R-systems

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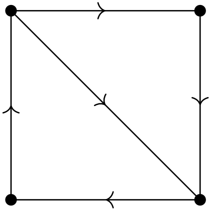
Joint work with Pavlo Pylyavskyy

Part 1: Definition

A system of equations

Let $G = (V, E)$ be a *strongly connected digraph*.



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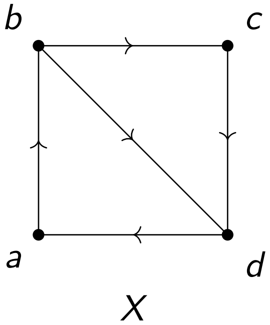

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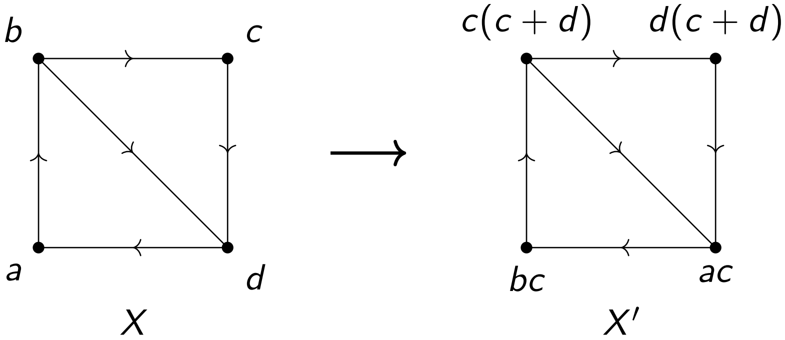
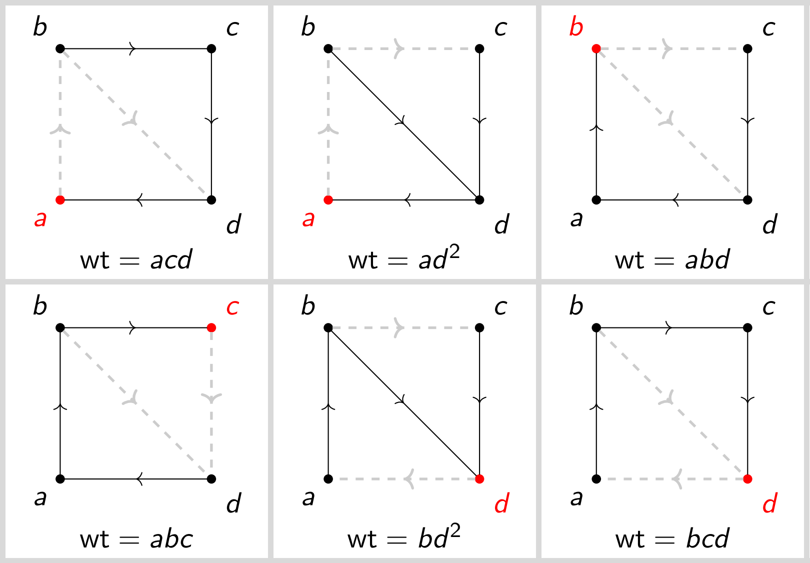


Solution

Theorem (G.-Pylyavskyy, 2017)

Let $G = (V, E)$ be a strongly connected digraph. Then there exists a birational map $\phi : \mathbb{P}^V(\mathbb{K}) \dashrightarrow \mathbb{P}^V(\mathbb{K})$ such that $X, X' \in \mathbb{P}^V(\mathbb{K})$ give a solution if and only if $X' = \phi(X)$.

Arborescence formula

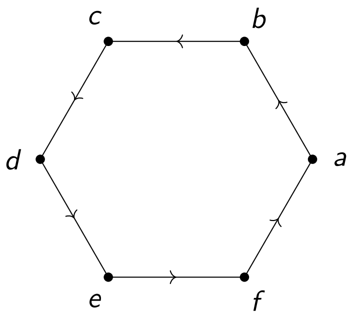


The R-system

Definition

Let $G = (V, E)$ be a strongly connected digraph. Then the *R-system associated with G* is a discrete dynamical system on $\mathbb{P}^V(\mathbb{K})$ that consists of iterating the map ϕ.

