Refined dual stable Grothendieck polynomials and generalized Bender-Knuth involutions

Gaku Liu

Joint work with Pavel Galashin and Darij Grinberg

MIT

FPSAC 2016
Grothendieck polynomials and their variations are $K$-theory analogues of Schubert and Schur polynomials.
Grothendieck polynomials and their variations are $K$-theory analogues of Schubert and Schur polynomials.

- **Grothendieck polynomials** (Lascoux-Schützenberger ’82): polynomial representatives of structure sheaves of Schubert varieties in the $K$-theory of flag manifolds
What is a dual stable Grothendieck polynomial?

Generalized BK involutions

Refined dual stable Grothendieck polynomials

History

Grothendieck polynomials and their variations are $K$-theory analogues of Schubert and Schur polynomials.

- **Grothendieck polynomials** (Lascoux-Schützenberger ’82): polynomial representatives of structure sheaves of Schubert varieties in the $K$-theory of flag manifolds

- **stable Grothendieck polynomials** (Fomin-Kirillov ’96): symmetric power series representatives of structure sheaves of Schubert varieties in the $K$-theory of the Grassmannian
Grothendieck polynomials and their variations are $K$-theory analogues of Schubert and Schur polynomials.

- **Grothendieck polynomials** (Lascoux-Schützenberger ’82): polynomial representatives of structure sheaves of Schubert varieties in the $K$-theory of flag manifolds

- **stable Grothendieck polynomials** (Fomin-Kirillov ’96): symmetric power series representatives of structure sheaves of Schubert varieties in the $K$-theory of the Grassmannian

- **dual stable Grothendieck polynomials** (Lam-Pylyavskyy ’07): symmetric functions which are the continuous dual basis to the stable Grothendieck polynomials with respect to the Hall inner product
Reverse plane partitions

A reverse plane partition (rpp) is a filling of a skew diagram $\lambda/\mu$ with positive integers such that entries are weakly increasing along rows and columns.
We define the *irredundant content* of an rpp $T$ to be the sequence $c(T) = (c_1, c_2, c_3, \ldots)$ where $c_i$ is the number of *columns* of $T$ which contain an $i$.

\[
\begin{array}{ccc}
1 & 1 & 3 \\
& 1 & 1 \\
& & 2 & 2 \\
& & & 1 & 3 & 4 \\
& & & & 2 & 3 \\
\end{array}
\]

$c(T) = (3, 3, 2, 1, 0, 0, \ldots)$
For each skew shape $\lambda/\mu$, define

$$g_{\lambda/\mu} = \sum_{T \text{ is an rpp of shape } \lambda/\mu} x^{c(T)}$$

where $x^{(c_1, c_2, c_3, \ldots)} = x_1^{c_1} x_2^{c_2} x_3^{c_3} \ldots$. 
For each skew shape $\lambda/\mu$, define

$$g_{\lambda/\mu} = \sum_{T \text{ is an rpp of shape } \lambda/\mu} x^c(T)$$

where $x^{(c_1, c_2, c_3, \ldots)} = x_1^{c_1} x_2^{c_2} x_3^{c_3} \ldots$.

The $g_{\lambda/\mu}$ are called dual stable Grothendieck polynomials.
Dual stable Grothendieck polynomials are symmetric

Theorem (Lam-Pylyavskyy '07)

For every $\lambda/\mu$, the power series $g_{\lambda/\mu}$ is symmetric in the $x_i$. 
Dual stable Grothendiecks are symmetric

Theorem (Lam-Pylyavskyy '07)

For every $\lambda/\mu$, the power series $g_{\lambda/\mu}$ is symmetric in the $x_i$.

Their proof uses Fomin-Greene operators—fundamentally combinatorial, but the combinatorics are mysterious.
Dual stable Grothendiecks are symmetric

Theorem (Lam-Pylyavskyy '07)

For every $\lambda/\mu$, the power series $g_{\lambda/\mu}$ is symmetric in the $x_i$.

