Braid variety cluster structures.

joint w/ T. Lam
M. Sherman-Bennett
D. Speyer

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Def A *quiver* Q is a directed graph without $\circlearrowleft$, $\rightleftarrows$, vertices partitioned into *frozen* and *mutable*.

Ex 

$\boxed{i}$ frozen
j mutable
arrows between frozens omitted

Cluster structures on varieties:

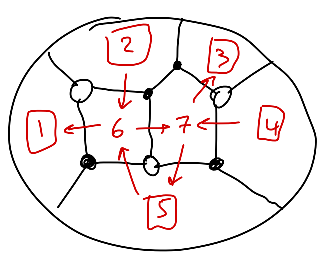
Interesting variety $X \to$ coordinate ring $\mathbb{C}[X]$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$
Quiver Q (vertices: $1,2,\dots,n$) $\to$ cluster algebra $\mathcal{A}(Q) \subset \mathbb{C}(x_1,\dots,x_n)$

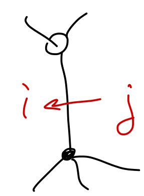
Usually: Q, X – known/easy, showing $\mathbb{C}[X] = \mathcal{A}(Q)$ – hard

Today: X – easy, Q – hard

Examples of cluster structures:

(1) X = open positroid variety [KLS'13, Postnikov'06]
Q = planar dual of a plabic graph G
plabic = planar, bicolored

Ex. G = 

arrows of Q: 

square moves  $\rightsquigarrow$ cluster mutations of Q.

Cluster structure: [G–Lam'19] ([SSBW'19], [Leclerc'14], [Muller–Speyer'14]

(2) X = Double Bruhat cell (Fomin - Zelevinsky '99)

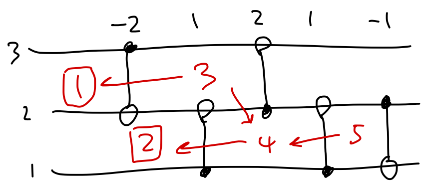
$v, w \in S_n \rightsquigarrow$ double braid word $\beta \in (\pm I)^{\ell(v)+\ell(w)}$ $\quad I = \{1, 2, \dots, n-1\}$

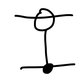
I and $-I$ commute

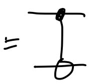
β = shuffle of (red. word for w on + indices) and (red. word for v on − indices)

Quiver: dual of plabic graph G_β

<u>Ex.</u> $v = s_2 s_1 \quad w = s_1 s_2 s_1 \in S_3$, $\quad \beta = -2\ 1\ 2\ 1\ -1$

G_β = 

i = 

$-j$ = 

<u>Braid varieties</u>

- include positroid varieties / double Bruhat cells as special cases.

- input:
  $\beta = i_1 i_2 \cdots i_m \in (\pm I)^m$
  $u \in S_n$ such that $u \leq \beta$

$\beta = i_1 \dots i_m \in (\pm I)^m$, $u \in S_n$

Def. Positive distinguished subexpr.

Set $u_m := u$. For $c = m, \dots, 2, 1$, set $u_{c-1} = \begin{cases} \min(u_c, u_c s_{i_c}), & i_c > 0 \\ \min(u_c, s_{|i_c|} u_c), & i_c < 0 \end{cases}$

Write $u \leq \beta$ if $u_0 = id$

Let $J := \{c : u_c = u_{c-1}\}$ - solid crossings

not in J - hollow crossings i_c

Def. Almost positive subexpr. Fix $d \in J$.

Set $u_m^{\langle d \rangle} = u$, $u_{c-1}^{\langle d \rangle} = \begin{cases} \min(u_c^{\langle d \rangle}, u_c^{\langle d \rangle} s_{i_c}), & i_c > 0 \\ \min(u_c^{\langle d \rangle}, s_{|i_c|} u_c^{\langle d \rangle}), & i_c < 0 \end{cases}$ for $c \neq d$, $u_{c-1}^{\langle d \rangle} = \begin{cases} \max(- \text{''} -) \\ \max(- \text{''} -) \end{cases}$ $c = d$.

