

Zamolodchikov periodicity and integrability

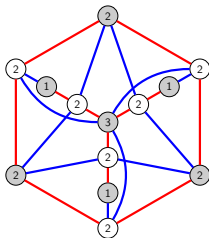
Pavel Galashin

MIT

galashin@mit.edu

University of California, Berkeley, October 16, 2017

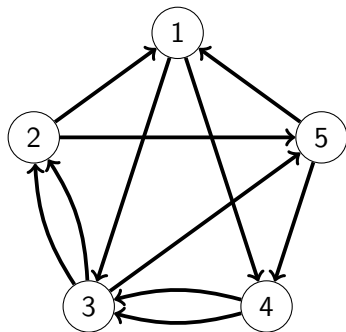
Joint work with Pavlo Pylyavskyy



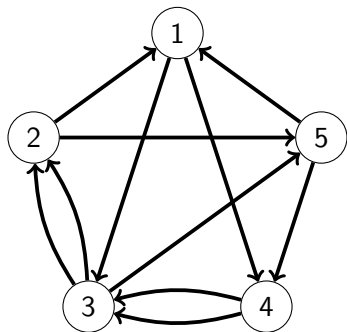
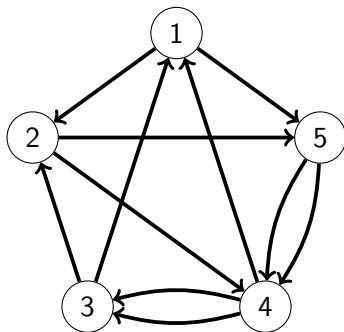
Part 1: T -systems

General T -systems (Nakanishi, 2011)

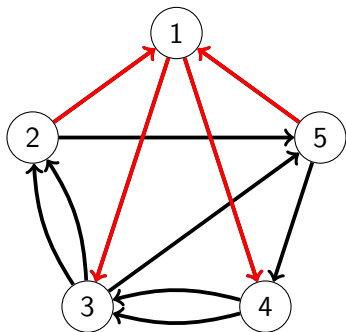
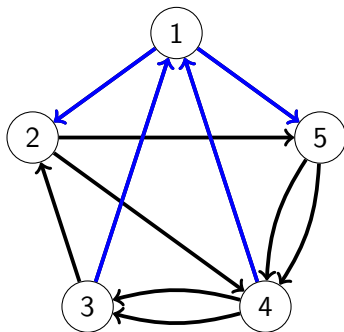
Q



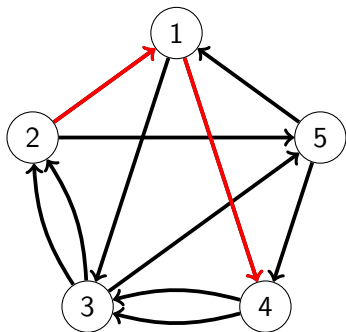
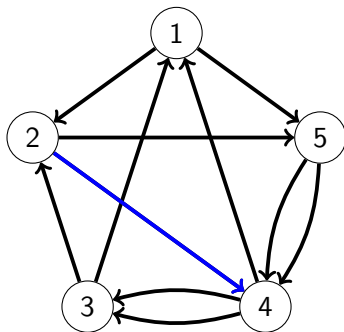
General T -systems (Nakanishi, 2011)

 Q  $\mu_1(Q)$ 

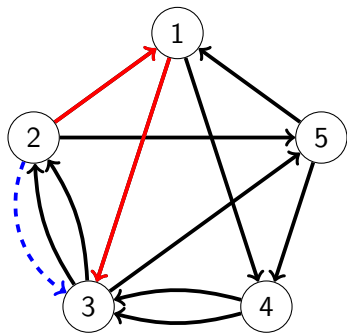
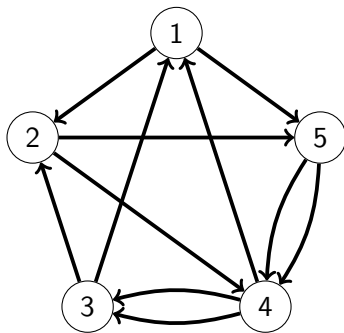
General T -systems (Nakanishi, 2011)

 Q  $\mu_1(Q)$ 

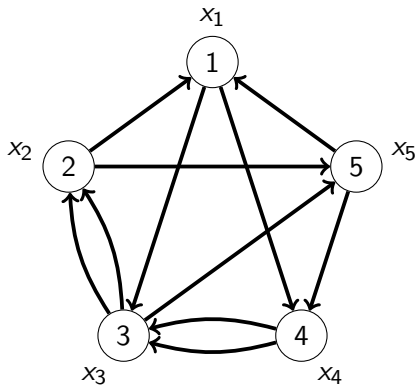
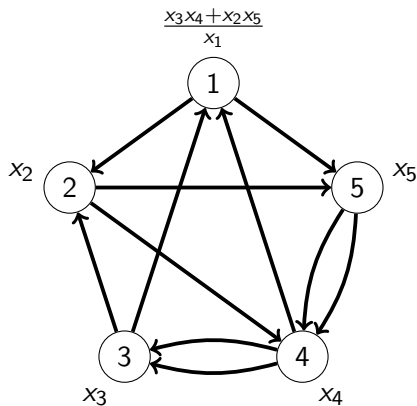
General T -systems (Nakanishi, 2011)

 Q  $\mu_1(Q)$ 

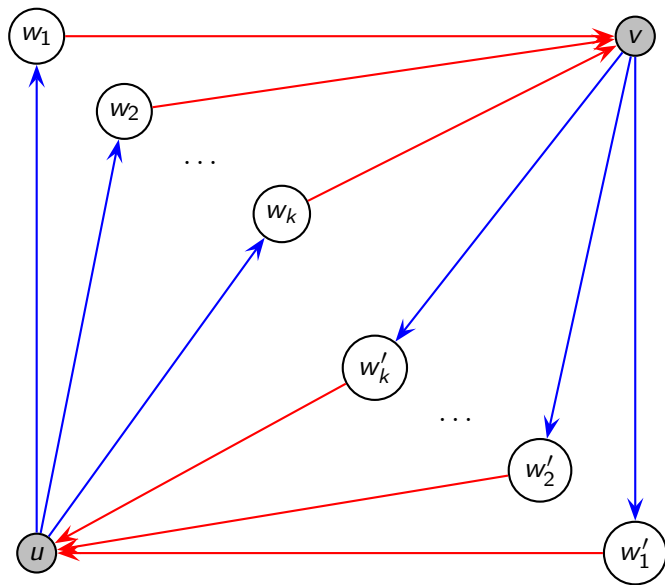
General T -systems (Nakanishi, 2011)

 Q  $\mu_1(Q)$ 

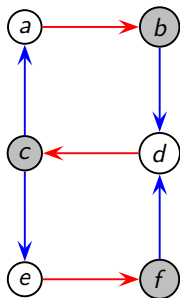
General T -systems (Nakanishi, 2011)

 Q  $\mu_1(Q)$ 

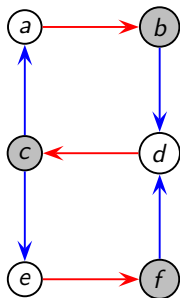
Bipartite **recurrent** quivers



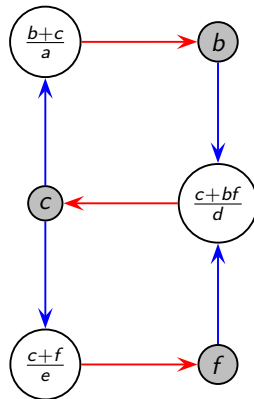
Bipartite T -system



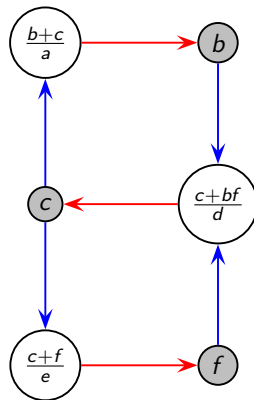
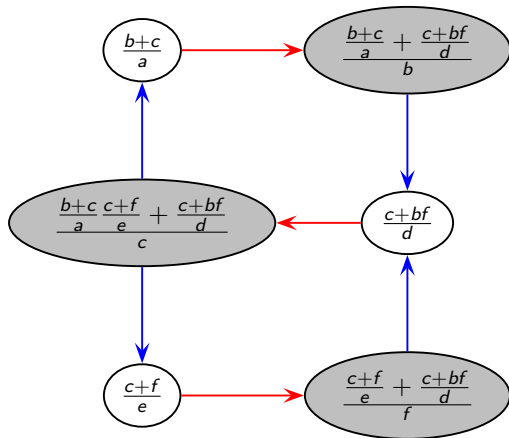
Bipartite T -system