Their proof uses Fomin-Greene operators—fundamentally combinatorial, but the combinatorics are mysterious.

Our result: A bijective proof of this theorem.
Theorem (Lam-Pylyavskyy ’07)

For every $\lambda/\mu$, the power series $g_{\lambda/\mu}$ is symmetric in the $x_i$. 

Their proof uses Fomin-Greene operators—fundamentally combinatorial, but the combinatorics are mysterious.

Our result: A bijective proof of this theorem.

- Bijection is a generalization of the Bender-Knuth involutions for semistandard tableaux.
Schur functions

A *semistandard Young tableau (SSYT)* is a filling of a skew diagram $\lambda/\mu$ with positive integers such that entries are weakly increasing along rows and *strictly* increasing down columns.
A semistandard Young tableau (SSYT) is a filling of a skew diagram $\lambda/\mu$ with positive integers such that entries are weakly increasing along rows and strictly increasing down columns.

For each skew shape $\lambda/\mu$, define the Schur function

$$s_{\lambda/\mu} = \sum_{T \text{ is a SSYT of shape } \lambda/\mu} x^{c(T)}.$$
A semistandard Young tableau (SSYT) is a filling of a skew diagram $\lambda/\mu$ with positive integers such that entries are weakly increasing along rows and strictly increasing down columns.

For each skew shape $\lambda/\mu$, define the Schur function

$$s_{\lambda/\mu} = \sum_{T \text{ is a SSYT of shape } \lambda/\mu} x^{c(T)}.$$

The Bender-Knuth involutions are a way to prove the $s_{\lambda/\mu}$ are symmetric.
Suffices to show that $s_{\lambda/\mu}$ is symmetric in the variables $x_i$ and $x_{i+1}$ for all $i$. 
Bender-Knuth involutions

Suffices to show that $s_{\lambda/\mu}$ is symmetric in the variables $x_i$ and $x_{i+1}$ for all $i$.

Let $\text{SSYT}(\lambda/\mu)$ be the set of all SSYT’s of shape $\lambda/\mu$. 
Suffices to show that $s_{\lambda/\mu}$ is symmetric in the variables $x_i$ and $x_{i+1}$ for all $i$.

Let $\text{SSYT}(\lambda/\mu)$ be the set of all SSYT’s of shape $\lambda/\mu$.

For each $i$, we define an involution $B_i : \text{SSYT}(\lambda/\mu) \rightarrow \text{SSYT}(\lambda/\mu)$ such that $c(B_i T) = s_i c(T)$, where $s_i$ is the permutation $(i \ i + 1)$. 
Generalized Bender-Knuth involutions

To prove $g_{\lambda/\mu}$ is symmetric, suffices to show it is symmetric in the variables $x_i$ and $x_{i+1}$ for all $i$. 
Generalized Bender-Knuth involutions

To prove $g_{\lambda/\mu}$ is symmetric, suffices to show it is symmetric in the variables $x_i$ and $x_{i+1}$ for all $i$.

Let $\text{RPP}(\lambda/\mu)$ be the set of all RPP’s of shape $\lambda/\mu$. 

Generalized Bender-Knuth involutions

To prove \( g_{\lambda/\mu} \) is symmetric, suffices to show it is symmetric in the variables \( x_i \) and \( x_{i+1} \) for all \( i \).

Let \( \text{RPP}(\lambda/\mu) \) be the set of all RPP’s of shape \( \lambda/\mu \).

