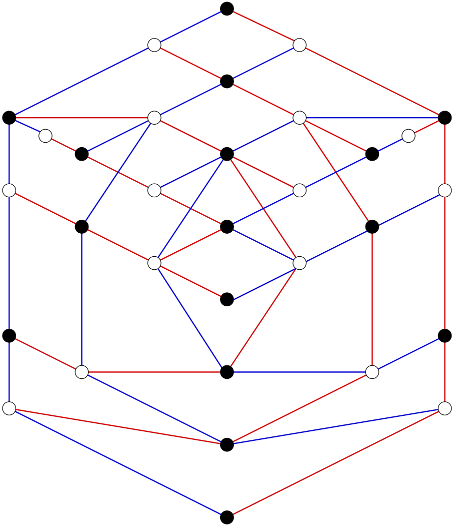
Let Q be a bipartite recurrent quiver. Then the following are equivalent.

1. *The T-system associated with Q is periodic.*
2. *Q is a finite $\boxtimes$ finite quiver.*

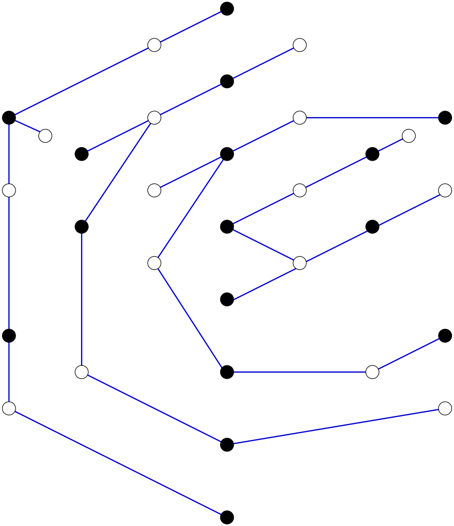
In addition, the T-system has period dividing

$$2(h + h').$$

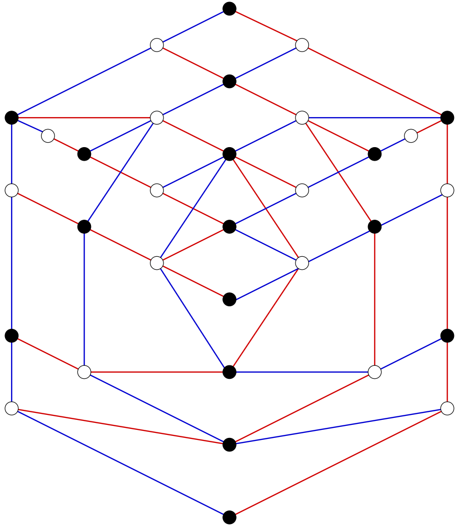
5 infinite families and 11 exceptional quivers (Stembridge)



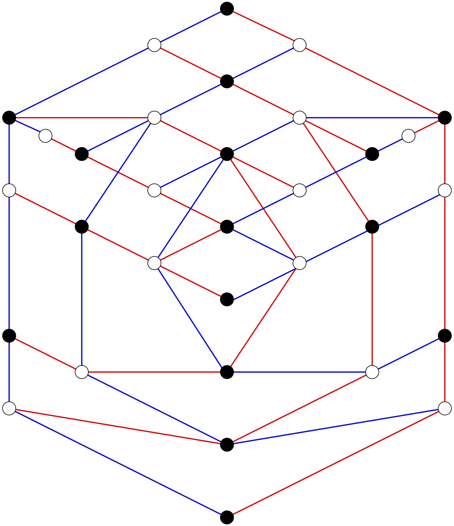
5 infinite families and 11 exceptional quivers (Stembridge)



5 infinite families and 11 exceptional quivers (Stembridge)



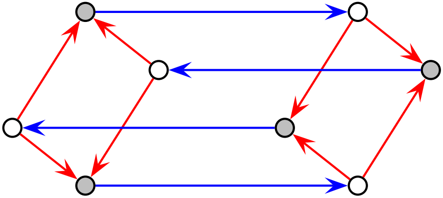
5 infinite families and 11 exceptional quivers (Stembridge)



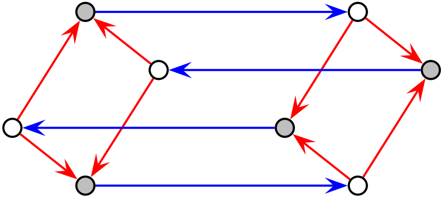
$$2(h + h') = 120$$

Part 3: Zamolodchikov integrability

Affine ⊠ finite quivers

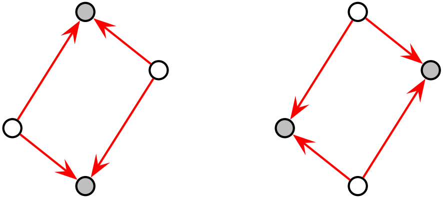


Affine ⊠ finite quivers



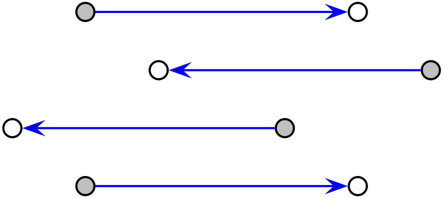
- Bipartite recurrent quiver

Affine $\boxtimes$ finite quivers



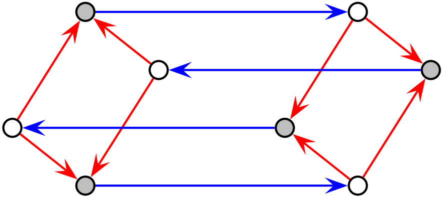
- Bipartite recurrent quiver
- All red components are **affine** Dynkin diagrams

Affine ⊠ finite quivers



- Bipartite recurrent quiver
- All red components are **affine** Dynkin diagrams
- All blue components are **finite** Dynkin diagrams