"mistake"

Call $d \in J$ mutable if $u_0^{\langle d \rangle} = id$, frozen otherwise. [BFZ '05] [Ingermanson '19]

"Grid minors": (Type A only!) $\Delta_{c,i} = \prod_{d \in J:\ u_c[i] \neq u_c^{\langle d \rangle}[i]} x_d$ $\quad [i] = \{1, \dots, i\}$ $\qquad \Delta_{c,-i} = \prod_{d \in J:\ u_c^{-1}[i] \neq (u_c^{\langle d \rangle})^{-1}[i]} x_d$

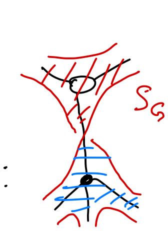
Quiver $Q_{u,\beta}$ For each $c \in J$, sum up half-arrows.

Alternative construction:

Def (Goncharov–Kenyon '13) G - plabic graph $\Rightarrow$ G - ribbon graph

clockwise/cclw edge order around black/white vertices.

Conjugate surface S_G:



Faces of G $\to$ some cycles inside S_G

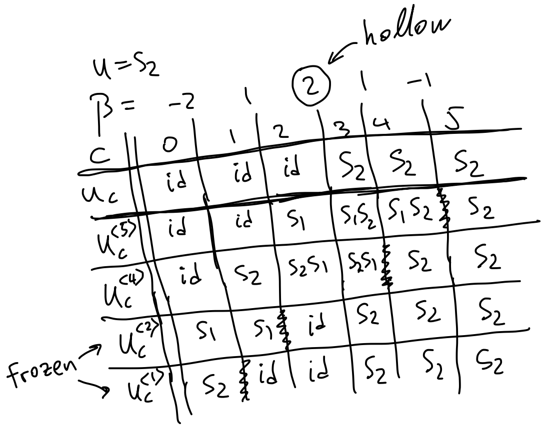
moves preserve S_G. $\boxed{Q_G \Leftrightarrow \text{intersection form of } S_G}$

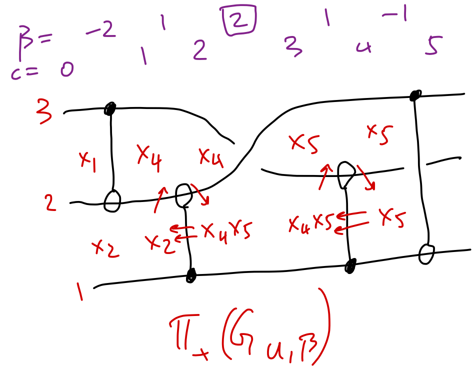
Our construction:

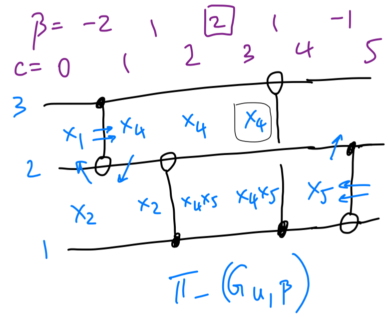
$\Pi_+(G_{u,\beta})$, $\Pi_-(G_{u,\beta})$ are projections of a 3D plabic graph $G_{u,\beta}$.

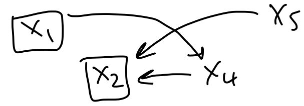
Regions where x_d appears: projections of a 2-disk D_d, ∂D_d - cycle in $G_{u,\beta}$.

Quiver $Q_{u,\beta}$ - intersection form of conjugate surface of $G_{u,\beta}$


u=s2
β= -2 1 2 1 -1
hollow
c 0 1 2 3 4 5
frozen


β= -2 1 2 1 -1
c= 0 1 2 3 4 5
x1 x4 x4 x5 x5
x2 x2 x4x5 x4x5 x5
Π+(Gu,β)


β= -2 1 2 1 -1
c= 0 1 2 3 4 5
x1 x4 x4 x4
x2 x2 x4x5 x4x5 x5
Π−(Gu,β)


x1 x5
x2 x4