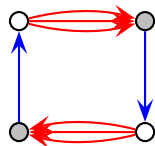
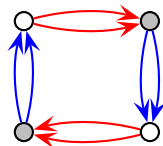
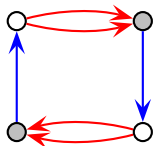
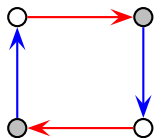
$\longrightarrow$



Bipartite T -system



Four classes of quivers



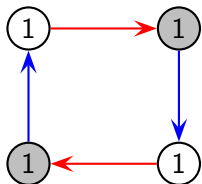
“finite $\boxtimes$ finite”

“affine $\boxtimes$ finite”

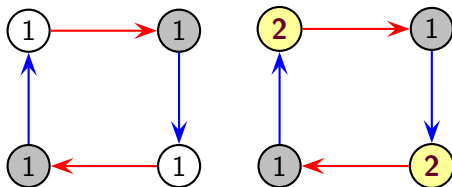
“affine $\boxtimes$ affine”

“wild”

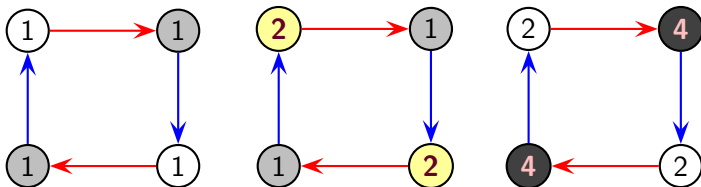
Example: finite $\boxtimes$ finite



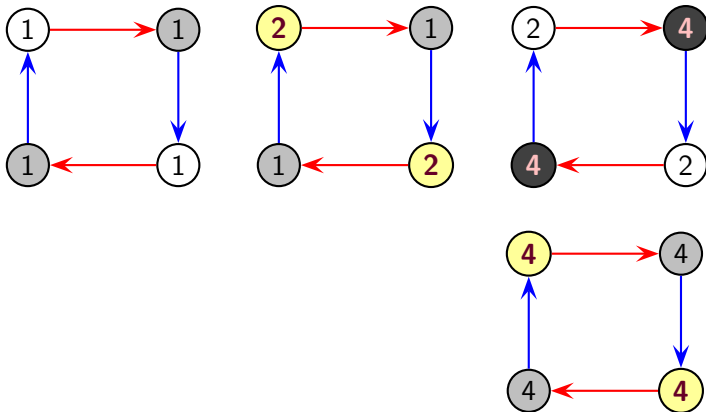
Example: finite $\boxtimes$ finite



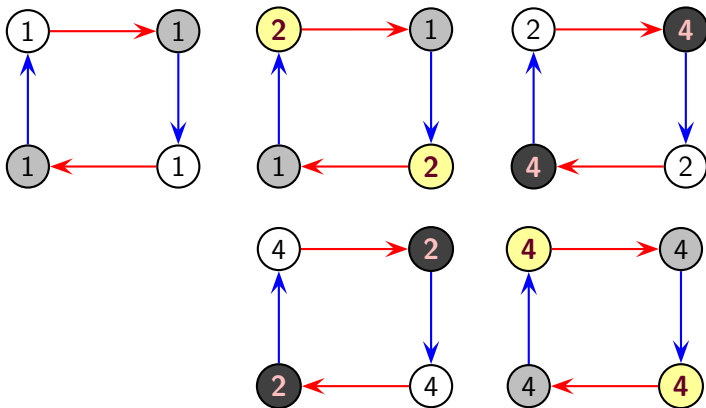
Example: finite $\boxtimes$ finite



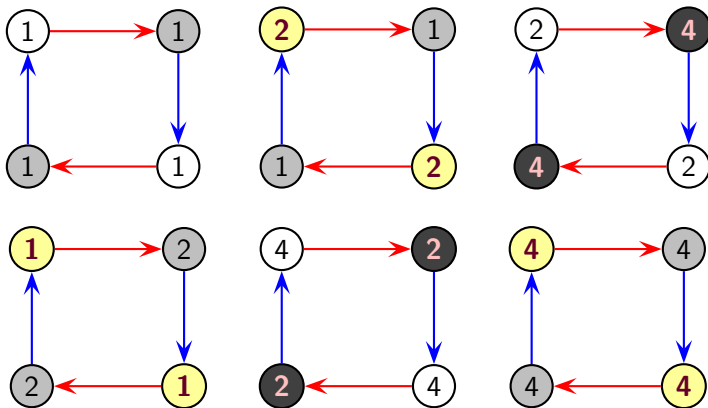
Example: finite $\boxtimes$ finite



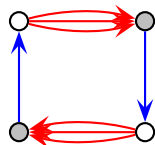
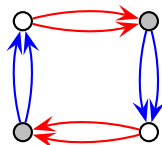
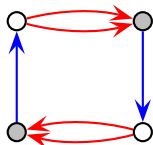
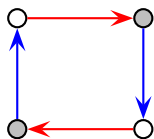
Example: finite $\boxtimes$ finite



Example: finite $\boxtimes$ finite



Four classes of quivers



“finite $\boxtimes$ finite”

“affine $\boxtimes$ finite”

“affine $\boxtimes$ affine”

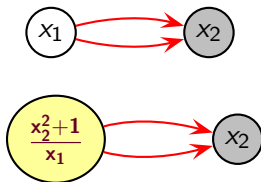
“wild”