Boring examples: a directed cycle



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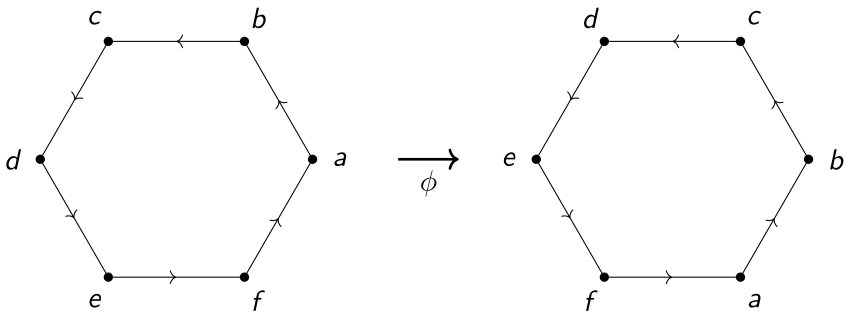

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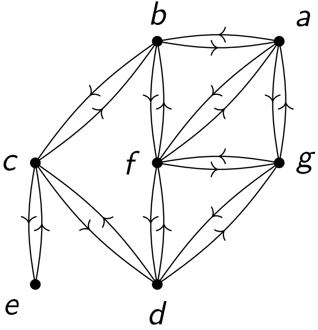

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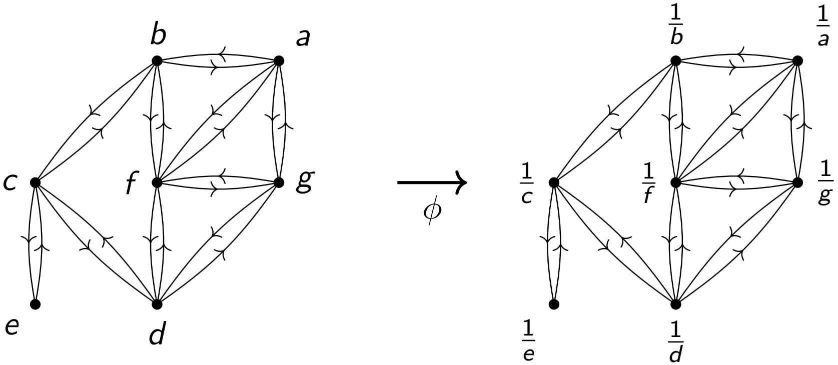

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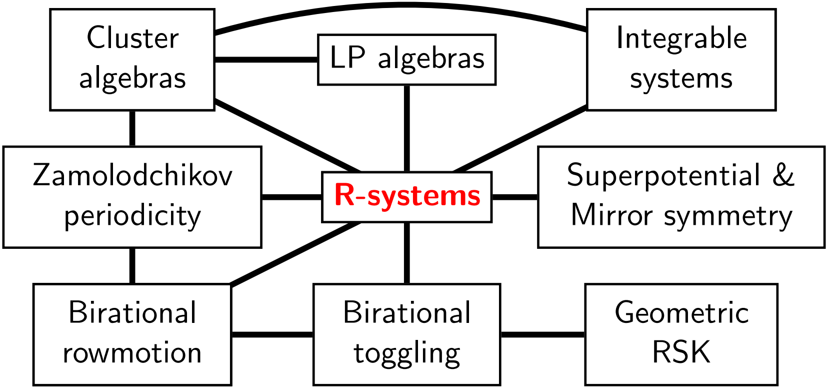
Cluster algebras
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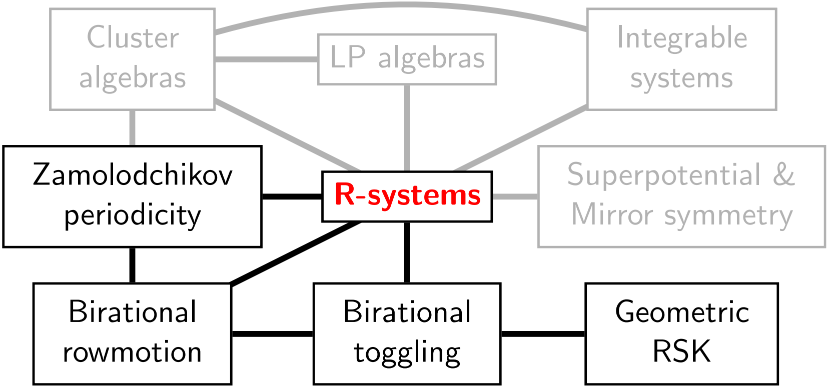
Birational toggling
Geometric RSK

Part 2: Toggle!