Affine ⊠ finite quivers

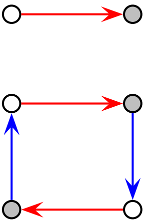


- Bipartite recurrent quiver
- All red components are **affine** Dynkin diagrams
- All blue components are **finite** Dynkin diagrams

$\uparrow$

"Affine ⊠ finite quiver"

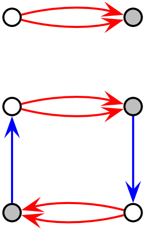
Four classes of quivers



"finite ⊠ finite"

periodic

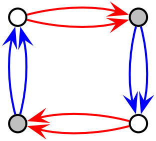
grows as $\exp(1)$



"affine ⊠ finite"

linearizable

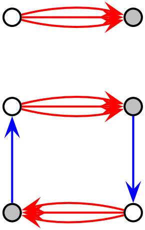
grows as $\exp(t)$



"affine ⊠ affine"

"integrable"?

grows as $\exp(t^2)$



"wild"

"non-integrable"?

grows as $\exp(\exp(t))$

A necessary condition

Theorem (G.-Pylyavskyy, 2016)

Let Q be a bipartite recurrent quiver. Then:

1. **IF** *the T-system associated with Q is linearizable,*
2.

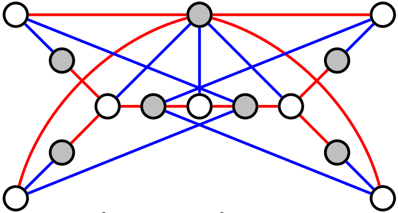
A necessary condition

Theorem (G.-Pylyavskyy, 2016)

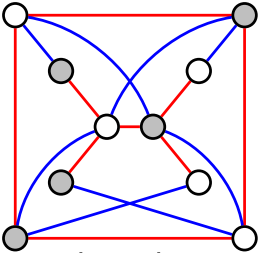
Let Q be a bipartite recurrent quiver. Then:

1. **IF** *the T-system associated with Q is linearizable,*
2. **THEN** *Q is either an affine $\boxtimes$ finite or finite $\boxtimes$ finite quiver.*

15 infinite families and 4 exceptional cases

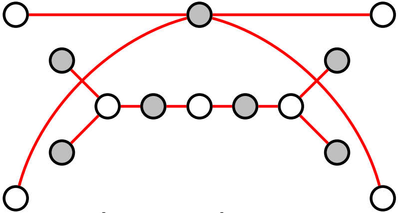


$\hat{D}_{n+1} * \hat{D}_{3n-1}$
for $n = 3$

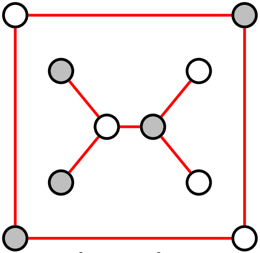


$\hat{A}_3 * \hat{D}_5$

15 infinite families and 4 exceptional cases

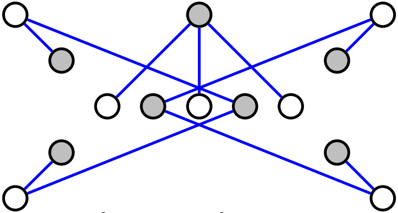


$\hat{D}_{n+1} * \hat{D}_{3n-1}$
for $n = 3$

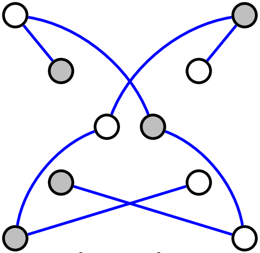


$\hat{A}_3 * \hat{D}_5$

15 infinite families and 4 exceptional cases

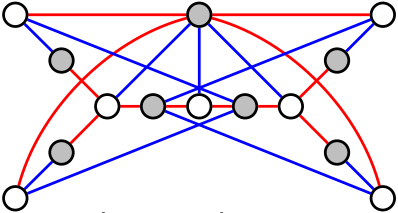


$\hat{D}_{n+1} * \hat{D}_{3n-1}$
for $n = 3$

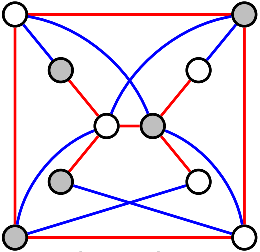


$\hat{A}_3 * \hat{D}_5$

15 infinite families and 4 exceptional cases

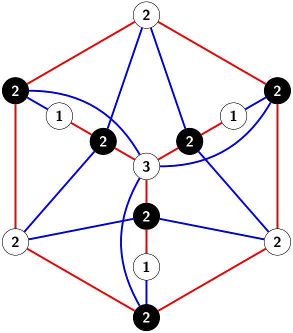


$\hat{D}_{n+1} * \hat{D}_{3n-1}$
for $n = 3$

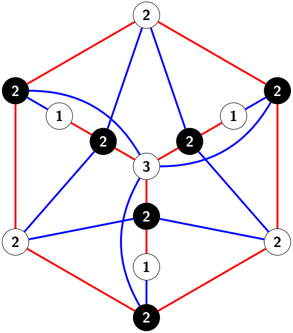


$\hat{A}_3 * \hat{D}_5$

Affine ⊠ affine quivers

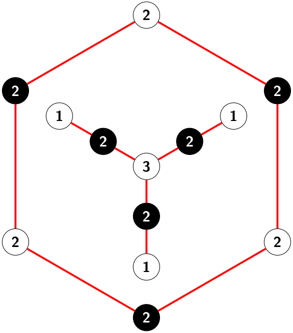


Affine ⊠ affine quivers



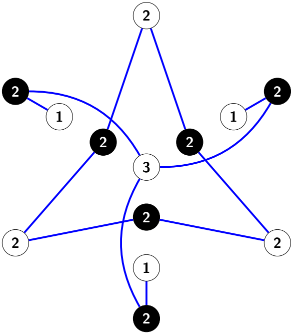
- Bipartite recurrent quiver

Affine ⊠ affine quivers



- Bipartite recurrent quiver
- All red components are **affine** Dynkin diagrams

Affine ⊠ affine quivers



- Bipartite recurrent quiver
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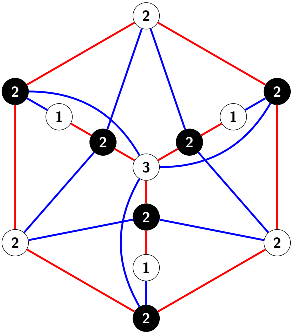
- All blue components are **affine** Dynkin diagrams

