Plabic graphs and zonotopal tilings

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Main result

**Theorem (G. (2017))**

\[(k, n)\text{-plabic graphs} \xrightleftharpoons[dual]{planar} \text{horizontal sections at level } k \text{ of fine zonotopal tilings of } \mathbb{Z}(n, 3)\]
Main result

Theorem (G. (2017))

\((k, n)\)-plabic graphs $\leftrightarrow$ planar dual

horizontal sections at level \(k\) of
fine zonotopal tilings of \(\mathcal{Z}(n, 3)\)

\(\mathcal{Z}(n, 3)\) for \(n = 5\)
Main result

**Theorem (G. (2017))**

$(k, n)$-plabic graphs $\leftrightarrow$ planar dual $\Rightarrow$ horizontal sections at level $k$ of fine zonotopal tilings of $\mathcal{Z}(n, 3)$

- Level 0: $\emptyset$
- Level 1: $1, 2, 3, 4, 5$
- Level 2: $12, 13, 14, 15, 23, 24, 34$
- Level 3: $123, 124, 125, 134, 135, 145, 234, 235, 245, 345$
- Level 4: $1234, 1235, 1245, 1345, 1455, 2345$
- Level 5: $12345$
Main result

Theorem (G. (2017))

\((k, n)\)-plabic graphs \(\xleftrightarrow{\text{planar dual}}\) horizontal sections at level \(k\) of fine zonotopal tilings of \(\mathcal{Z}(n, 3)\)

level = 5

level = 4

level = 3

level = 2

level = 1

level = 0

\(\mathcal{Z}(n, 3)\) for \(n = 5\)
Main result

Theorem (G. (2017))

\((k, n)\)-plabic graphs \(\leftrightarrow\) dual planar

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$$(k, n)$$-plabic graphs $\xleftrightarrow{\text{planar}} \text{dual} \quad \text{horizontal sections at level } k \text{ of fine zonotopal tilings of } \mathcal{Z}(n, 3)$$

a (2, 5)-plabic graph

$\mathcal{Z}(n, 3)$ for $n = 5$
Part 1: Zonotopal tilings
Zonotopes

Definition (Minkowski sum)

\[ A, B \subset \mathbb{R}^d, \quad A + B := \{ a + b \mid a \in A, b \in B \}. \]
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Definition

Vector configuration:

\[ \mathbf{V} = (v_1, v_2, \ldots, v_n), \quad \text{where} \quad v_i \in \mathbb{R}^d. \]
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*Zonotope*:

\[ \mathcal{Z}_\mathbf{V} := [0, v_1] + [0, v_2] + \cdots + [0, v_n] \subset \mathbb{R}^d. \]
Two-dimensional zonotopes

\[ V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \]

\[ Z_V = \begin{pmatrix} 0 \\ v_1 \\ v_2 \end{pmatrix} \]

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Cyclic zonotopes

Definition

*Cyclic vector configuration:* \( C(n, d) := (v_1, v_2, \ldots, v_n) \), where

\[ v_i = (1, r_i, r_i^2, \ldots, r_i^{d-1}) \] for some \( 0 < r_1 < r_2 < \cdots < r_n \in \mathbb{R} \).
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\]

\[
\mathbf{C}(4, 2) = (v_1, v_2, v_3, v_4)
\]

\[
\mathbf{C}(6, 3) = (v_1, v_2, v_3, v_4, v_5, v_6)
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*Cyclic zonotope:* $Z(n, d) := ZC(n, d)$.

\[ C(4, 2) = \]

\[ C(6, 3) = \]
A *zonotopal tiling* of $\mathbb{Z}_V$ is a polyhedral subdivision $\Delta$ of $\mathbb{Z}_V$ into smaller zonotopes.
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A piece $\mathcal{Z}_{V'}$ is a **parallelotope** if the vectors in $V'$ form a basis of $\mathbb{R}^d$. 

![Diagram of zonotopal tiling](image-url)
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**Definition**
Vertices of zonotopal tilings

Fact

*Number of vertices in a fine zonotopal tiling of $\mathbb{Z}_V$ equals the number $\text{Ind}(V)$ of linearly independent subsets of $V$.***
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\text{Ind}(C(n, d)) = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{d}.
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Number of vertices in a fine zonotopal tiling of $\mathbb{Z}^n$ equals the number $\text{Ind}(\mathcal{V})$ of linearly independent subsets of $\mathcal{V}$.