Map



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Birational toggling

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Definition (Kirillov (2001), Einstein-Propp (2013))

Birational toggle operation:

$$X'_v X_v = \left(\sum_{v \lessdot w} X_w \right) \left(\sum_{u \lessdot v} \frac{1}{X_u} \right)^{-1}.$$

Birational rowmotion for the product of two chains

Theorem (Grinberg-Roby, 2015)

For $P = [n] \times [k]$, birational rowmotion is periodic with period $n + k$.

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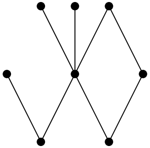
Proposition (Glick, 2016)

There is a simple monomial transformation that shows that the two theorems above are equivalent.

Birational rowmotion $\subseteq$ R-systems



P

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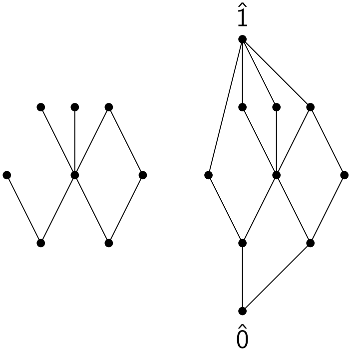

P $\hat{P}$

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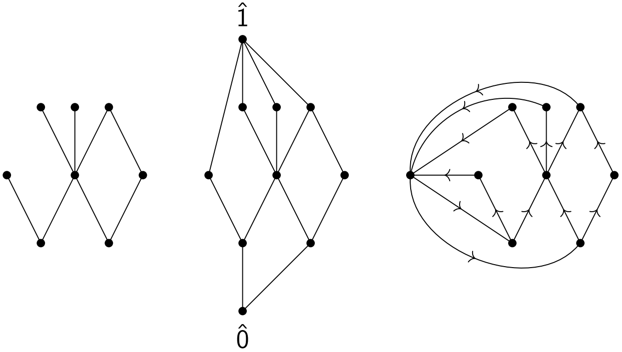

P $\hat{P}$ $G(P)$

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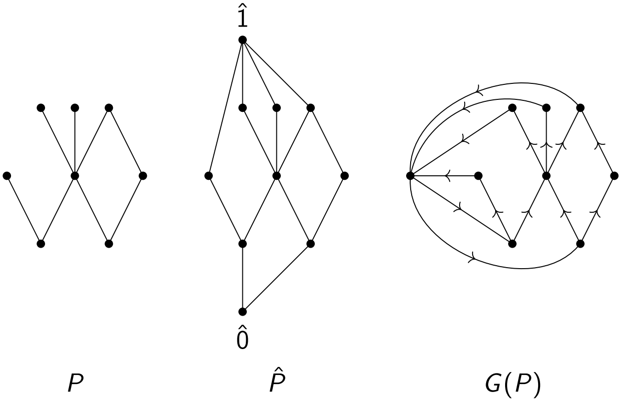

Proposition (G.-Pylyavskyy, 2017)

Birational rowmotion on P $=$ R-system associated with $G(P)$.

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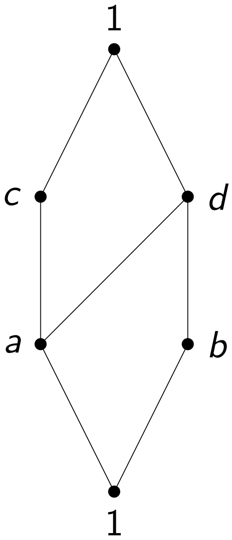