Affine ⊠ affine quivers

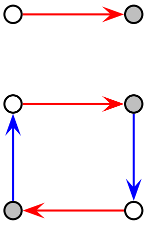


- Bipartite recurrent quiver
- All red components are **affine** Dynkin diagrams
- All blue components are **affine** Dynkin diagrams

↑

"Affine ⊠ affine quiver"

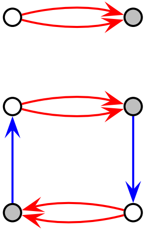
Four classes of quivers



"finite $\boxtimes$ finite"

periodic

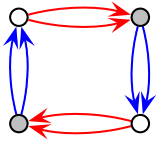
grows as $\exp(1)$



"affine $\boxtimes$ finite"

linearizable

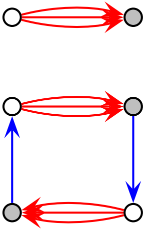
grows as $\exp(t)$



"affine $\boxtimes$ affine"

"integrable"?

grows as $\exp(t^2)$



"wild"

"non-integrable"?

grows as $\exp(\exp(t))$

A necessary condition

Theorem (G.-Pylyavskyy, 2017)

Let Q be a bipartite recurrent quiver. Then:

1. **IF** *the T-system associated with Q has zero algebraic entropy,*
2.

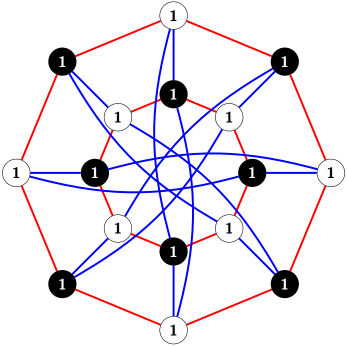
A necessary condition

Theorem (G.-Pylyavskyy, 2017)

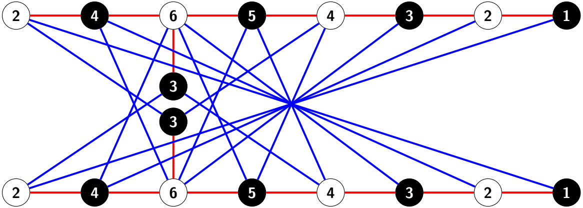
Let Q be a bipartite recurrent quiver. Then:

1. **IF** *the T-system associated with Q has zero algebraic entropy,*
2. **THEN** *Q is either an affine $\boxtimes$ affine or an affine $\boxtimes$ finite or a finite $\boxtimes$ finite quiver.*

30+ infinite families and 13 exceptional cases

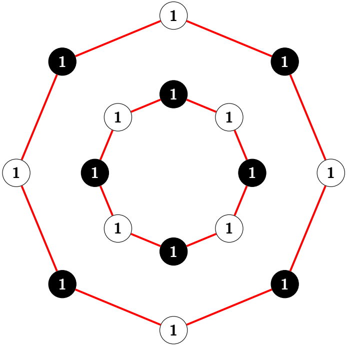


$\hat{A}_{2n-1} * \hat{A}_{2n-1}$ for $n = 4$



$\hat{E}_8 * \hat{E}_8$

30+ infinite families and 13 exceptional cases



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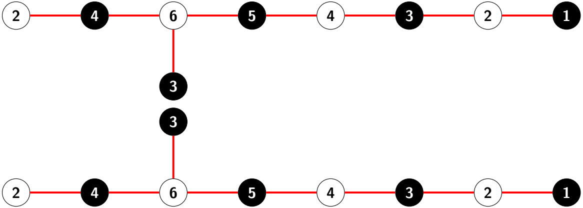

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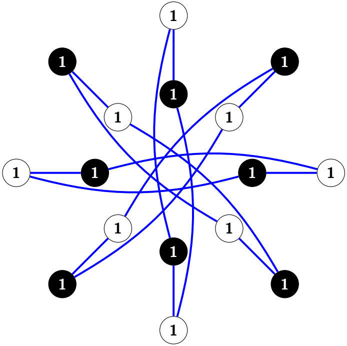
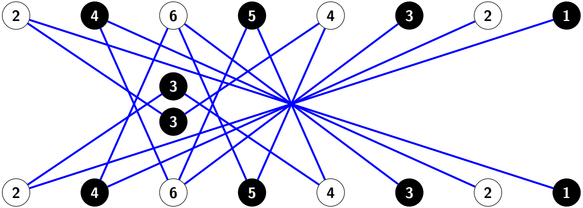


$$\hat{A}_{2n-1} * \hat{A}_{2n-1} \text{ for } n = 4$$



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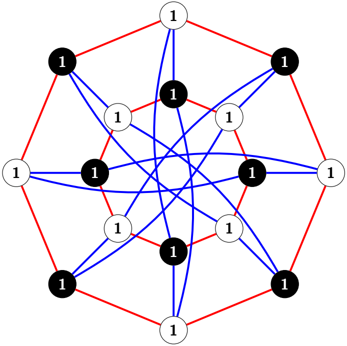