periodic

Example: affine $\boxtimes$ finite



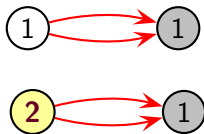
Example: affine $\boxtimes$ finite



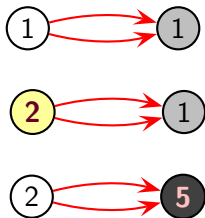
Example: affine $\boxtimes$ finite



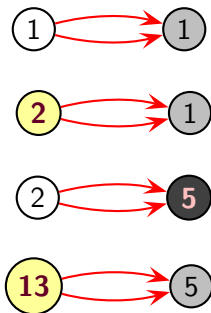
Example: affine $\boxtimes$ finite



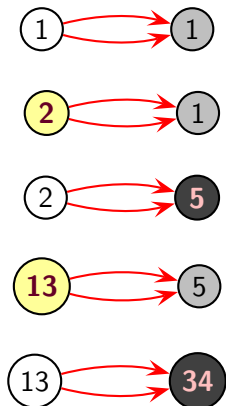
Example: affine $\boxtimes$ finite



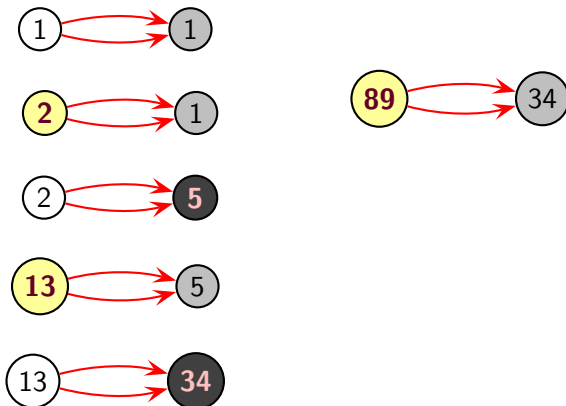
Example: affine $\boxtimes$ finite



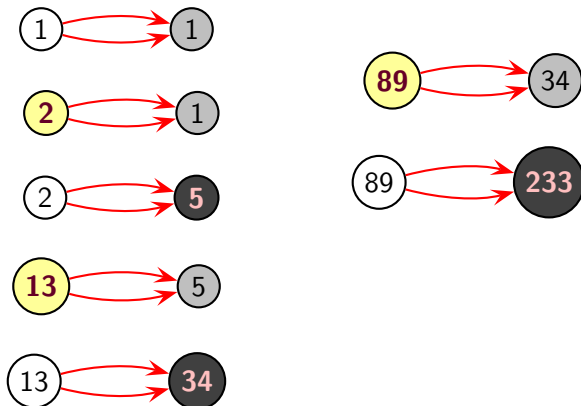
Example: affine $\boxtimes$ finite



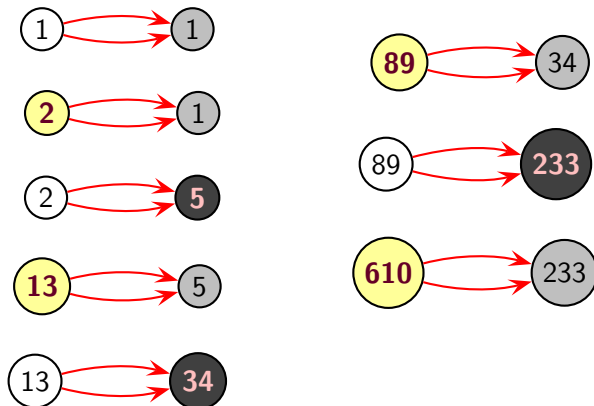
Example: affine $\boxtimes$ finite



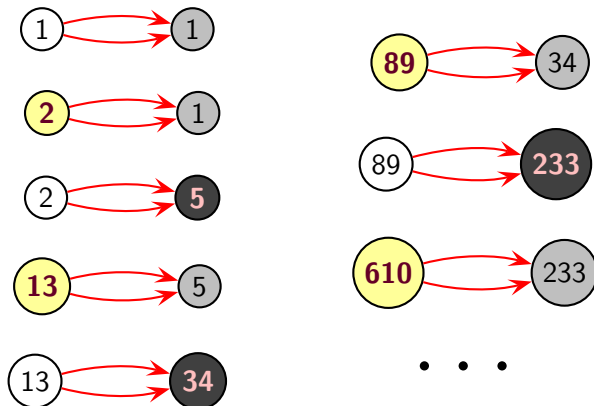
Example: affine $\boxtimes$ finite



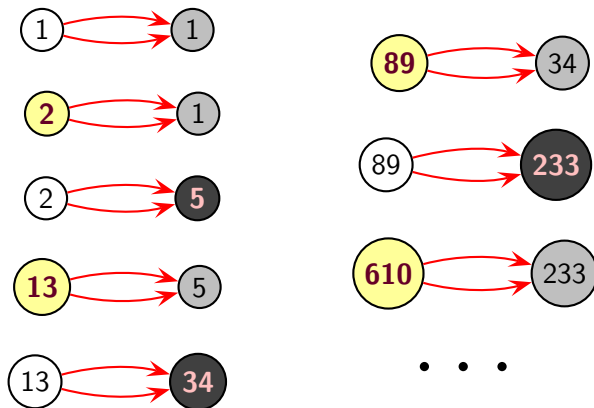
Example: affine $\boxtimes$ finite



Example: affine $\boxtimes$ finite

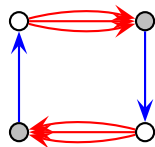
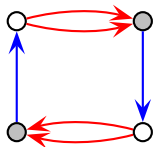
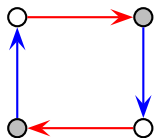
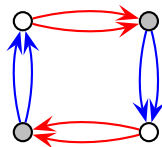


Example: affine $\boxtimes$ finite



$$x_{n+1} = 3x_n - x_{n-1}$$

Four classes of quivers



“finite $\boxtimes$ finite”

“affine $\boxtimes$ finite”

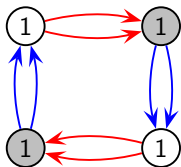
“affine $\boxtimes$ affine”

“wild”

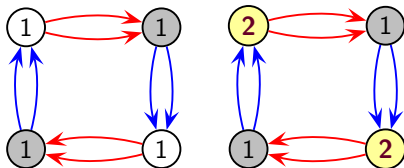
periodic

linearizable

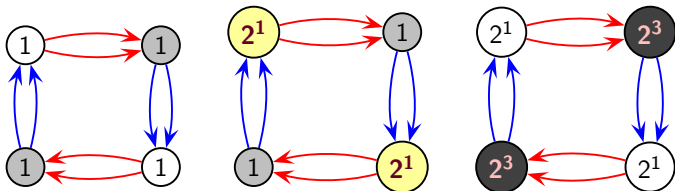
Example: affine $\boxtimes$ affine



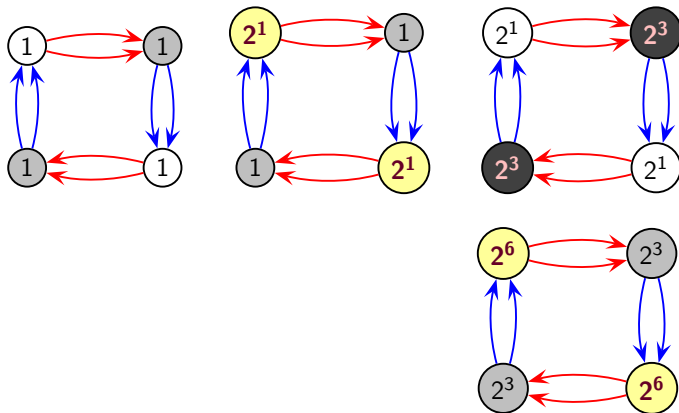
Example: affine $\boxtimes$ affine



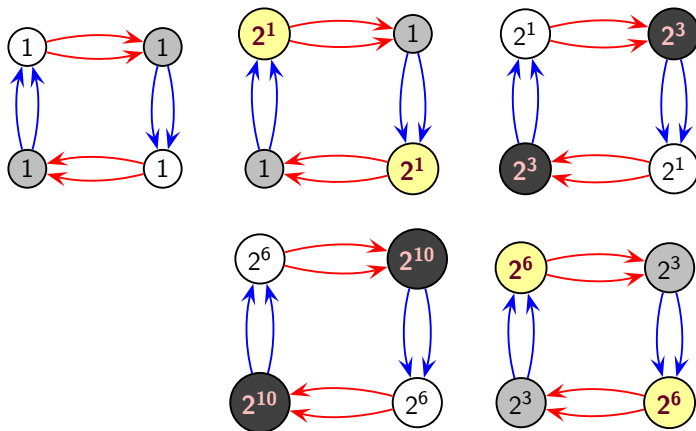
Example: affine $\boxtimes$ affine



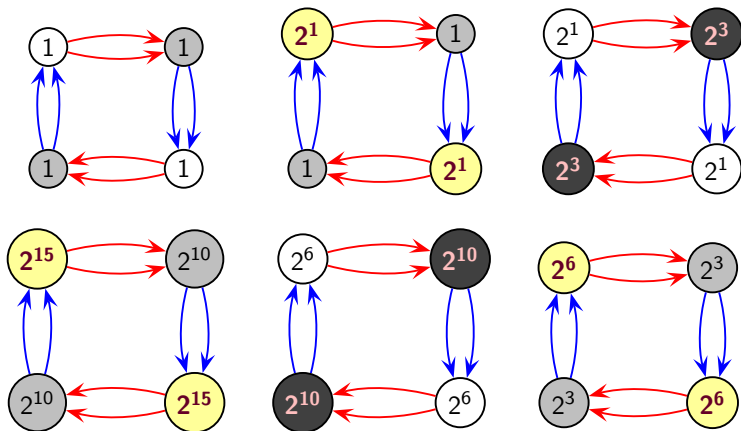
Example: affine $\boxtimes$ affine



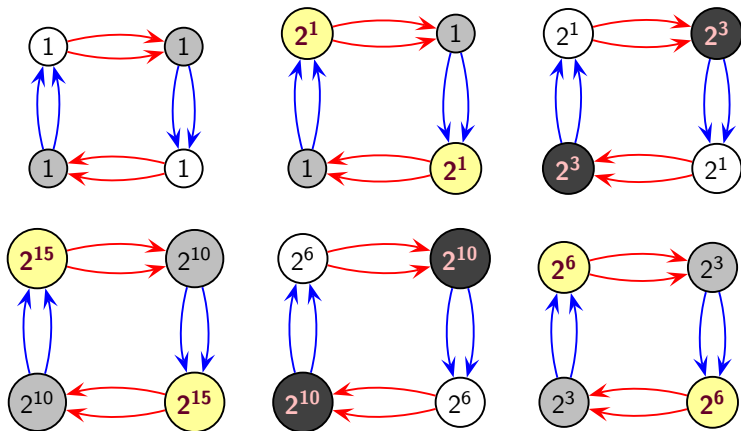
Example: affine $\boxtimes$ affine



Example: affine $\boxtimes$ affine



Example: affine $\boxtimes$ affine



$$2^{\binom{n}{2}}$$

Four classes of quivers



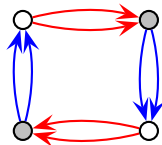
“finite $\boxtimes$ finite”

periodic



“affine $\boxtimes$ finite”

linearizable



“affine $\boxtimes$ affine”

grows as
 $\exp(t^2)$

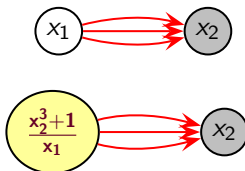


“wild”