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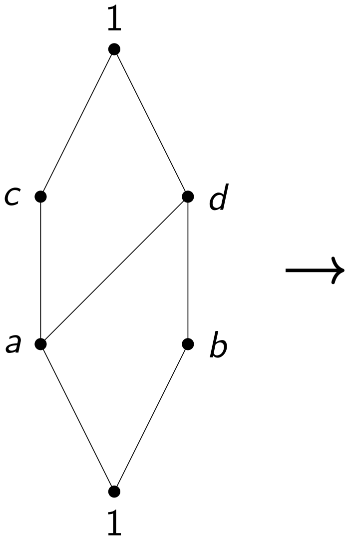

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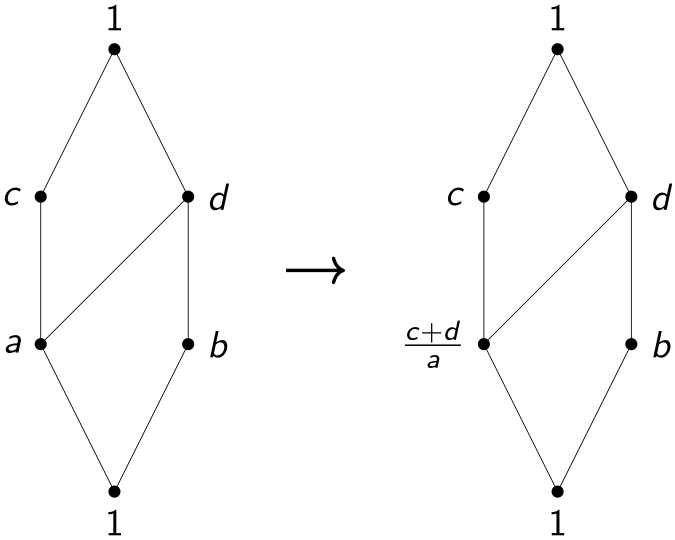

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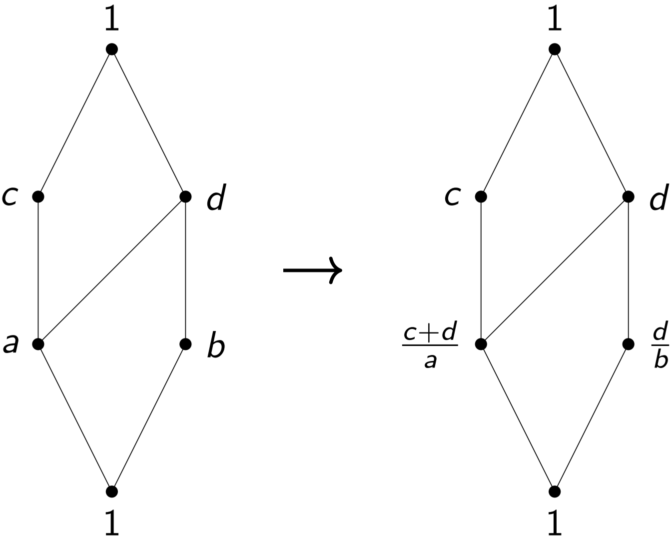

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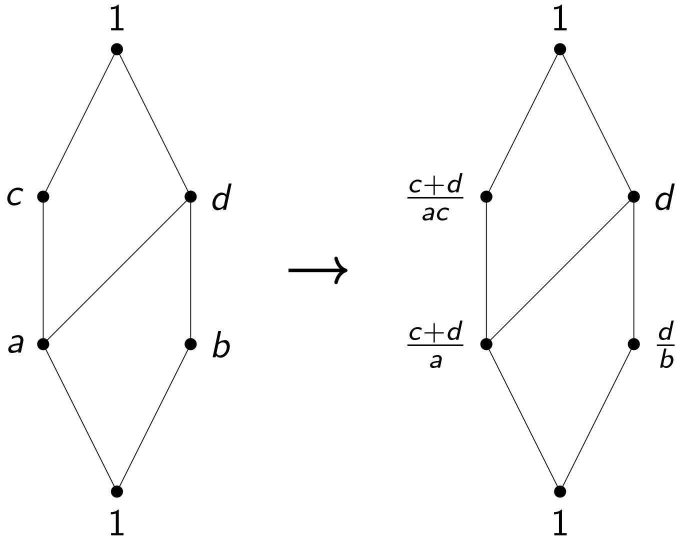