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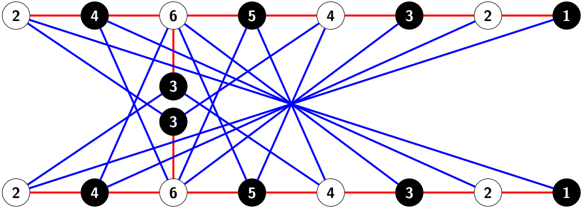
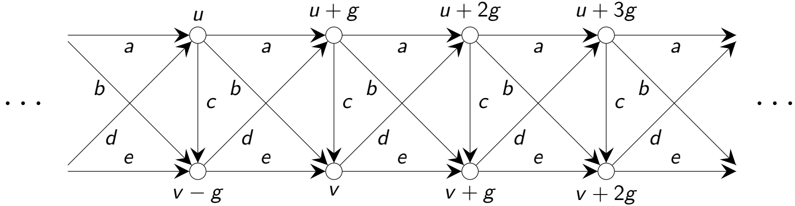


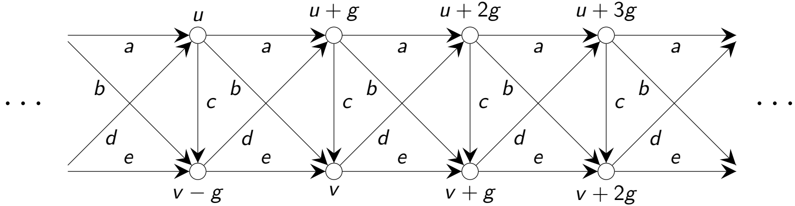
$\hat{E}_8 * \hat{E}_8$

Part 4: Linear recurrences in cylindrical networks

A cylindrical network

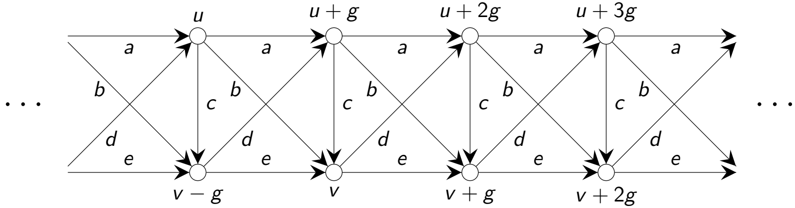


Sequence



$$f(\ell) = N(u, v + \ell g) = \sum_{P: u \to \cdots \to v + \ell g} \mathrm{wt}(P), \quad \ell \in \mathbb{Z}$$

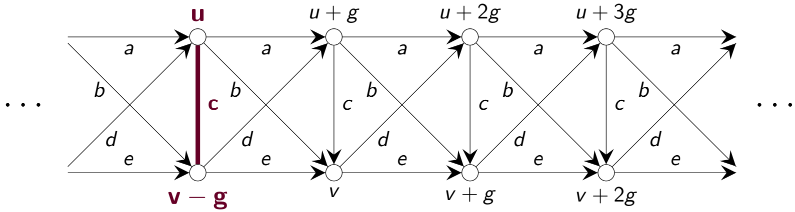
Sequence



$$f(\ell) = N(u, v+\ell g) = \sum_{P: u \to \cdots \to v+\ell g} \mathrm{wt}(P), \quad \ell \in \mathbb{Z}$$

$$\ldots, f(-3), f(-2) = 0$$

Sequence



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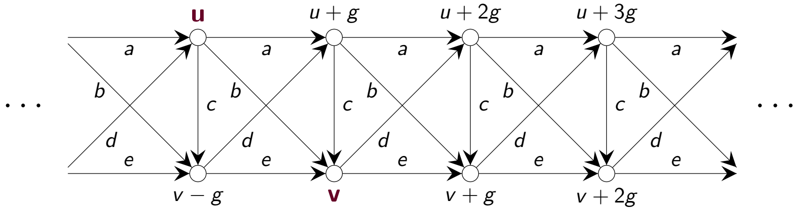
$$\ldots, f(-3), f(-2) = 0$$

$$f(-1) = N(\mathbf{u}, \mathbf{v} - \mathbf{g}) = \mathbf{c}$$

Sequence



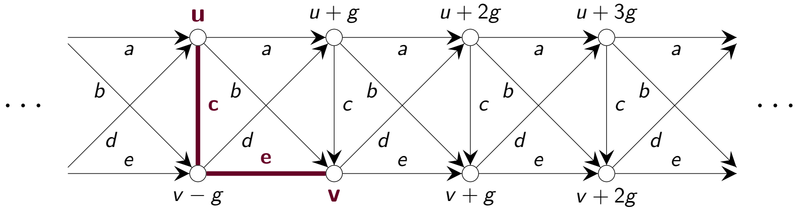
$$f(\ell) = N(u, v + \ell g) = \sum_{P: u \to \cdots \to v + \ell g} \mathrm{wt}(P), \quad \ell \in \mathbb{Z}$$

$$\ldots, f(-3), f(-2) = 0$$

$$f(-1) = N(u, v - g) = c$$

$$f(0) = N(\mathbf{u}, \mathbf{v}) = ce + c^2 d + ac + b$$

Sequence



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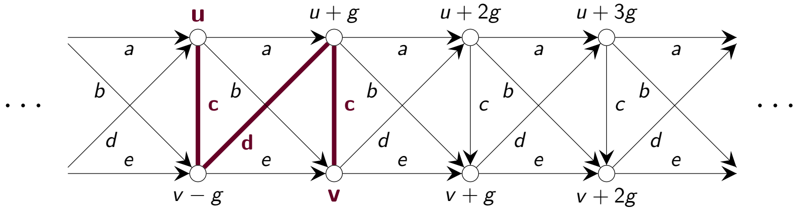
Sequence



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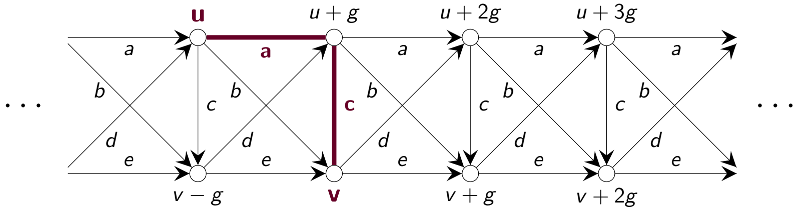