Example: wild



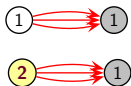
Example: wild



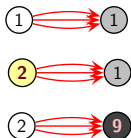
Example: wild



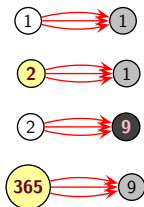
Example: wild



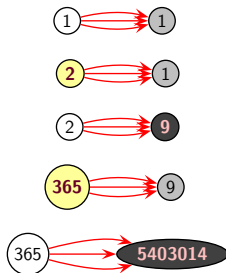
Example: wild



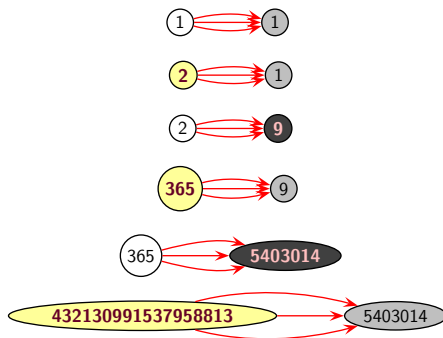
Example: wild



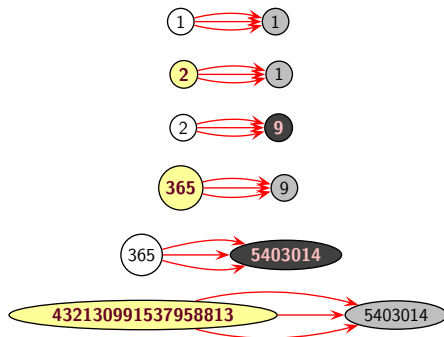
Example: wild



Example: wild



Example: wild



A003818 $a(1)=a(2)=1, a(n+1) = (a(n)^3 + 1)/a(n-1).$

1, 1, 2, 9, 365, 5403014, 432130991537958813,

14935169284101525874491673463268414536523593057 ([list](#); [graph](#); [refs](#); [list](#))

OFFSET 1, 3

COMMENTS The term $a(9)$ has 121 digits. - [Harvey P. Dale](#),

Four classes of quivers



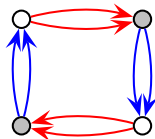
"finite $\boxtimes$ finite"

periodic



"affine $\boxtimes$ finite"

linearizable



"affine $\boxtimes$ affine"

grows as
 $\exp(t^2)$



"wild"

grows as
 $\exp(\exp(t))$

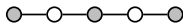
Part 2: The master conjecture

ADE Dynkin diagrams

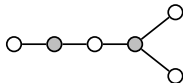
Name

Finite diagram

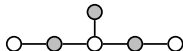
A_n



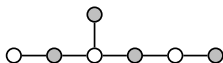
D_n



E_6



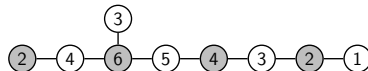
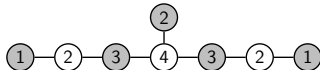
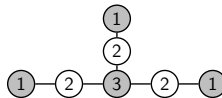
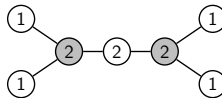
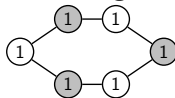
E_7



E_8



Affine diagram



Name

$\hat{A}_{n-1}$

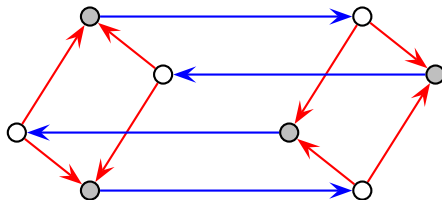
$\hat{D}_{n-1}$

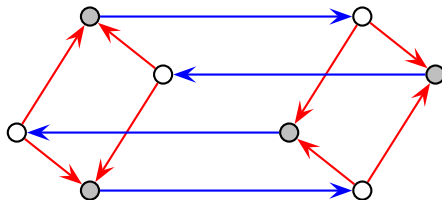
$\hat{E}_6$

$\hat{E}_7$

$\hat{E}_8$

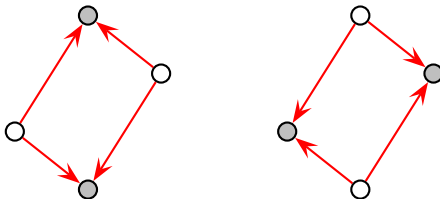
Affine $\boxtimes$ finite quivers





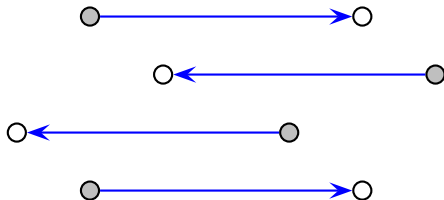
- Bipartite recurrent quiver

Affine $\boxtimes$ finite quivers



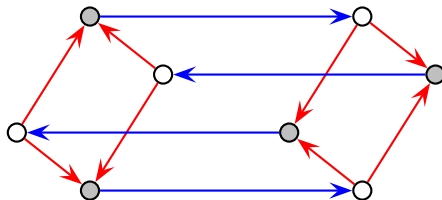
- Bipartite recurrent quiver
- All red components are **affine** Dynkin diagrams

Affine $\boxtimes$ finite quivers



- Bipartite recurrent quiver
- All red components are **affine** Dynkin diagrams
- All blue components are **finite** Dynkin diagrams

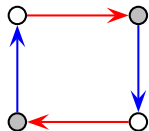
Affine $\boxtimes$ finite quivers



- Bipartite recurrent quiver
- All red components are **affine** Dynkin diagrams
- All blue components are **finite** Dynkin diagrams

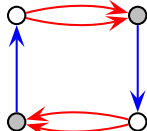
↑
“**Affine** $\boxtimes$ **finite** quiver”

Four classes of quivers



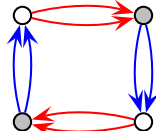
“finite $\boxtimes$ finite”

periodic



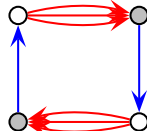
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 $\exp(t^2)$



“wild”

grows as
 $\exp(\exp(t))$

Master conjecture

Conjecture (G.-Pylyavskyy, 2016)

Master conjecture

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- $\text{finite} \boxtimes \text{finite} \iff \text{periodic}$

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Conjecture (G.-Pylyavskyy, 2016)

- *finite* $\boxtimes$ *finite* $\iff$ *periodic*
- *affine* $\boxtimes$ *finite* $\iff$ *linearizable, but not periodic*
- *affine* $\boxtimes$ *affine* $\iff$ *grows as $\exp(t^2)$*
- *wild* $\iff$ *grows as $\exp(\exp(t))$*

Results

Theorem (G.-Pylyavskyy, 2016)

Periodic $\iff$ *finite* $\boxtimes$ *finite*

Theorem (G.-Pylyavskyy, 2016)

Periodic $\iff$ *finite* $\boxtimes$ *finite*

Theorem (G.-Pylyavskyy, 2016)

Linearizable $\implies$ *affine* $\boxtimes$ *finite or finite* $\boxtimes$ *finite*

Theorem (G.-Pylyavskyy, 2016)

Periodic $\iff$ finite $\boxtimes$ finite

Theorem (G.-Pylyavskyy, 2016)

Linearizable $\implies$ affine $\boxtimes$ finite or finite $\boxtimes$ finite

Theorem (G.-Pylyavskyy, 2017)

Grows slower than $\exp(\exp(t)) \implies$ affine $\boxtimes$ affine, affine $\boxtimes$ finite, or finite $\boxtimes$ finite

Results

Theorem (G.-Pylyavskyy, 2016)

Periodic $\iff$ finite $\boxtimes$ finite

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Linearizable $\implies$ affine $\boxtimes$ finite or finite $\boxtimes$ finite

Theorem (G.-Pylyavskyy, 2017)

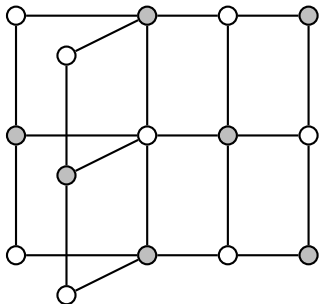
Grows slower than $\exp(\exp(t)) \implies$ affine $\boxtimes$ affine, affine $\boxtimes$ finite, or finite $\boxtimes$ finite