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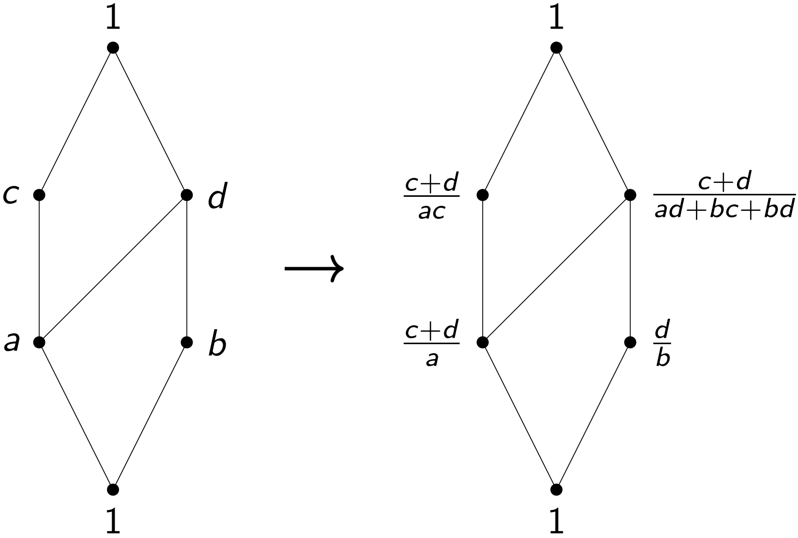

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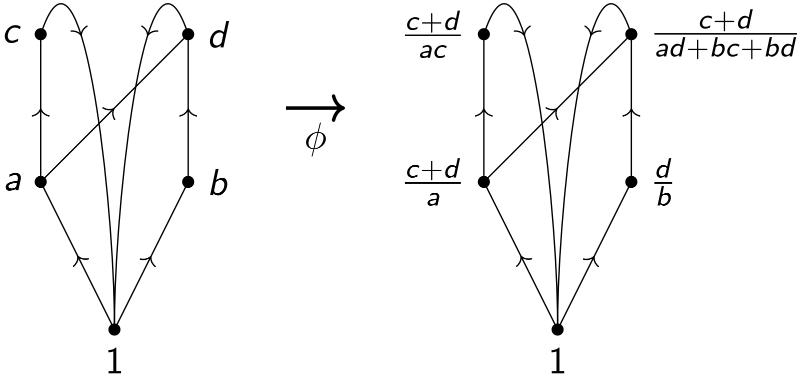

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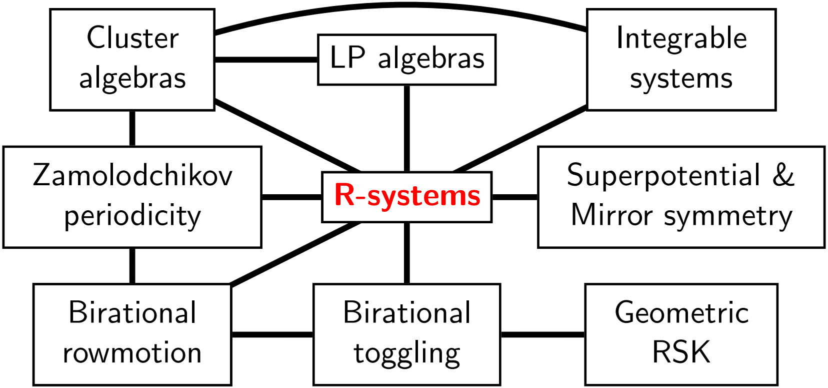

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Part 3: Singularity confinement

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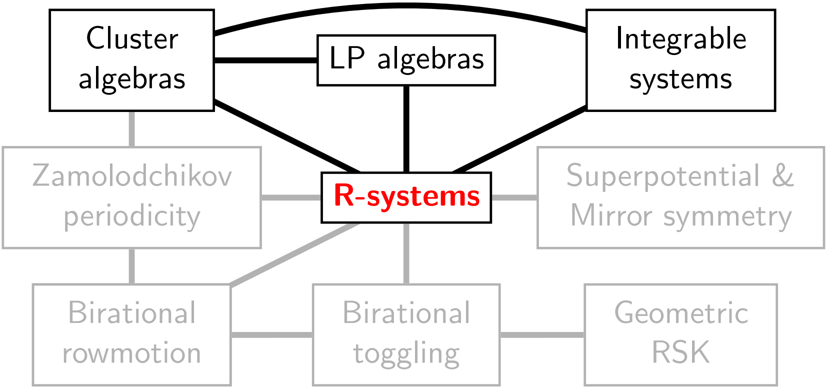

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The Laurent phenomenon

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(Formerly M0857)

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Theorem (Fomin-Zelevinsky, 2002)

For each $n > 4$, τ_n is a Laurent polynomial in $\alpha, \beta, \tau_1, \tau_2, \tau_3, \tau_4$.