For each \( i \), we define an involution \( B_i : \text{RPP}(\lambda/\mu) \rightarrow \text{RPP}(\lambda/\mu) \) such that \( c(B_i T) = s_i c(T) \), where \( s_i \) is the permutation \((i \ i + 1)\).
Three types of columns

Restricting an rpp to cells with entries 1 or 2, we have three types of columns:
Three types of columns

Restricting an rpp to cells with entries 1 or 2, we have three types of columns:

- **1-pure**: Contains 1’s and no 2’s.
Three types of columns

Restricting an rpp to cells with entries 1 or 2, we have three types of columns:

- **1-pure**: Contains 1’s and no 2’s.
- **mixed**: Contains both 1’s and 2’s.
Restricting an rpp to cells with entries 1 or 2, we have three types of columns:

- **1-pure**: Contains 1’s and no 2’s.
- **mixed**: Contains both 1’s and 2’s.
- **2-pure**: Contains 2’s and no 1’s.
Defintion of $B_1$

Let $T \in \text{RPP}(\lambda/\mu)$. Construct $B_1(T)$ from $T$ as follows.
Let $T \in \text{RPP}(\lambda/\mu)$. Construct $B_1(T)$ from $T$ as follows.

1. Change all 1-pure columns to 2-pure columns and all 2-pure columns to 1-pure columns (of the same size).
Defintion of $B_1$

Let $T \in \text{RPP}(\lambda/\mu)$. Construct $B_1(T)$ from $T$ as follows.

1. Change all 1-pure columns to 2-pure columns and all 2-pure columns to 1-pure columns (of the same size).
2. “Resolve desents” one at a time until none remain.
Defintion of $B_1$ 

Let $T \in \text{RPP}(\lambda/\mu)$. Construct $B_1(T)$ from $T$ as follows.

1. Change all 1-pure columns to 2-pure columns and all 2-pure columns to 1-pure columns (of the same size).
2. “Resolve descents” one at a time until none remain.

A “descent” is a pair of adjacent columns which contain a 2 immediately to the left of a 1.
Resolving descents: Example
Resolving descents: Example
Definition of $B_1$

Let $T \in \text{RPP}(\lambda/\mu)$. Construct $B_1(T)$ from $T$ as follows.

1. Change all 1-pure columns to 2-pure columns and all 2-pure columns to 1-pure columns (of the same length).
2. “Resolve descents” one at a time until none remain.
Let $T \in \text{RPP}(\lambda/\mu)$. Construct $B_1(T)$ from $T$ as follows.

1. Change all 1-pure columns to 2-pure columns and all 2-pure columns to 1-pure columns (of the same length).
2. “Resolve descents” one at a time until none remain.

How do we know that this process will terminate?
Let $T \in \text{RPP}(\lambda/\mu)$. Construct $B_1(T)$ from $T$ as follows.

1. Change all 1-pure columns to 2-pure columns and all 2-pure columns to 1-pure columns (of the same length).
2. “Resolve descents” one at a time until none remain.
   - How do we know that this process will terminate?
   - Look at positions of 1-pure and 2-pure columns.
Defintion of $B_1$

Let $T \in \text{RPP}(\lambda/\mu)$. Construct $B_1(T)$ from $T$ as follows.

1. Change all 1-pure columns to 2-pure columns and all 2-pure columns to 1-pure columns (of the same length).
2. “Resolve descents” one at a time until none remain.
   - How do we know that this process will terminate?
     - Look at positions of 1-pure and 2-pure columns.
   - How do we know the end result is unique?
Let $S$ be the set of all intermediate tableaux that can be achieved during the above algorithm.
A lemma

Let $S$ be the set of all intermediate tableaux that can be achieved during the above algorithm.

For $T, T' \in S$, write $T \xrightarrow{u} T'$ if $T'$ is obtained from $T$ by resolving a descent in columns $u, u + 1$. 
A lemma

Let $S$ be the set of all intermediate tableaux that can be achieved during the above algorithm.

For $T, T' \in S$, write $T \xrightarrow{u} T'$ if $T'$ is obtained from $T$ by resolving a descent in columns $u, u + 1$.

Write $T \xrightarrow{*} T'$ if $T'$ can be obtained from $T$ through a sequence of descent resolutions.
Let $S$ be the set of all intermediate tableaux that can be achieved during the above algorithm.

For $T, T' \in S$, write $T \xrightarrow{u} T'$ if $T'$ is obtained from $T$ by resolving a descent in columns $u, u + 1$.