What is left:

Conjecture (G.-Pylyavskyy, 2017)

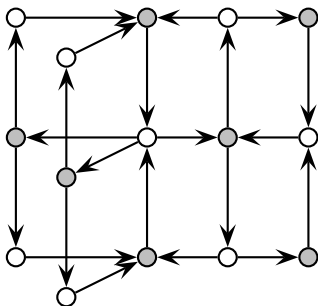
- *affine $\boxtimes$ finite $\implies$ linearizable*
- *affine $\boxtimes$ affine $\implies$ grows as $\exp(t^2)$*

Tensor product



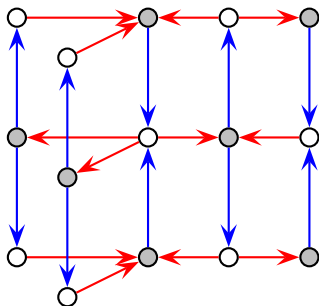
$$D_5 \otimes A_3$$

Tensor product



$$D_5 \otimes A_3$$

Tensor product



$$D_5 \otimes A_3$$

Theorem (B. Keller, 2013)

*Tensor product of **finite** Dynkin diagrams $\implies$ the T -system is periodic.*

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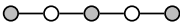
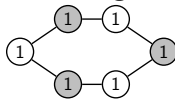
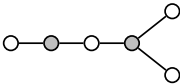
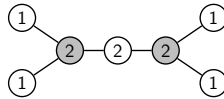
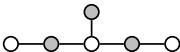
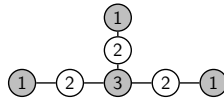
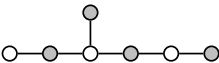
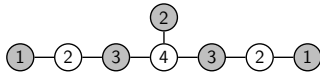
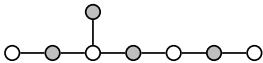
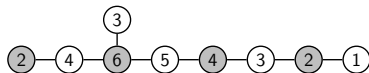
- Zamolodchikov (1991): $\Lambda \otimes A_1$ (conjectured);
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- Fomin-Zelevinsky (2003): $\Lambda \otimes A_1$;

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- Fomin-Zelevinsky (2003): $\Lambda \otimes A_1$;
- Volkov (2005): $A_n \otimes A_m$;

ADE Dynkin diagrams

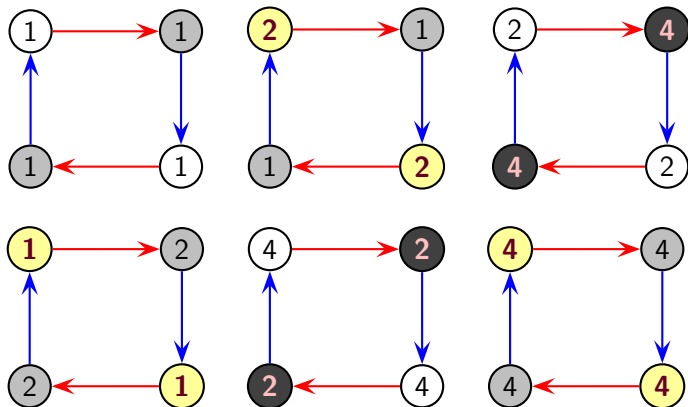
Name	Finite diagram	h	Affine diagram	Name
A_n		$n + 1$		$\hat{A}_{n-1}$
D_n		$2n - 2$		$\hat{D}_{n-1}$
E_6		12		$\hat{E}_6$
E_7		18		$\hat{E}_7$
E_8		30		$\hat{E}_8$

Theorem (B. Keller, 2013)

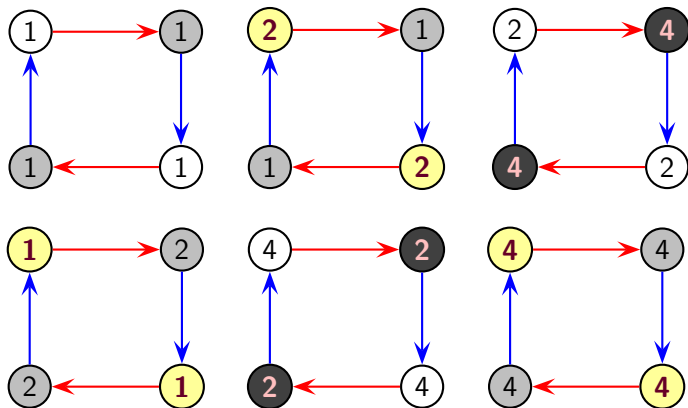
*Tensor product of **finite** Dynkin diagrams $\implies$ the T -system is periodic
with period dividing*

$$2(h + h').$$

Example: $A_2 \otimes A_2$

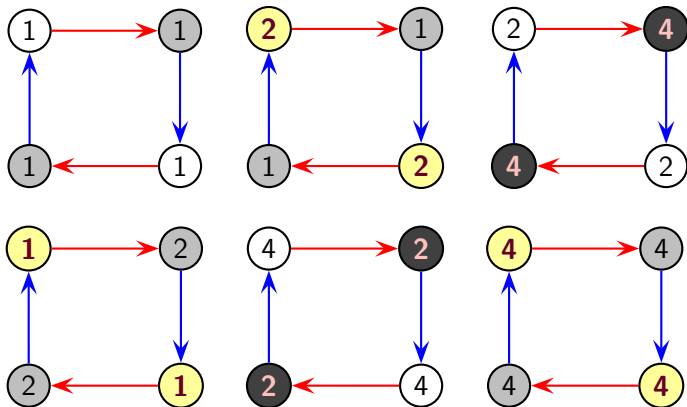


Example: $A_2 \otimes A_2$



6 steps!

Example: $A_2 \otimes A_2$



~~12~~
6 steps!

Results

Theorem (G.-Pylyavskyy, 2016)

Periodic $\iff$ **finite** $\boxtimes$ **finite**

Theorem (G.-Pylyavskyy, 2016)

Linearizable $\implies$ *affine* $\boxtimes$ *finite* or *finite* $\boxtimes$ *finite*

Theorem (G.-Pylyavskyy, 2017)

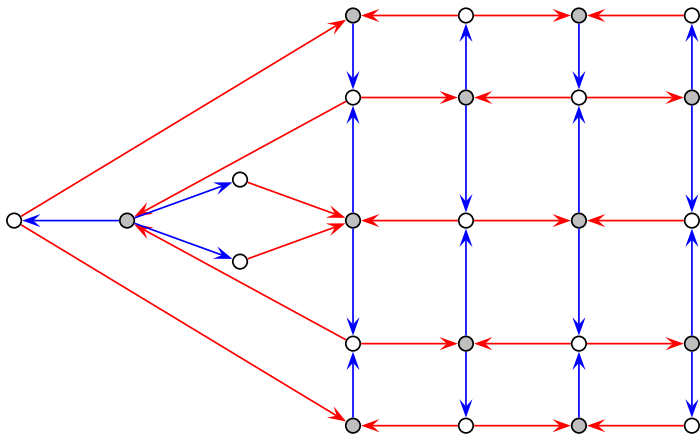
Grows slower than $\exp(\exp(t)) \implies$ *affine* $\boxtimes$ *affine*, *affine* $\boxtimes$ *finite*, or *finite* $\boxtimes$ *finite*

What is left:

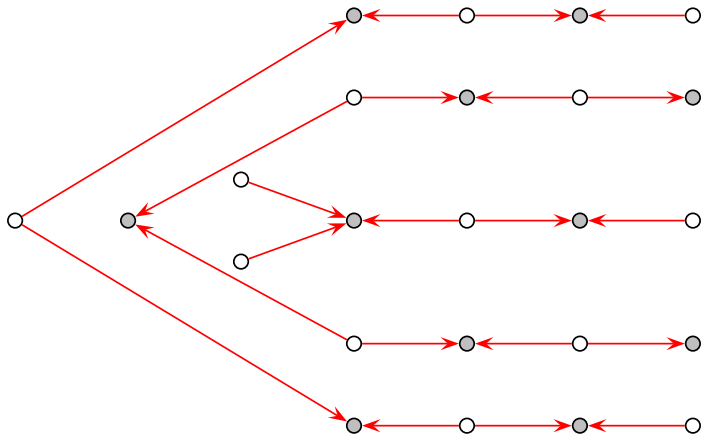
Conjecture (G.-Pylyavskyy, 2017)

