Parity Duality for the Amplituhedron

joint w/ T. Lam

1. Totally Nonnegative Grassmannian.

Def. $Gr(k,n) = \{ M \in Mat_{k \times n}(\mathbb{R}) \mid rk M = k \} / (\text{row operations})$

$\dim Gr(k,n) = k(n-k)$

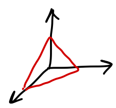
Plücker coord: Δ_I = $k \times k$ minor on columns I.

$I \subset \{1,2,\dots,n\}$, $|I| = k$

Def (Lusztig/Postnikov)

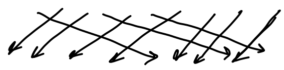
$$Gr_{\geq 0}(k,n) = \{ V \in Gr(k,n) \mid \Delta_I(V) \geq 0 \}$$

Ex. $Gr(1,n) = \mathbb{R}\mathbb{P}^{n-1}$, $Gr_{\geq 0}(1,n) = \{ (x_1 : x_2 : \dots : x_n) \mid x_i \geq 0 \} \cong \Delta^{n-1}$ simplex



Cell decomposition of $Gr_{\geq 0}(k,n)$:

dim	"Matroid"	# cells	$\tilde{S}_n(-k, n-k)$
$k(n-k)$	All $\Delta_I > 0$	1	id
$k(n-k)-1$	$\Delta_{\{i, i+1, \dots, i+k-1\}} = 0$ (mod n), rest > 0	n	s_i, $i = 1, \dots, n$
...	some $\Delta_I = 0$, rest > 0	...	...
0	all but one $\Delta_I = 0$	$\binom{n}{k}$	$f_I(i) = \begin{cases} i+n-k, & i \in I \\ i-k, & i \notin I \end{cases}$



Thm (Rietsch/Postnikov): each cell (some $\Delta_I = 0$, rest > 0) is an open ball

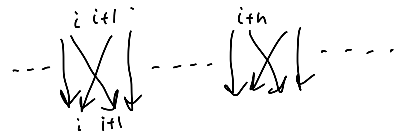
Thm. (G.-Karp-Lam)

- $Gr_{\geq 0}(k,n)$ is a closed ball
- each cell closure (some $\Delta_I = 0$, rest ≥ 0) is a closed ball.

2. Affine permutations.

Def: $\tilde{S}_n := \left\{ f : \mathbb{Z} \to \mathbb{Z} \text{ bijection} \;\middle|\; \begin{array}{l} (1)\ f(i+n) = f(i) \quad \forall i \\ (2)\ \sum_{i=1}^{n} (f(i) - i) = 0 \end{array} \right\}$

$\tilde{S}_n$ is a Coxeter group gen'd by $s_i =$



Def. For $a, b \geq 0$, $\tilde{S}_n(-a, b) := \{ f \in \tilde{S}_n \mid i - a \leq f(i) \leq i + b \;\; \forall i \}$.

Thm. (Postnikov/Knutson-Lam-Speyer):
Boundary cells of $Gr_{\geq 0}(k,n)$ are in natural bijection with $\tilde{S}_n(-k, n-k)$.

$$Gr_{\geq 0}(k,n) = \bigsqcup_{f \in \tilde{S}_n(-k,n-k)} \Pi_f^{>0}$$

$$\operatorname{codim} \Pi_f^{>0} = \operatorname{inv}(f) = \#\{ i < j \mid f(i) > f(j),\ 1 \leq i \leq n \} \qquad \Pi_f^{>0} \cong \mathbb{R}^{k(n-k) - \ell(f)}$$

$\uparrow$ Coxeter length

$\boxed{k+l+m=n, \quad m\text{-even}}$

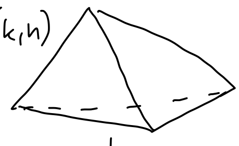
3. Amplituhedron (Arkani-Hamed & Trnka)

Def. Fix $Z \in Gr_{>0}(k+m, n)$.