Write $T \xrightarrow{*} T'$ if $T'$ can be obtained from $T$ through a sequence of descent resolutions.

**Lemma**

If $T, T_u, \text{ and } T_v \in S$ such that $T \xrightarrow{u} T_u$ and $T \xrightarrow{v} T_v$, then there exists $T' \in S$ such that $T_u \xrightarrow{*} T'$ and $T_v \xrightarrow{*} T'$. 
Proof of lemma

Lemma

If $T$, $T_u$, and $T_v \in S$ such that $T \xrightarrow{u} T_u$ and $T \xrightarrow{v} T_v$, then there exists $T' \in S$ such that $T_u \xrightarrow{*} T'$ and $T_v \xrightarrow{*} T'$. 

Proof: If $|u - v| \geq 2$, then the result is easy. Assume $u = v - 1$. Columns $u$, $u + 1$, $u + 2$ must look like:

```
1 1
2 2
```
Proof of lemma

Lemma

If $T$, $T_u$, and $T_v \in S$ such that $T \xrightarrow{u} T_u$ and $T \xrightarrow{v} T_v$, then there exists $T' \in S$ such that $T_u \xrightarrow{*} T'$ and $T_v \xrightarrow{*} T'$.

Proof: If $|u - v| \geq 2$, then the result is easy.
Proof of lemma

**Lemma**

If $T$, $T_u$, and $T_v \in S$ such that $T \xrightarrow{u} T_u$ and $T \xrightarrow{v} T_v$, then there exists $T' \in S$ such that $T_u \xrightarrow{*} T'$ and $T_v \xrightarrow{*} T'$.

Proof: If $|u - v| \geq 2$, then the result is easy.

Assume $u = v - 1$. Columns $u$, $u + 1$, $u + 2$ must look like:

```
  1 1
  2
  2
```
Resolving descents: End result is unique

**Proposition**

For each $T \in S$, there is a unique $T' \in \text{RPP}(\lambda/\mu)$ such that $T \rightarrow T'$. 
Resolving descents: End result is unique

Proposition

For each $T \in S$, there is a unique $T' \in \text{RPP}(\lambda/\mu)$ such that $T \to T'$.

Proof: Let $\ell : S \to \mathbb{N}$ be a function such that if $T_1 \overset{u}{\rightarrow} T_2$, then $\ell(T_1) < \ell(T_2)$. 
Resolving descents: End result is unique

**Proposition**

For each $T \in S$, there is a unique $T' \in \text{RPP}(\lambda/\mu)$ such that $T \xrightarrow{*} T'$.

**Proof:** Let $\ell : S \rightarrow \mathbb{N}$ be a function such that if $T_1 \xrightarrow{u} T_2$, then $\ell(T_1) < \ell(T_2)$.

We use backward induction on $\ell(T)$. Suppose $T \notin \text{RPP}(\lambda/\mu)$. Suppose $T \xrightarrow{u} T_u$ and $T \xrightarrow{v} T_v$. 

Resolving descents: End result is unique

**Proposition**

For each $T \in S$, there is a unique $T' \in \text{RPP}(\lambda/\mu)$ such that $T \rightarrow T'$.

Proof: Let $\ell : S \rightarrow \mathbb{N}$ be a function such that if $T_1 \rightarrow T_2$, then $\ell(T_1) < \ell(T_2)$.

We use backward induction on $\ell(T)$. Suppose $T \notin \text{RPP}(\lambda/\mu)$. Suppose $T \rightarrow T_u$ and $T \rightarrow T_v$.

By induction, there are unique $T'_u, T'_v \in \text{RPP}(\lambda/\mu)$ such that $T_u \Rightarrow T'_u, T_v \Rightarrow T'_v$. 
Resolving descents: End result is unique

Proposition

For each \( T \in S \), there is a unique \( T' \in \text{RPP}(\lambda/\mu) \) such that \( T \rightarrow T' \).