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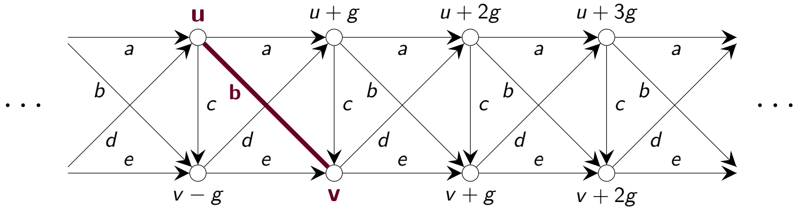

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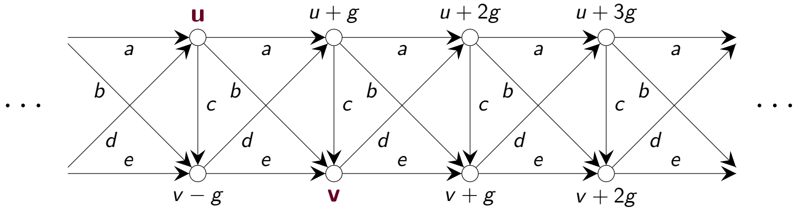

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Linear recurrences in cylindrical networks

Theorem (G.-Pylyavskyy, 2017)

The sequence $f(\ell)$ satisfies a linear recurrence for all sufficiently large ℓ.

Linear recurrences in cylindrical networks

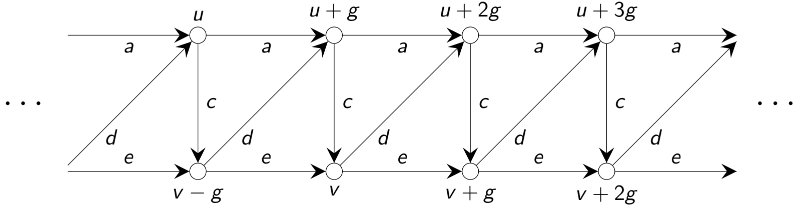
Theorem (G.-Pylyavskyy, 2017)

The sequence $f(\ell)$ satisfies a linear recurrence for all sufficiently large ℓ. The characteristic polynomial $Q(t)$ of the recurrence is given by

$$Q(t) = t^d - J_1 t^{d-1} + \cdots + (-1)^d J_d,$$

where the recurrence coefficients J_r have a nice combinatorial interpretation.

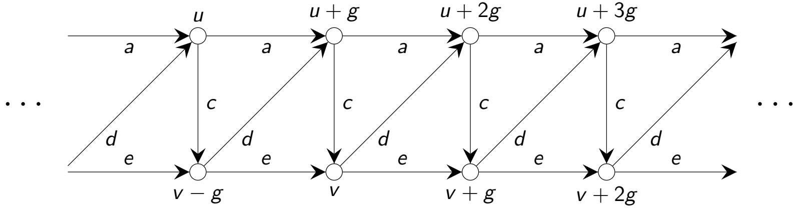
Consider a **planar cylindrical network**:



$$J_r = \sum_{\mathbf{C} \text{ is an } r\text{-cycle}} \mathrm{wt}(\mathbf{C})$$

The meaning of the recurrence coefficients

Consider a **planar cylindrical network**:



$$J_r = \sum_{\mathbf{C} \text{ is an } r\text{-cycle}} \mathrm{wt}(\mathbf{C})$$

$$J_1 = a + cd + e$$

The meaning of the recurrence coefficients

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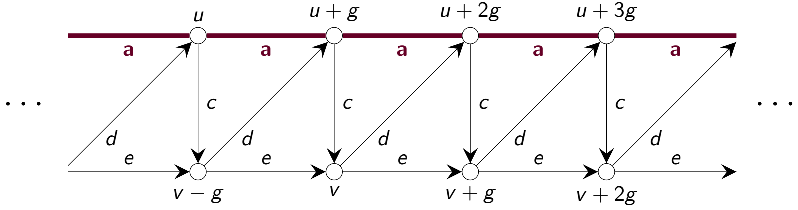

$$J_r = \sum_{\mathbf{C} \text{ is an } r\text{-cycle}} \mathrm{wt}(\mathbf{C})$$

$$J_1 = \mathbf{a} + cd + e$$

The meaning of the recurrence coefficients

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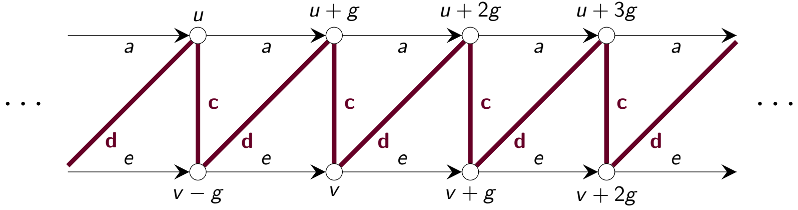
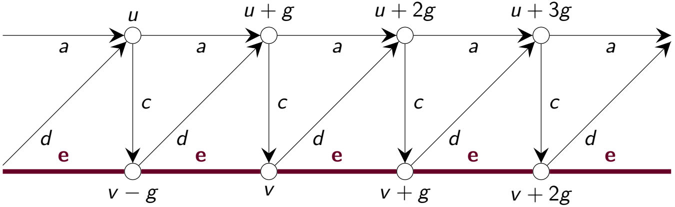


$$J_r = \sum_{\mathbf{C} \text{ is an } r\text{-cycle}} \mathrm{wt}(\mathbf{C})$$

$$J_1 = a + \mathbf{cd} + e$$

The meaning of the recurrence coefficients

Consider a **planar cylindrical network**:



$$J_r = \sum_{\mathbf{C} \text{ is an } r\text{-cycle}} \mathrm{wt}(\mathbf{C})$$