- $\bullet$ *affine* $\boxtimes$ *finite* $\implies$ *linearizable*
- $\bullet$ *affine* $\boxtimes$ *affine* $\implies$ *grows as* $\exp(t^2)$

Finite $\boxtimes$ finite quivers

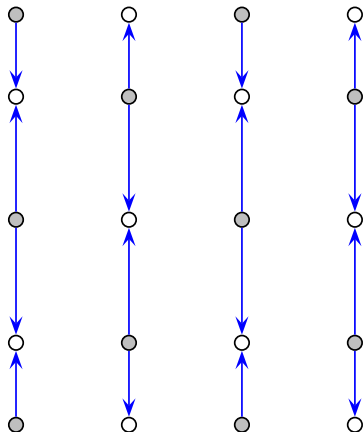
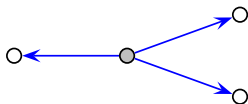


Finite $\boxtimes$ finite quivers



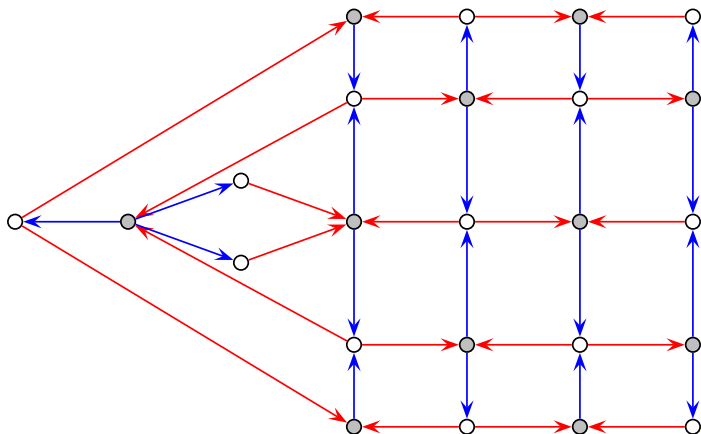
$$h = 9 + 1 = 12 - 2 = 10;$$

Finite $\boxtimes$ finite quivers



$$h = 9 + 1 = 12 - 2 = 10; \quad \mathbf{h' = 5 + 1 = 8 - 2 = 6;}$$

Finite $\boxtimes$ finite quivers

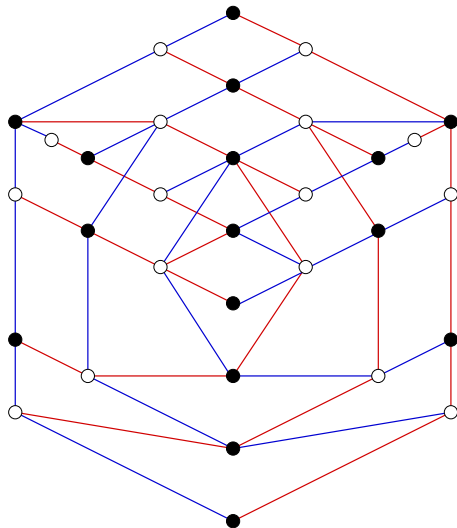


$$h = 9 + 1 = 12 - 2 = 10; \quad h' = 5 + 1 = 8 - 2 = 6; \quad \text{Period} = \mathbf{32}$$

Part 3: The classification

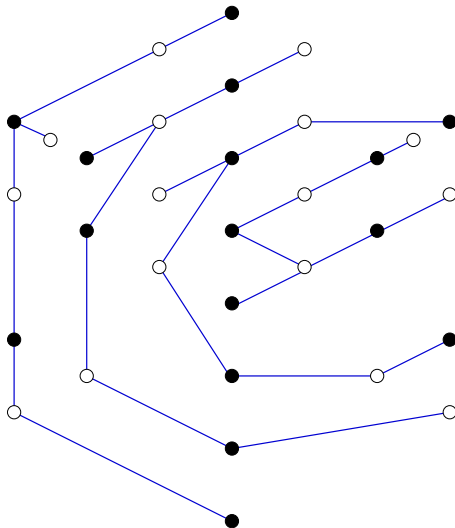
Finite $\boxtimes$ finite classification (Stembridge, 2010)

5 infinite families and 11 exceptional quivers



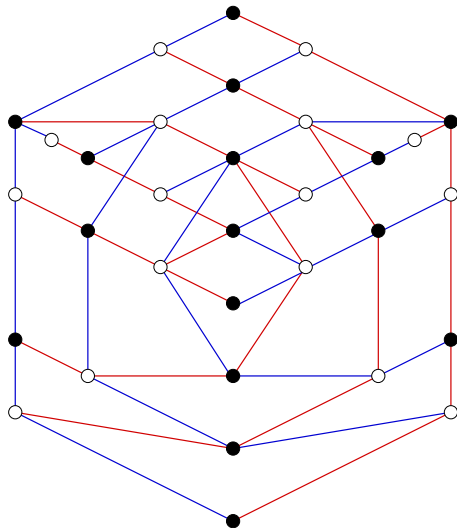
Finite $\boxtimes$ finite classification (Stembridge, 2010)

5 infinite families and 11 exceptional quivers

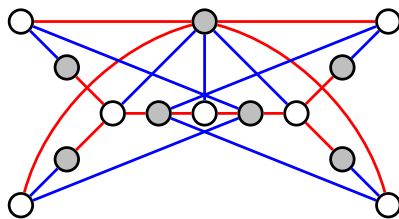


Finite $\boxtimes$ finite classification (Stembridge, 2010)

5 infinite families and 11 exceptional quivers

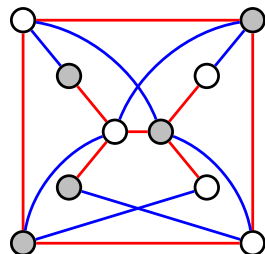


15 infinite families and 4 exceptional cases



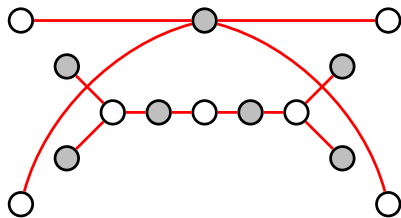
$$\hat{D}_{n+1} * \hat{D}_{3n-1}$$

for $n = 3$



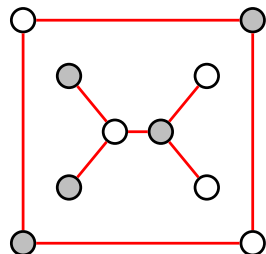
$$\hat{A}_3 * \hat{D}_5$$

15 infinite families and 4 exceptional cases



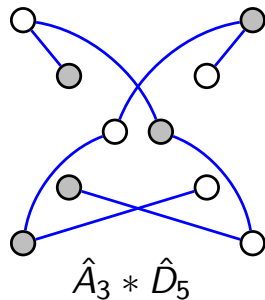
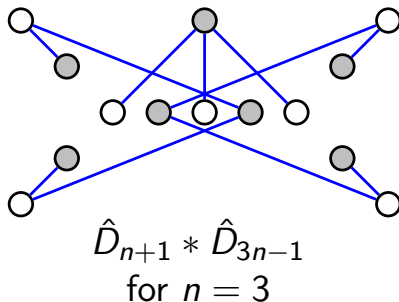
$$\hat{D}_{n+1} * \hat{D}_{3n-1}$$

for $n = 3$

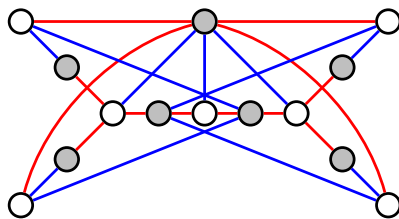


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15 infinite families and 4 exceptional cases