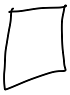
$$\mathcal{A}_{klmn}(Z) := \{ V \cdot Z^T \mid V \in Gr_{\geq 0}(k,n) \} \subset Gr(k, k+m).$$

$$k\underbrace{\boxed{\quad V \quad}}_{n} \cdot n\underbrace{\boxed{Z^T}}_{k+m}$$

Fact: $V \cdot Z^T$ has rank k

$Gr_{\geq 0}(k,n)$ 

$\downarrow \cdot Z^T$

$\mathcal{A}_{klmn}(Z)$  $\subset Gr(k, k+m)$

Ex. $k=1$ $\quad Gr_{\geq 0}(k,n) \cong \Delta^{n-1}$,

$$\mathcal{A}_{klmn}(Z) = \text{cyclic polytope } C(n,m) \subset \mathbb{P}^m = Gr(1, 1+m)$$

4. Triangulations of $\mathcal{A}_{klmn}(Z)$.

$\dim \Pi_f^{>0} = k(n-k) - \mathrm{inv}(f) = k(l+m) - \mathrm{inv}(f)$

$\dim \mathcal{A}_{klmn}(Z) = km$

Thus, $\dim \Pi_f^{>0} = \dim \mathcal{A}_{klmn}(Z) \iff \boxed{\mathrm{inv}(f) = kl}$

Def. $f \in \tilde{S}_n(-k, n-k)$ is *Z-admissible* if $\mathrm{inv}(f) = kl$ and

$Z|_{\Pi_f^{>0}}$ is injective.

Lemma $f \in \tilde{S}_n(-k, \underbrace{n-k}_{l+m})$ is Z-adm. $\Rightarrow f \in \tilde{S}_n(-k, l)$

Def. $f_1, \dots, f_N \in \tilde{S}_n(-k, n-k)$ form a *Z-triangulation* if

- each f_i is Z-adm
- images $Z(\Pi_{f_i}^{>0})$ do not overlap
- closures of images cover entire $\mathcal{A}_{klmn}(Z)$.

Physics conjecture. Combinatorics of $\mathcal{A}_{klmn}(Z)$ does not depend on $Z \in Gr_{>0}(k+m, n)$.

5. Main Theorem.

<u>Def.</u> - f is <u>$k\ell mn$-admissible</u> if it is Z-adm. for all Z.

- $f_1, \dots, f_N$ form a <u>$k\ell mn$-triangulation</u> if they form a Z-triangulation for all Z.

<u>Thm</u> (G-Lam)

(1) $f \in \tilde{S}_n(-k,\ell)$ is $k\ell mn$-adm.

$$\Updownarrow$$

$f^{-1} \in \tilde{S}_n(-\ell,k)$ is ℓkmn-adm.

(2) $f_1, \dots, f_N$ form a $k\ell mn$-triang

$$\Updownarrow$$

$f_1^{-1}, \dots, f_N^{-1}$ form a ℓkmn-triang.

Note: $\dim \Pi_f^{>0} = km$

$\dim \Pi_{f^{-1}}^{>0} = \ell m$

<u>Ex</u> triangs of $\ell=1$ ampl $\xleftrightarrow{1:1}$ triangs of cyclic polytope $C(n,m)$.

6. Combinatorial motivation.

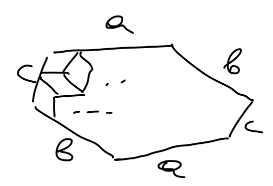
<u>Conj.</u> (Karp-Williams-Zhang)

$\#\{\text{cells in a triangulation of } A_{k\ell mn}(Z)\} = M(k,\ell,\frac{m}{2})$

where $M(a,b,c) = \prod_{p=1}^{a} \prod_{q=1}^{b} \prod_{r=1}^{c} \frac{p+q+r-1}{p+q+r-2} = \#\{\text{rhombus tilings of } a\times b\times c \text{ hexagon}\}$

$\uparrow$

symmetric in a,b,c.



7. Proof idea.

Twist map (Marsh–Scott, Muller–Speyer)

$\tau : \mathrm{Mat}^\circ(k,n) \to \mathrm{Mat}^\circ(k,n)$

Mat°: $\Delta_{1\cdots k} \neq 0$, $\Delta_{2\cdots k+1} \neq 0$, $\vdots$, $\Delta_{n 1 \cdots k-1} \neq 0$

$\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} \to \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$