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\]

\[V = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \]

\[\Delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 1 \end{pmatrix} \]

\[
\text{Ind}(V) = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} = 11, \quad |\text{Vert}(\Delta)| = 11.
\]
Question

*Which collections of subsets of $[n]$ can appear as $\text{Vert}(\Delta)$, where $\Delta$ is a fine zonotopal tiling of $\mathbb{Z}(n,2)$?*
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Definition (Leclerc–Zelevinsky (1998))

$S, T \subset [n]$ are strongly separated if there is no $i < j < k$ such that

$$i, k \in S \setminus T \text{ and } j \in T \setminus S$$

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Strongly separated:  

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
S \setminus T & & & & & T \setminus S & & & \\
\end{array}
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\[
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1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
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\end{array}
\]

$\mathcal{D} \subset 2^{[n]}$ is a strongly separated collection if all $S, T \in \mathcal{D}$ are strongly separated.
Proposition (Leclerc–Zelevinsky (1998))

The map $\Delta \mapsto \text{Vert}(\Delta)$ is a bijection between:

- fine zonotopal tilings $\Delta$ of $\mathbb{Z}^2(\mathbb{n}, 2)$,
- and maximal by inclusion strongly separated collections $D \subset 2^{\{n\}}$.

Corollary (Leclerc–Zelevinsky (1998))

Purity phenomenon: every maximal by inclusion strongly separated collection $D \subset 2^{\{n\}}$ is also maximal by size:

$|D| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2}$.
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Q: How many fine zonotopal tilings?
Chord separation

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When $|S| = |T|$, both definitions are due to Leclerc–Zelevinsky.
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\[
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\hline
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$S \setminus T$ | $T \setminus S$
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$S, T \subset [n]$ are *chord separated* if there is no $i < j < k < \ell$ such that