Proof: Let \( \ell : S \rightarrow \mathbb{N} \) be a function such that if \( T_1 \overset{u}{\rightarrow} T_2 \), then \( \ell(T_1) < \ell(T_2) \).

We use backward induction on \( \ell(T) \). Suppose \( T \notin \text{RPP}(\lambda/\mu) \).

Suppose \( T \overset{u}{\rightarrow} T_u \) and \( T \overset{v}{\rightarrow} T_v \).

By induction, there are unique \( T_u' \), \( T_v' \in \text{RPP}(\lambda/\mu) \) such that \( T_u \overset{*}{\rightarrow} T_u' \), \( T_v \overset{*}{\rightarrow} T_v' \).

By the Lemma, we must have \( T_u' = T_v' \).
Resolving descents: End result is unique

Proposition

For each \( T \in S \), there is a unique \( T' \in \text{RPP}(\lambda/\mu) \) such that \( T \overset{*}{\to} T' \).

Proof: Let \( \ell : S \to \mathbb{N} \) be a function such that if \( T_1 \overset{u}{\to} T_2 \), then \( \ell(T_1) < \ell(T_2) \).

We use backward induction on \( \ell(T) \). Suppose \( T \not\in \text{RPP}(\lambda/\mu) \). Suppose \( T \overset{u}{\to} T_u \) and \( T \overset{v}{\to} T_v \).

By induction, there are unique \( T'_u, T'_v \in \text{RPP}(\lambda/\mu) \) such that \( T_u \overset{*}{\to} T'_u, T_v \overset{*}{\to} T'_v \).

By the Lemma, we must have \( T'_u = T'_v \).

Since this holds for any \( u, v \), the Proposition is proved.
Newman’s Lemma

Note about the above proof: We are implicitly basing our argument on Newman’s lemma (or the diamond lemma): A terminating rewriting system is confluent if it locally confluent.
Let $T \in \text{RPP}(\lambda/\mu)$. Construct $B_1(T)$ from $T$ as follows.

1. Change all 1-pure columns to 2-pure columns and all 2-pure columns to 1-pure columns (of the same length).
2. “Resolve descents” one at a time until none remain.
   - How do we know that this process will terminate?
     - Look at positions of 1-pure and 2-pure columns.
   - How do we know the end result is unique?
Let $T \in \text{RPP}(\lambda/\mu)$. Construct $B_1(T)$ from $T$ as follows.

1. Change all 1-pure columns to 2-pure columns and all 2-pure columns to 1-pure columns (of the same length).
2. “Resolve descents” one at a time until none remain.
   - How do we know that this process will terminate?
     - Look at positions of 1-pure and 2-pure columns.
   - How do we know the end result is unique?
     - We do.
Define $B_1$

Let $T \in \text{RPP}(\lambda/\mu)$. Construct $B_1(T)$ from $T$ as follows.

1. Change all 1-pure columns to 2-pure columns and all 2-pure columns to 1-pure columns (of the same length).
2. “Resolve descents” one at a time until none remain.
   - How do we know that this process will terminate?
     - Look at positions of 1-pure and 2-pure columns.
   - How do we know the end result is unique?
     - We do.

Easy to check that $B_1 : \text{RPP}(\lambda/\mu) \rightarrow \text{RPP}(\lambda/\mu)$ is an involution.
Defintion of $B_1$

Let $T \in \text{RPP}(\lambda/\mu)$. Construct $B_1(T)$ from $T$ as follows.

1. Change all 1-pure columns to 2-pure columns and all 2-pure columns to 1-pure columns (of the same length).
2. “Resolve descents” one at a time until none remain.
   - How do we know that this process will terminate?
     - Look at positions of 1-pure and 2-pure columns.
   - How do we know the end result is unique?
     - We do.

Easy to check that $B_1 : \text{RPP}(\lambda/\mu) \to \text{RPP}(\lambda/\mu)$ is an involution.