Let $u_{i+n} := u_i$. $\quad v_i \in \mathbb{R}^k$ is defined by

$\langle v_i, u_i \rangle = 1$

$\langle v_i, u_{i+1} \rangle = \cdots = \langle v_i, u_{i+k-1} \rangle = 0.$

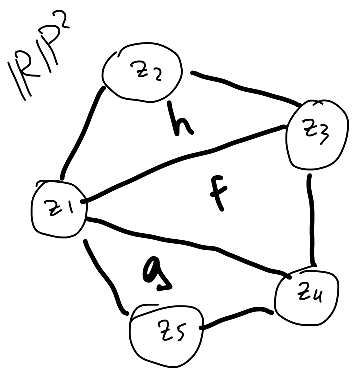
Key lemma:

$$\begin{array}{c|c|} k & V \\ \hline \ell & Z^\perp \\ \end{array}_{\,n} \xrightarrow{\tau} \begin{array}{c|c|} k & \tilde{Z}^\perp \\ \hline \ell & \tilde{V} \\ \end{array}_{\,n}$$

If $V \in \Pi_f^{>0} \subset \mathrm{Gr}_{\geq 0}(k,n)$, $Z \in \mathrm{Gr}_{>0}(k+m,n)$ $\Rightarrow$ $\tilde{Z} \in \mathrm{Gr}_{>0}(\ell+m,n)$, $\tilde{V} \in \Pi_{f^{-1}}^{>0} \subset \mathrm{Gr}_{\geq 0}(\ell,n)$.

"stacked twist map".

Example:



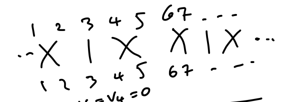
$\mathcal{A}_{1225}(Z)$ = pentagon

$\boxed{k=1, \ell=2, m=2, n=5}$

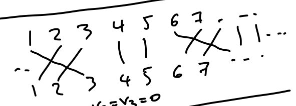
$\tilde{S}_5(-1,4)$ $\quad$ inv $= k\ell = 2$

$\dim \mathcal{A}_{1225}(Z) = km = 2$

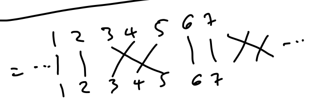
clearly form a triang

$f =$


$v_2 = v_4 = 0$

$g =$


$v_2 = v_3 = 0$

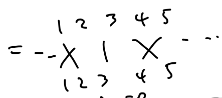
$h =$


$v_4 = v_5 = 0$

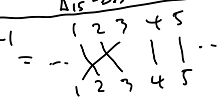
$\boxed{k=2, \ell=1, m=2, n=5}$

$\tilde{S}_5(-2,3)$, inv $= k\ell = 2$

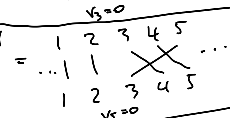
$\dim \mathcal{A}_{2125}(Z) = km = 4$

$f^{-1} =$


$\Delta_{15} = \Delta_{23} = 0$

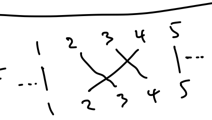
$g^{-1} =$


$v_3 = 0$

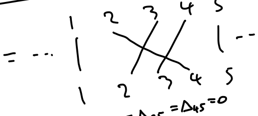
$h^{-1} =$


$v_5 = 0$

presumably (?) form a triangulation

$\pi^{-1} =$


$\notin \tilde{S}_5(-1,4)$

$\pi =$


$\in \tilde{S}_5(-2,3)$

not Z-adm!

$\Delta_{34} = \Delta_{35} = \Delta_{45} = 0$

$\underbrace{\begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{bmatrix}}_{5,\ \in\ \mathrm{Gr}_{\geq 0}(1,5)}{}_{1} \cdot \underbrace{\begin{bmatrix} - z_1 - \\ - z_2 - \\ - z_3 - \\ - z_4 - \\ - z_5 - \end{bmatrix}}_{3}{}_{5} = {}_1\underbrace{\begin{bmatrix} a : b : c \end{bmatrix}}_{3} \in \mathbb{RP}^2$

$\underbrace{\begin{bmatrix} | & | & | & | & | \\ v_1 & v_2 & v_3 & v_4 & v_5 \\ | & | & | & | & | \end{bmatrix}}_{5,\ \in\ \mathrm{Gr}_{\geq 0}(2,5)}{}_{2} \cdot \underbrace{\begin{bmatrix} - z_1 - \\ - z_2 - \\ - z_3 - \\ - z_4 - \\ - z_5 - \end{bmatrix}}_{4}{}_{5} = {}_2\underbrace{\begin{bmatrix} \ ? \ \end{bmatrix}}_{4} \in \mathrm{Gr}(2,4)$