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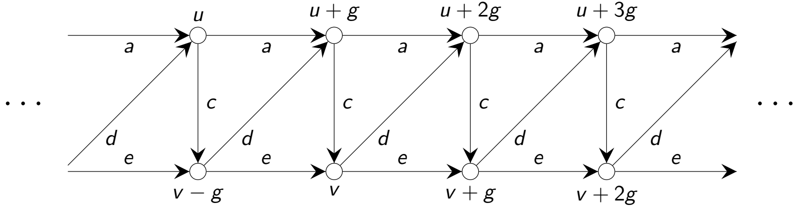
The meaning of the recurrence coefficients

Consider a **planar cylindrical network**:



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The meaning of the recurrence coefficients

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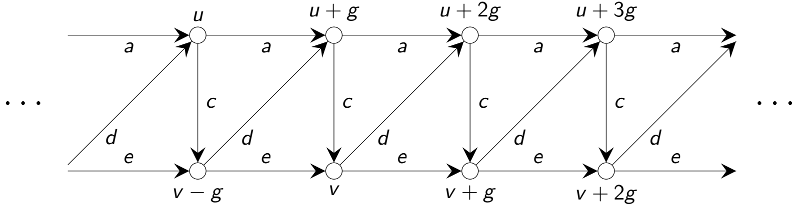

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The meaning of the recurrence coefficients

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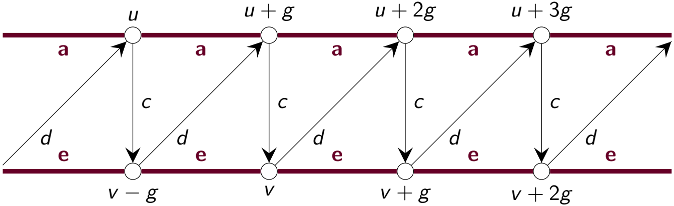

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The meaning of the recurrence coefficients

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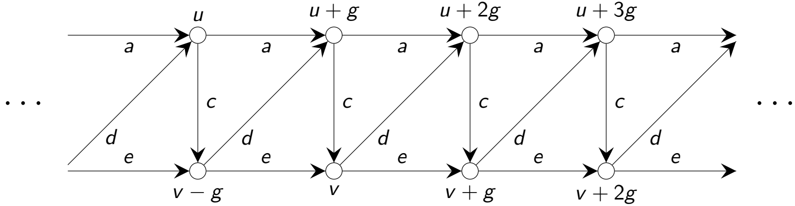

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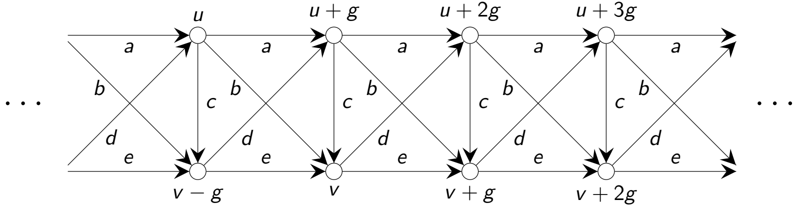
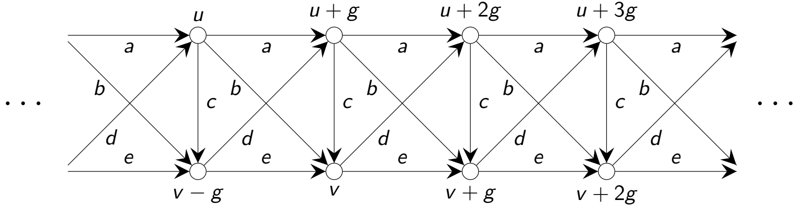
r-paths



Fix two **non-permutable** r-vertices $\mathbf{u} = (u_1, u_2, \dots, u_r)$ and $\mathbf{v} = (v_1, v_2, \dots, v_r)$ and define

$$f(\ell) = N(\mathbf{u}, \mathbf{v} + \ell g) = \sum_{\mathbf{P}:\mathbf{u}\to\cdots\to\mathbf{v}+\ell g} \mathrm{wt}(\mathbf{P}), \quad \ell \in \mathbb{Z}.$$



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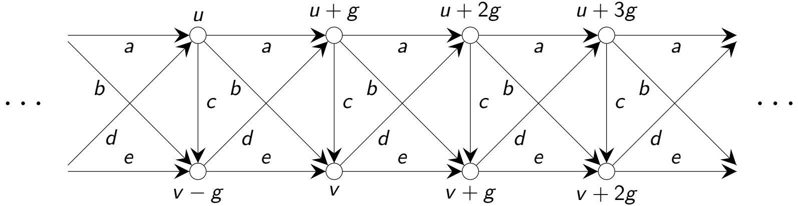
Theorem (G.-Pylyavskyy, 2017)

The sequence $f(\ell)$ satisfies a linear recurrence for all sufficiently large ℓ.

r-paths



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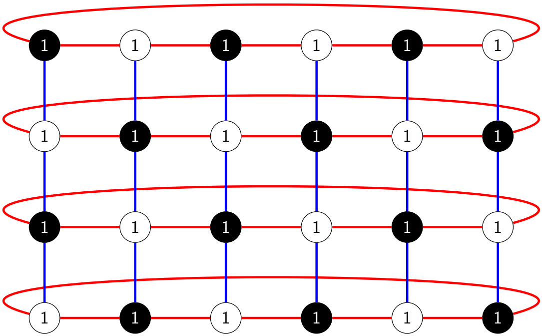
Theorem (G.-Pylyavskyy, 2017)