Singularity confinement

Consider a mapping of the plane $(x_{n-1}, x_n) \mapsto (x_n, x_{n+1})$ given by

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substitute $x_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}$

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$$\tau_4 = \alpha x_2 + \beta$$

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$$\tau_9 = \alpha^3\beta^3 x_1^6 x_2^8 + \cdots + \alpha\beta^8$$

Moral

Sequence with Laurent property

↓ monomial transformation

Integrable mapping with singularity confinement

Moral

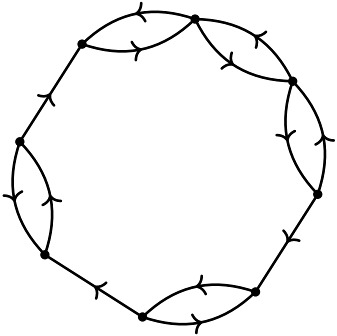
Sequence with Laurent property

$\downarrow$ monomial transformation

Integrable mapping with singularity confinement

Lots of R-systems exhibit singularity confinement!

Examples: subgraphs of a bidirected cycle



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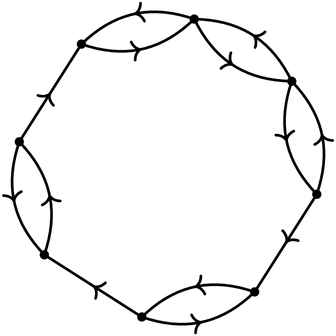
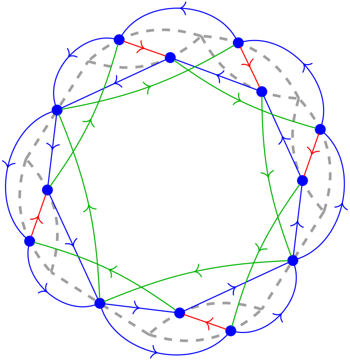
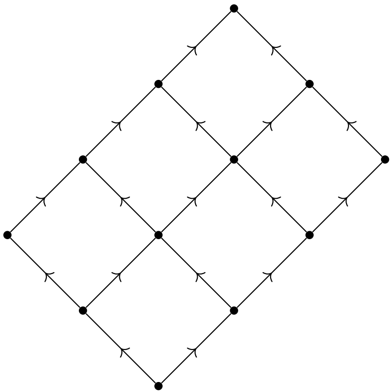




Controlled by a cluster algebra

Examples: rectangle posets (Grinberg-Roby)



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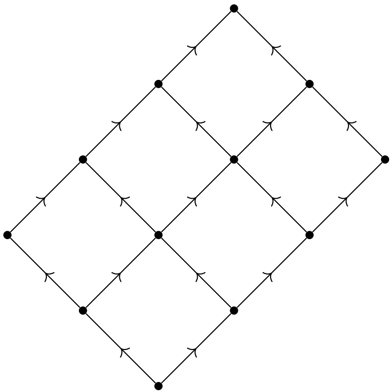
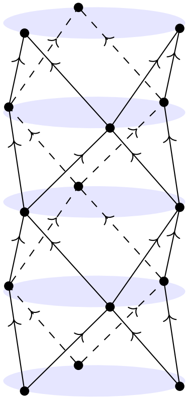