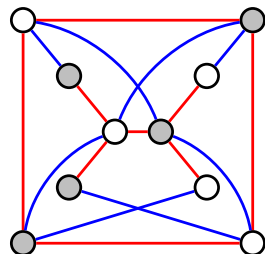


15 infinite families and 4 exceptional cases



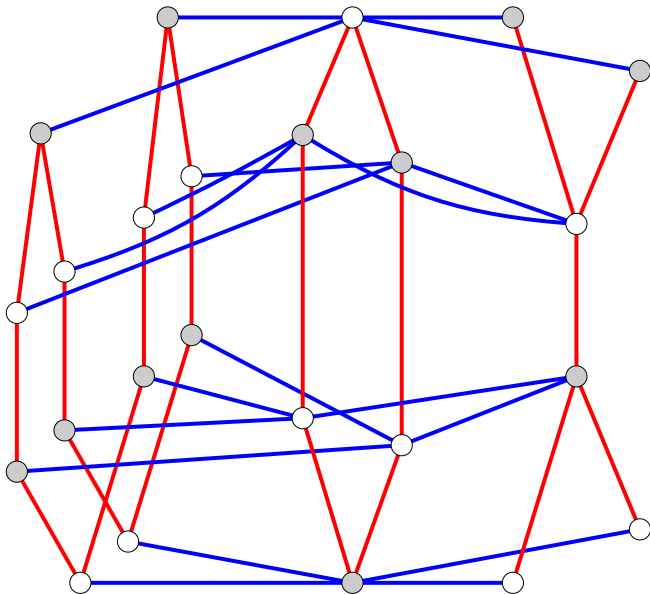
$$\hat{D}_{n+1} * \hat{D}_{3n-1}$$

for $n = 3$

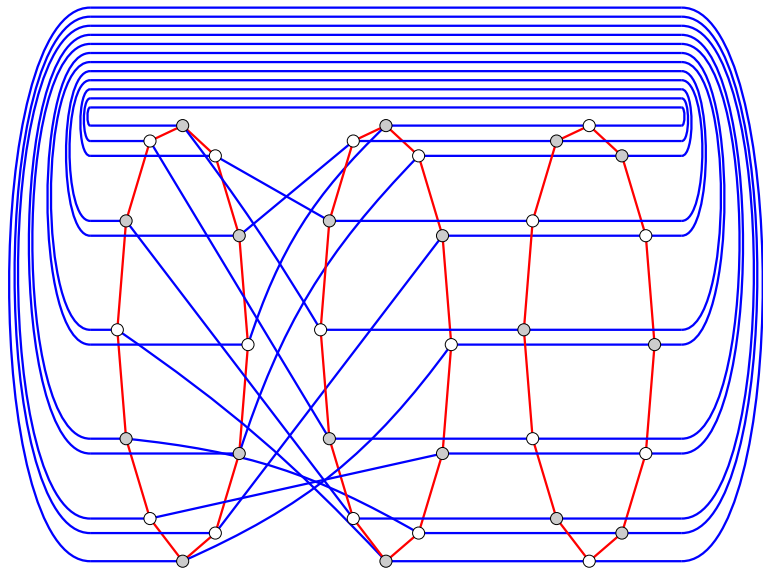


$$\hat{A}_3 * \hat{D}_5$$

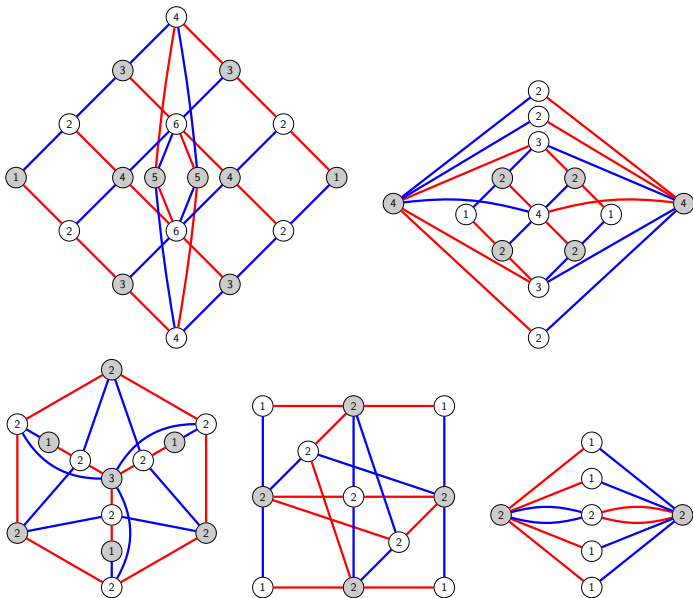
Affine $\boxtimes$ affine quivers



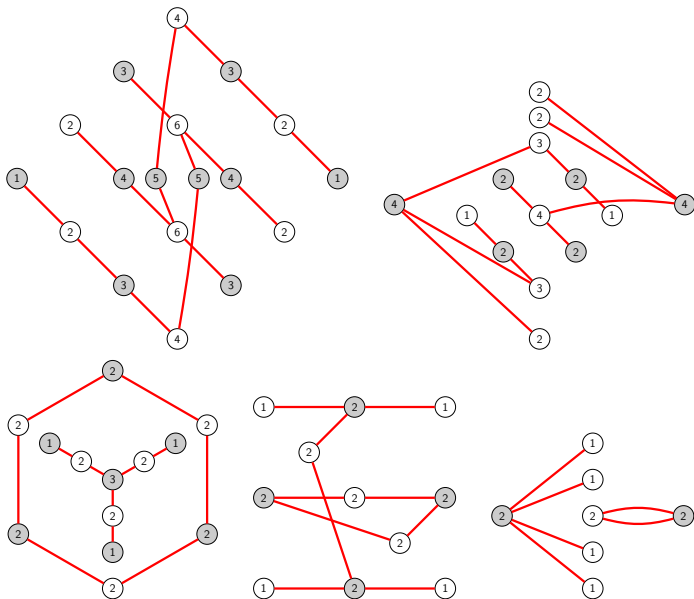
Toric quivers



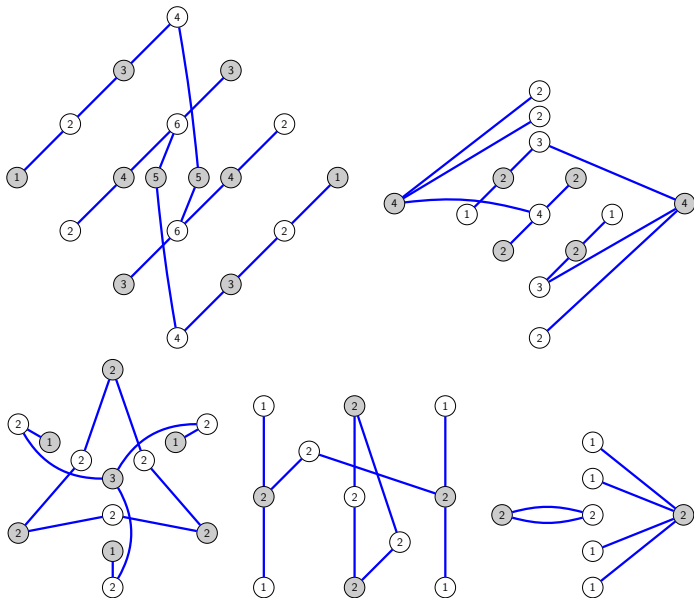
Affine $\boxtimes$ affine classification: 40 infinite, 13 exceptional



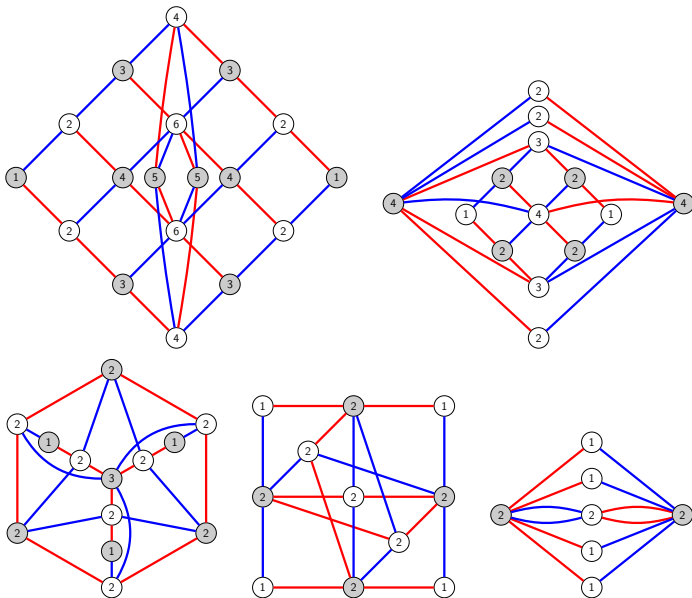
Affine $\boxtimes$ affine classification: 40 infinite, 13 exceptional



Affine $\boxtimes$ affine classification: 40 infinite, 13 exceptional



Affine $\boxtimes$ affine classification: 40 infinite, 13 exceptional



Results

Theorem (G.-Pylyavskyy, 2016)

Periodic $\iff$ *finite* $\boxtimes$ *finite*

Theorem (G.-Pylyavskyy, 2016)