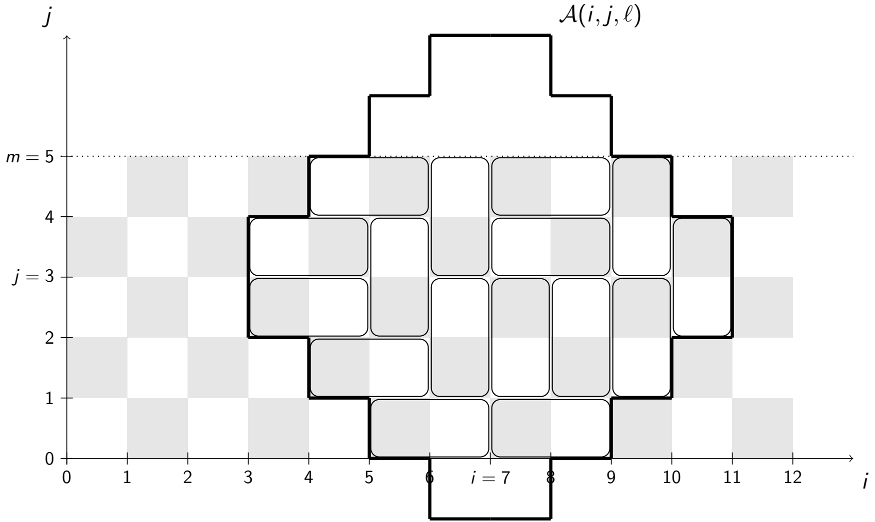
The sequence $f(\ell)$ satisfies a linear recurrence for all sufficiently large ℓ. Planar cylindrical network $\Rightarrow$ characteristic polynomial equals $Q^{[\mathbf{e}_r]}(t)$.

What does this have to do with T-systems?

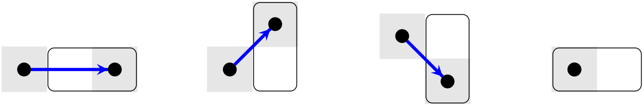
The quiver of type $\hat{A}_{2n-1} \otimes A_m$



Speyer's formula via domino tilings



From tilings to r-paths



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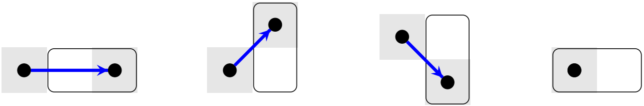
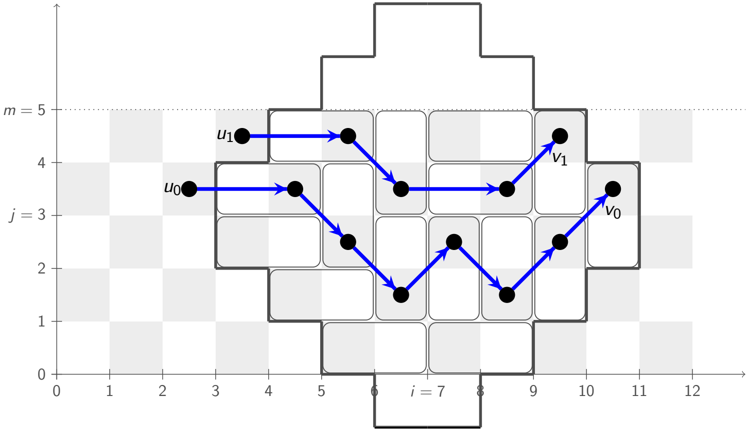




Linearizability for $\hat{A}_{2n-1} \otimes A_m$

Theorem (G.-Pylyavskyy, 2017)

The T-system for $\hat{A}_{2n-1} \otimes A_m$ is linearizable.

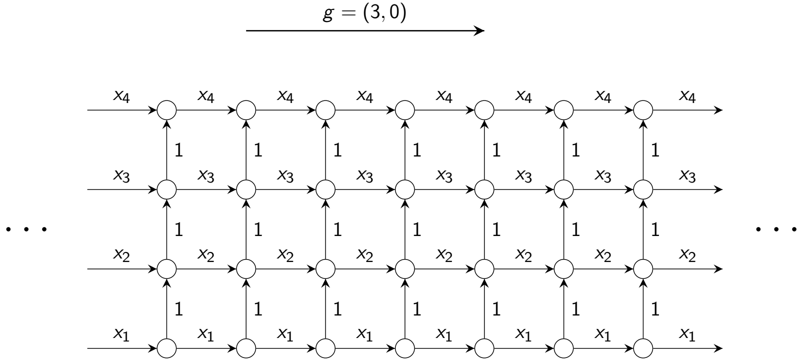
Linearizability for $\hat{A}_{2n-1} \otimes A_m$

Theorem (G.-Pylyavskyy, 2017)

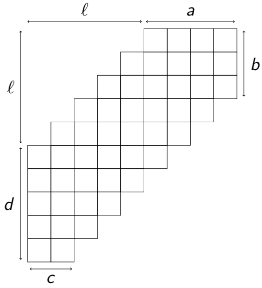
The T-system for $\hat{A}_{2n-1} \otimes A_m$ is linearizable.
The recurrence coefficients are generating functions over domino tilings of the cylinder with fixed **Thurston height**.

Part 5: some pictures

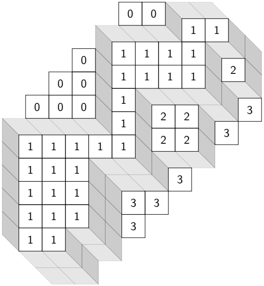
Schur functions

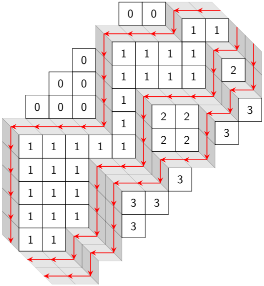


Plane partitions

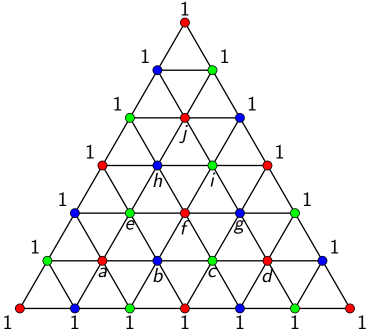


					0	0	1	1
				1	1	1	1	2
			0	1	1	1	1	3
		0	0	1	2	2	3	
	0	0	0	1	2	2		
1	1	1	1	1	3			
1	1	1	3	3				
1	1	1	3					
1	1	1						
1	1							