Controlled by a Y-system

Examples: cylindric posets



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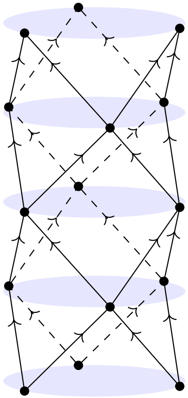
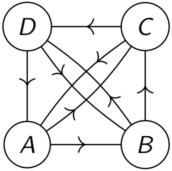
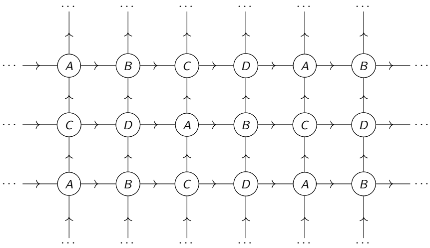


Controlled by an LP algebra

Examples: toric digraphs





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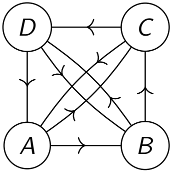

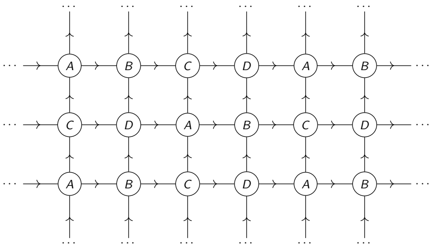

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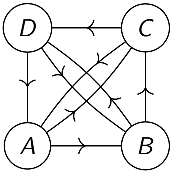

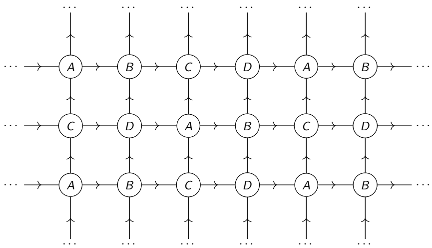

Controlled by ??? $R_v(t) = \frac{\tau_v(t-1)}{\tau_v(t)}$;

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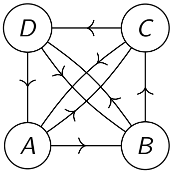

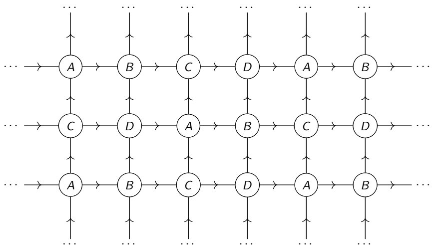

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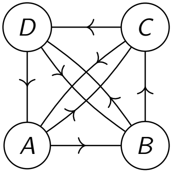

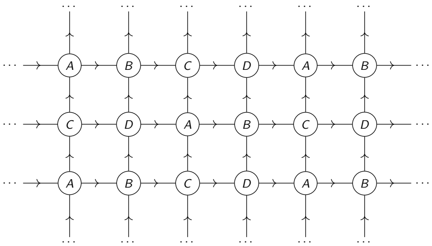

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Coefficients

Definition

Coefficient-free R-system:

$$X_v X_v' = \left(\sum_{v \to w} X_w\right) \left(\sum_{u \to v} \frac{1}{X_u'}\right)^{-1}.$$

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R-system with coefficients:

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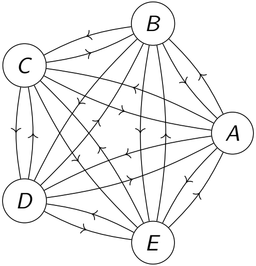
Example: the universal R-system



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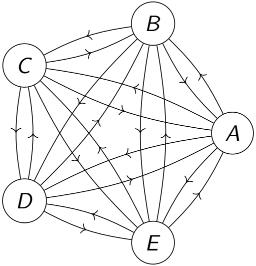

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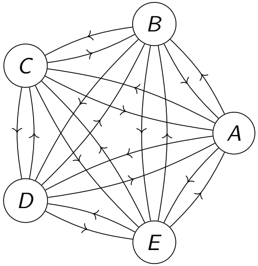

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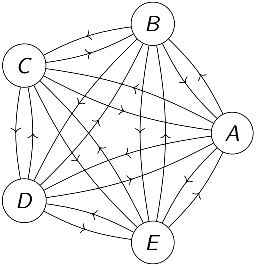

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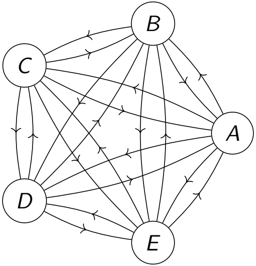

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