Thus, $g_{\lambda/\mu}$ is symmetric.
Generalized Bender-Knuth involutions

The $B_i$ are the unique extensions of the Bender-Knuth involutions (to rpp) that satisfies a certain “locality” condition (see the last section of our paper).
Generalized Bender-Knuth involutions

The $B_i$ are the unique extensions of the Bender-Knuth involutions (to rpp) that satisfies a certain “locality” condition (see the last section of our paper).

The $B_i$ also give some additional structure to $RPP(\lambda/\mu)$ beyond the above symmetry: they preserve some of the behavior between adjacent rows of an rpp.
The statistic $\text{ceq}$

For $T \in \text{RPP}(\lambda/\mu)$, define $\text{ceq}(T) = (q_1, q_2, q_3, \ldots)$ where $q_i$ is the number of vertically adjacent pairs of cells in rows $i, i + 1$ of $T$ with equal entries.

\[
\begin{array}{ccc}
1 & 1 & 3 \\
1 & 1 & \\
2 & 2 & \\
1 & 3 & 4 \\
2 & 3 & \\
\end{array}
\]

$\text{ceq}(T) = (2, 0, 0, 1, 0, 0, \ldots)$
For each skew shape $\lambda/\mu$, define

$$\tilde{g}_{\lambda/\mu} = \sum_{T \in RPP(\lambda/\mu)} t^{ceq(T)} x^c(T)$$

where $t^{(q_1,q_2,q_3,...)} = t_1^{q_1} t_2^{q_2} t_3^{q_3} \cdots$. 
Refined dual stable Grothendieck polynomials

For each skew shape $\lambda/\mu$, define

$$\tilde{g}_{\lambda/\mu} = \sum_{T \in RPP(\lambda/\mu)} t^{\text{ceq}}(T) \chi^c(T)$$

where $t(q_1,q_2,q_3,...) = t_1^{q_1} t_2^{q_2} t_3^{q_3} \cdots$.

If $t = 1$, then $\tilde{g}_{\lambda/\mu} = g_{\lambda/\mu}$.
If $t = 0$, then $\tilde{g}_{\lambda/\mu} = s_{\lambda/\mu}$. 
For each skew shape $\lambda/\mu$, define

$$\tilde{g}_{\lambda/\mu} = \sum_{T \in \text{RPP}(\lambda/\mu)} t^{\text{ceq}(T)} x^c(T)$$

where $t(q_1, q_2, q_3, \ldots) = t_1^{q_1} t_2^{q_2} t_3^{q_3} \cdots$.

If $t = 1$, then $\tilde{g}_{\lambda/\mu} = g_{\lambda/\mu}$.
If $t = 0$, then $\tilde{g}_{\lambda/\mu} = s_{\lambda/\mu}$.

From the previous proof, $\tilde{g}_{\lambda/\mu}$ is symmetric in $x$. 
An example and a conjecture

Example: If $\lambda/\mu$ is a single column with $n$ cells, then

$$\tilde{g}_{\lambda/\mu} = e_n(t_1, t_2, \ldots, t_{n-1}, x_1, x_2, \ldots).$$

Conjecture (Grinberg):

$$\tilde{g}_{\lambda'/\mu'} = \det \left( e_{\lambda i} - \mu j - i + j \left( t_{\mu j} + 1, \ldots, t_{\lambda i - 1}, x_1, x_2, \ldots \right) \right).$$
An example and a conjecture

Example: If $\lambda/\mu$ is a single column with $n$ cells, then

$$\tilde{g}_{\lambda/\mu} = e_n(t_1, t_2, \ldots, t_{n-1}, x_1, x_2, \ldots).$$

Conjecture (Grinberg):

$$\tilde{g}_{\lambda'/\mu'} = \det \left( e_{\lambda_i - \mu_j - i + j}(t_{\mu_j+1}, \ldots, t_{\lambda_i-1}, x_1, x_2, \ldots) \right)_{i,j=1}^{\ell(\lambda)}$$
Thank you!
What is a dual stable Grothendieck polynomial? Generalized BK involutions Refined dual stable Grothendieck polynomials

References