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Theorem (G.-Pylyavskyy, 2017)

Grows slower than $\exp(\exp(t)) \implies$ affine $\boxtimes$ affine, affine $\boxtimes$ finite, or finite $\boxtimes$ finite

What is left:

Conjecture (G.-Pylyavskyy, 2017)

- *affine* $\boxtimes$ *finite* $\implies$ *linearizable*
- *affine* $\boxtimes$ *affine* $\implies$ *grows as $\exp(t^2)$*

Results

Theorem (G.-Pylyavskyy, 2016)

Periodic $\iff$ finite $\boxtimes$ finite

Theorem (G.-Pylyavskyy, 2016)

Linearizable $\implies$ affine $\boxtimes$ finite or finite $\boxtimes$ finite

Theorem (G.-Pylyavskyy, 2017)

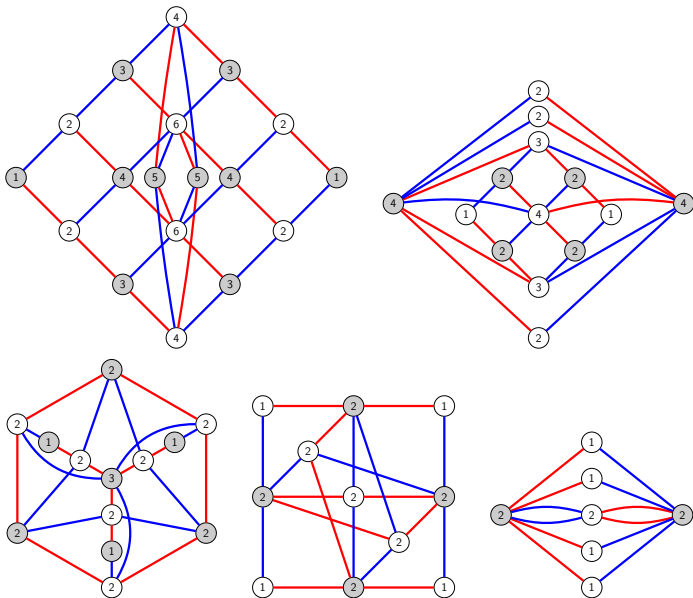
Grows slower than $\exp(\exp(t)) \implies$ affine $\boxtimes$ affine, affine $\boxtimes$ finite, or finite $\boxtimes$ finite

What is left:

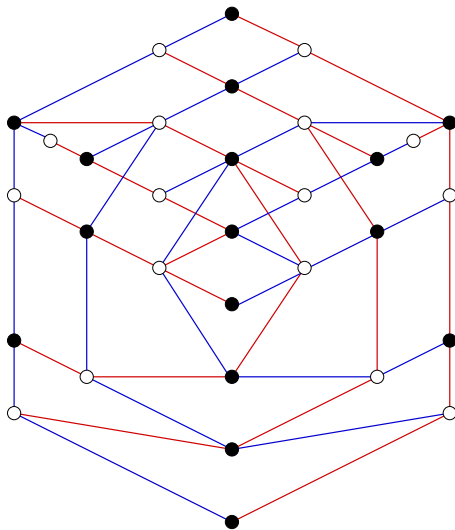
Conjecture (G.-Pylyavskyy, 2017)

- **affine $\boxtimes$ finite $\implies$ linearizable**
- **affine $\boxtimes$ affine $\implies$ grows as $\exp(t^2)$**

Affine $\boxtimes$ affine classification: 40 infinite, 13 exceptional



Stembridge's classification (2010)



"Ocneanu graphs" (2001)

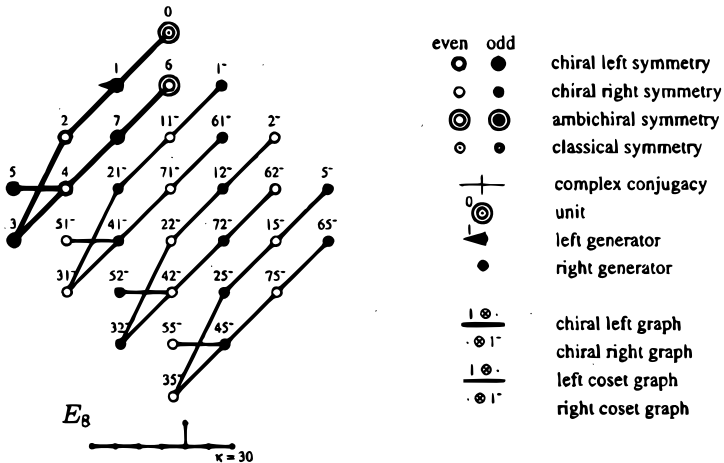
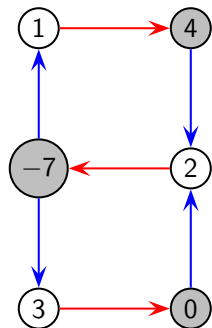


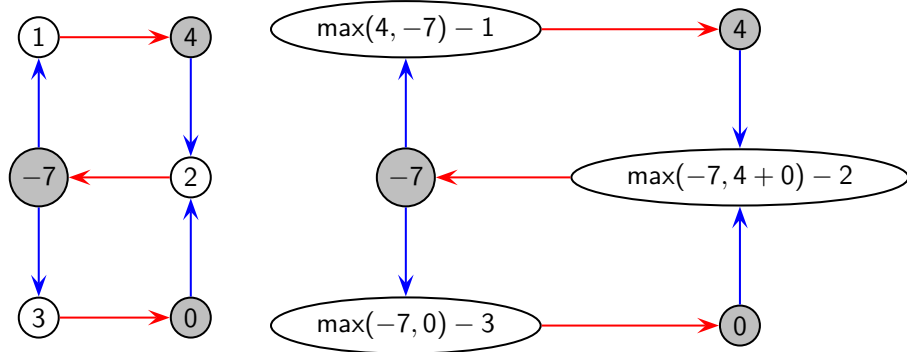
Fig. 23. Quantum symmetries for Coxeter graphs E_8 .

Part 4: Tropical T -systems

Tropical T -system



Tropical T -system



Tropical problems

Conjecture (G.-Pylyavskyy, 2017)

- *affine* $\boxtimes$ *finite* $\implies$ *tropical T -system grows linearly*
- *affine* $\boxtimes$ *affine* $\implies$ *tropical T -system grows quadratically*

Tropical problems

Conjecture (G.-Pylyavskyy, 2017)

- *affine $\boxtimes$ finite $\implies$ tropical T -system grows linearly*
- *affine $\boxtimes$ affine $\implies$ tropical T -system grows quadratically*

Problem

affine $\boxtimes$ finite quivers: show that the number of “blue mutations” is finite.

Tropical problems

Conjecture (G.-Pylyavskyy, 2017)

- *affine* $\boxtimes$ *finite* $\implies$ *tropical T-system grows linearly*
- *affine* $\boxtimes$ *affine* $\implies$ *tropical T-system grows quadratically*

Problem

affine $\boxtimes$ *finite* quivers: show that the number of “blue mutations” is finite.

“blue mutation”:
$$\sum_{(u,v)} \lambda_u > \sum_{(v,w)} \lambda_w.$$

Tropical problems

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- *affine* $\boxtimes$ *finite* $\implies$ *tropical T-system grows linearly*
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Problem

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“blue mutation”:
$$\sum_{(u,v)} \lambda_u > \sum_{(v,w)} \lambda_w.$$

Problem

affine $\boxtimes$ *affine* quivers: show that the “acceleration” is bounded.

Tropical problems

Conjecture (G.-Pylyavskyy, 2017)

- *affine* $\boxtimes$ *finite* $\implies$ *tropical T-system grows linearly*
- *affine* $\boxtimes$ *affine* $\implies$ *tropical T-system grows quadratically*

Problem

affine $\boxtimes$ *finite* quivers: show that the number of “blue mutations” is finite.

$$\text{“blue mutation”}: \sum_{(u,v)} \lambda_u > \sum_{(v,w)} \lambda_w.$$

Problem

affine $\boxtimes$ *affine* quivers: show that the “acceleration” is bounded.

$$\text{“acceleration”} = \left| \sum_{(u,v)} \lambda_u - \sum_{(v,w)} \lambda_w \right|.$$

Bibliography

Slides: <http://math.mit.edu/~galashin/slides/berkeley.pdf>

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Y-systems and generalized associahedra.
Ann. of Math. (2), 158(3):977–1018, 2003.

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The periodicity conjecture for pairs of Dynkin diagrams.
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Thank you!

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