Cube recurrence in a triangle



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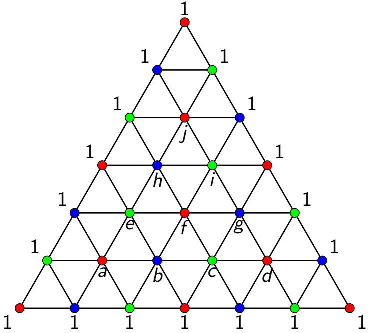
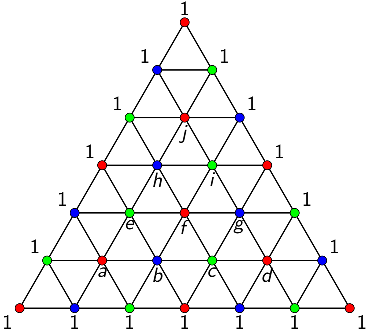


Mutating at f gives

$$f' = \frac{eg + bi + ch}{f}.$$

Cube recurrence in a triangle



Mutating at f gives

$$f' = \frac{eg + bi + ch}{f}.$$

Mutate at all red, then all green, then all blue, then all red, etc...

Cube recurrence in a triangle: periodicity

Theorem (G., 2017)

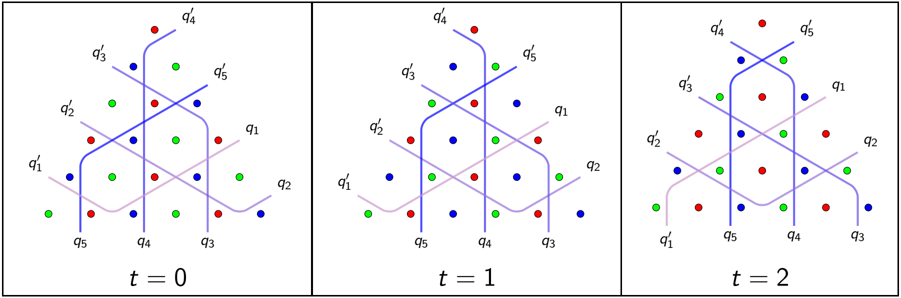
The cube recurrence in a triangle of size m is periodic with period $6m$.

Cube recurrence in a triangle: periodicity

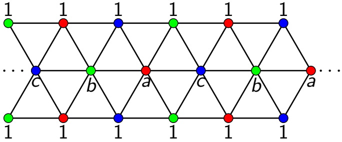
Theorem (G., 2017)

The cube recurrence in a triangle of size m is periodic with period $6m$.

Proof:



Cube recurrence in a cylinder



Cube recurrence in a cylinder: linearizability

Theorem (G., 2017)

The cube recurrence in a cylinder is linearizable.

Cube recurrence in a cylinder: linearizability

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The recurrence coefficients are generating functions for **cylindric groves**.

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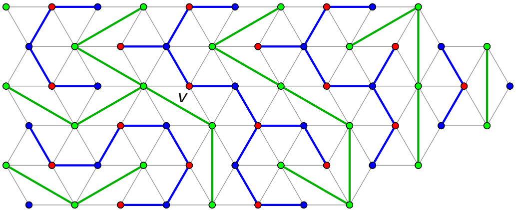

Cube recurrence in a cylinder: linearizability

Theorem (G., 2017)

The cube recurrence in a cylinder is linearizable.
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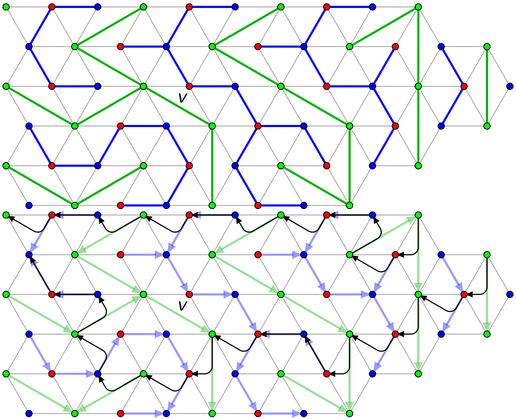
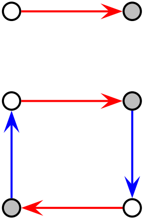
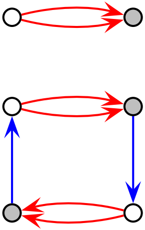
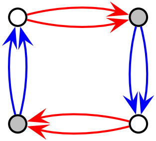
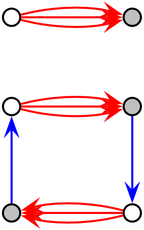


Zamolodchikov phenomena for the cube recurrence

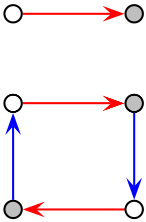
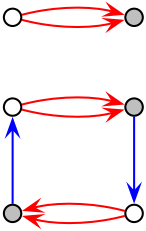
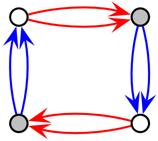
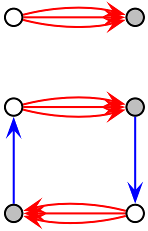
Theorem (G., 2017)

1. *The cube recurrence in a triangle is periodic.*
2. *The cube recurrence in a cylinder is linearizable.*
3. *The cube recurrence in a torus grows quadratic exponentially.*

Four classes of quivers

			
“finite $\boxtimes$ finite”	“affine $\boxtimes$ finite”	“affine $\boxtimes$ affine”	“wild”
periodic	linearizable	“integrable”?	“non-integrable”?
grows as $\exp(1)$	grows as $\exp(t)$	grows as $\exp(t^2)$	grows as $\exp(\exp(t))$

Four classes of quivers

			
"finite ⊠ finite"	"affine ⊠ finite"	"affine ⊠ affine"	"wild"
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disc	**cylinder**	**torus**	

Bibliography

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Bernhard Keller.
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three more papers coming very soon!

Thank you!

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