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```
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S \ T
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T \ S
```
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**Chord separation**

Strongly separated:

1 2 3 4 5 6 7 8 9

$S \setminus T$

Chord separated:

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$S \setminus T$

$T \setminus S$
Example for $n = 4$

*Chord separation:* no $i < j < k < \ell$ such that $i, k \in S \setminus T$, $j, \ell \in T \setminus S$ or vice versa.
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Q: How many fine zonotopal tilings?
A: Two.
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Part 2: Plabic graphs
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- turns right at each black vertex
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A plabic graph is *reduced* if it contains:

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- No strand intersects itself
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\((k, n)\)-plabic graphs

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\[ 1 \quad 2 \quad 3 \quad 4 \quad 5 \]
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Postnikov (2007): each \((k, n)\)-plabic graph has \(k(n - k) + 1\) faces.
Scott (2005): label each face of a \((k, n)\)-plabic graph by a \(k\)-element set:

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Conjecture (Leclecrc–Zelevinsky (1998), Scott (2005))

Every maximal by inclusion chord separated collection \( D \subset \binom{[n]}{k} \) has size

\[
k(n - k) + 1.
\]
Conjecture (Leclerc–Zelevinsky (1998), Scott (2005))

Every maximal by inclusion chord separated collection $\mathcal{D} \subset \binom{[n]}{k}$ has size $k(n - k) + 1$.

Conjecture (Leclecrc–Zelevinsky (1998), Scott (2005))

Every maximal by inclusion chord separated collection $\mathcal{D} \subset \binom{[n]}{k}$ has size $k(n - k) + 1$.


Theorem (Oh–Postnikov–Speyer (2011))

The map $G \mapsto \text{Faces}(G)$ is a bijection* between:

- $(k, n)$-plabic graphs, and
- maximal by inclusion chord separated collections $\mathcal{D} \subset \binom{[n]}{k}$.
Corollary (Oh–Postnikov–Speyer (2011))

Every maximal by inclusion chord separated collection \( \mathcal{D} \subset \binom{[n]}{k} \) has size

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Corollary (Oh–Postnikov–Speyer (2011))

Every maximal by inclusion chord separated collection $\mathcal{D} \subset \binom{[n]}{k}$ has size

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Theorem (G. (2017))

The map $\Delta \mapsto \text{Vert}(\Delta)$ is a bijection between:

- fine zonotopal tilings $\Delta$ of $\mathcal{Z}(n, 3)$, and
- maximal by inclusion chord separated collections $\mathcal{D} \subset 2^{[n]}$. 
Contradiction?

**Corollary (Oh–Postnikov–Speyer (2011))**

*Every maximal by inclusion chord separated collection* $\mathcal{D} \subset \binom{[n]}{k}$ *has size*

$$k(n - k) + 1.$$ 

**Theorem (G. (2017))**

*The map* $\Delta \mapsto \text{Vert}(\Delta)$ *is a bijection between:*

- *fine zonotopal tilings* $\Delta$ *of* $\mathbb{Z}(n, 3)$, *and*
- *maximal by inclusion chord separated collections* $\mathcal{D} \subset 2^{[n]}$.

**Corollary (G. (2017))**

*Every maximal by inclusion chord separated collection* $\mathcal{D} \subset 2^{[n]}$ *has size*

$$\text{Ind}(\mathcal{C}(n, 3)) = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}.$$
Corollary (Oh–Postnikov–Speyer (2011))

Every maximal by inclusion chord separated collection $\mathcal{D} \subset \binom{[n]}{k}$ has size

$$k(n - k) + 1.$$

Luckily for us,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} = \sum_{k=0}^{n} (k(n - k) + 1).$$

Corollary (G. (2017))

Every maximal by inclusion chord separated collection $\mathcal{D} \subset 2^{[n]}$ has size

$$\text{Ind}(\mathcal{C}(n, 3)) = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}.$$
Part 3: Putting it all together

level = 0
∅

level = 1
1
2
3
4
5

level = 2
12
34
5

level = 3
12
34
5

level = 4
12
34
5

level = 5
12345
Sections of tiles

\[ z = |S| + 1 \]

\[ z = |S| + 2 \]

\[ z = |S| + 3 \]
Sections of tiles

\[ z = |S| + 3 \]

\[ z = |S| + 2 \]

\[ z = |S| + 1 \]

\[ z = |S| \]
Example: $n = 4$
Example: $n = 4$
Example: $n = 4$
Example: $n = 4$
Example: $n = 4$
Example: $n = 4$
Example: \( n = 4 \)
Theorem (Postnikov (2007))

Any two \((k, n)\)-plabic graphs are connected by a sequence of moves:

\( (M1) \) \hspace{1cm} (M2) \hspace{1cm} (M3)
Moves and flips

Theorem (Postnikov (2007))

Any two \((k, n)\)-plabic graphs are connected by a sequence of moves:

- **(M1)**
- **(M2)**
- **(M3)**

A flip of a fine zonotopal tiling of \(\mathcal{Z}(n, 3)\) consists of replacing one tiling of \(\mathcal{Z}(4, 3)\) with another.
Moves and flips

Theorem (Postnikov (2007))

*Any two* $(k, n)$-plabic graphs *are connected by a sequence of moves:*

A *flip* of a fine zonotopal tiling of $\mathcal{Z}(n, 3)$ consists of replacing one tiling of $\mathcal{Z}(4, 3)$ with another.

Theorem (Ziegler (1993))

*Any two fine zonotopal tilings of $\mathcal{Z}(n, 3)$ are connected by a sequence of flips.*
Moves = sections of flips

\[
\begin{align*}
S &= \text{sections of flips} \\
S_{ab} &\quad S_{bc} &\quad S_{cd} &\quad S_{d} \\
S_{ab} &\quad S_{bc} &\quad S_{c} &\quad S_{d} \\
S_{ab} &\quad S_{b} &\quad S_{c} &\quad S_{d} \\
S_{ab} &\quad S_{b} &\quad S_{c} &\quad S_{d} \\
S_{ab} &\quad S_{b} &\quad S_{c} &\quad S_{d} \\
S_{ab} &\quad S_{b} &\quad S_{c} &\quad S_{d} \\
\end{align*}
\]
Pseudoplane arrangements

S \subset S c \subset S b \subset S a \subset S b c \subset S a c \subset S a b \subset S a b c

∅ 1 2 3 4 5 1 2 5 1 4 5 2 3 4 3 4 5 1 2 3 4 5

Plabic graphs and zonotopal tilings

FPSAC 2018, 07/19/2018
Pseudoplane arrangements

\[ S_{ab}C \]
\[ S_{ab} \]
\[ S_{bC} \]
\[ S_{aC} \]
\[ S_{b} \]
\[ S_{c} \]
\[ S_{a} \]
\[ S \]
Pseudoplane arrangements
**Main result**

**Theorem (G. (2017))**

\[(k, n)\text{-plabic graphs} \xleftrightarrow{\text{planar dual}} \text{horizontal sections at } z = k \text{ of fine zonotopal tilings of } \mathbb{Z}(n, 3)\]
Main result

**Theorem (G. (2017))**

\[ (k, n)\text{-plabic graphs} \xleftrightarrow{\text{planar \, dual}} \text{horizontal sections at } z = k \text{ of fine zonotopal tilings of } \mathcal{Z}(n,3) \]

**Purity** \(\implies\) every \((k, n)\)-plabic graph arises this way
Main result

**Theorem (G. (2017))**

$$(k, n)$$-plabic graphs $\xleftarrow{\text{planar}}^{\text{dual}} \xrightarrow{\text{horizontal sections at } z = k} \text{fine zonotopal tilings of } \mathcal{Z}(n, 3)$$

**Purity** $\implies$ every $(k, n)$-plabic graph arises this way  
**Moves** = horizontal sections of flips
Main result

Theorem (G. (2017))

\[(k, n)\text{-plabic graphs} \leftrightarrow \text{planar dual} \quad \text{horizontal sections at } z = k \text{ of fine zonotopal tilings of } \mathcal{Z}(n, 3)\]

Purity $\implies$ every $(k, n)$-plabic graph arises this way

Moves = horizontal sections of flips

Strands = horizontal sections of pseudoplanes.
Pavel Galashin.
Plabic graphs and zonotopal tilings.

Pavel Galashin and Alexander Postnikov.
Purity and separation for oriented matroids

Alexander Postnikov.
Total positivity, Grassmannians, and networks.

Suho Oh, Alexander Postnikov, and David E. Speyer.
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Günter M. Ziegler.
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Thank you!
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