# Curves inside DAHA & EHA.

joint w/ T. Lam

$$\begin{tikzcd}
Sk(T^2-D^2) \arrow[rr, "{[MS21]}"] \arrow[d, "f.h."] & & S\ddot{H}_N \arrow[d, "{[SV11]}"] \\
Sk(T^2) \arrow[r, "{[MS17]}"] \arrow[d, "f.h."] & \mathcal{E}_{q,1/q} \arrow[d, "\text{act on } 1"] & \mathcal{E}_{q,t} \arrow[l, "t=1/q"'] \arrow[d, "\text{act on } 1"] \\
Sk(A^2) \arrow[r, "{[Tur88]\,[AM98]}"] \arrow[d, "f.h."] & \Lambda_q \arrow[d, "\text{plethysm}"] & \Lambda_{q,t} \arrow[l, "t=1/q"'] \arrow[d, "\text{plethysm}"] \\
Sk(D^2) \arrow[r, "{[HOMFLY85]\,[PT87]}"] & \mathbb{C}(a,q) & \mathbb{C}(a,q,t) \arrow[l, "t=1/q"']
\end{tikzcd}$$

Schur positive for convex curves? → $\Lambda_{q,t}$

KR homology for some convex curves? → $\mathbb{C}(a,q,t)$

Legend:

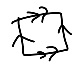
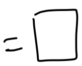
$Sk$ = Skein algebra

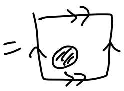
$S\ddot{H}_N$ = spherical DAHA

$\mathcal{E}_{q,t}$ = EHA

$\Lambda$ = symm. functions

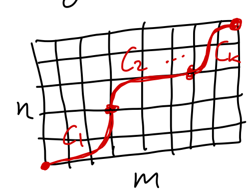
f.h. = fill hole

$T^2 =$  $D^2 =$ 

$A^2 =$  $T^2-D^2 =$ 

## ① Overview

**Def.** A (monotone) curve $C$ is a plot of a strictly monotone function $f:[0,m]\to[0,n]$ s.t. $f(0)=0$, $f(m)=n$.



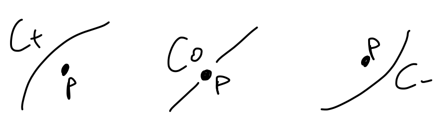

Notation: $C = C_1C_2\cdots C_k$

$k(C) = \#\{\text{lattice pts on } C\} - 1$

$C$-primitive $\iff k(C)=1$.

Main result #1. $C \longrightarrow \boxed{\text{topological construction}} \longrightarrow F_C \in \Lambda_{q,t}$ ← Studied by Negut, ...

Main result #2: Skein relation: for $p\in\mathbb{Z}^2$, if $C_+, C_-, C_0$ agree outside small neighborhood of $p \Rightarrow \boxed{F_{C_+} = qt\,F_{C_-} + F_{C_0}.}$




## Motivation:

curve ⟶ link

[GL20] positroid ⟶ link

[GL21] repetition-free positroid ⟶ convex curve ⟶ link

($C$ is convex if $f:[0,m]\to[0,n]$ is a convex function)

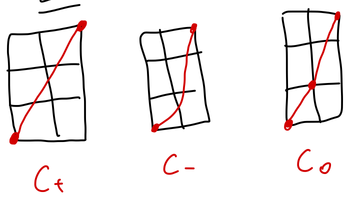
Prop [GL21] Coef. of $s_{(m)}$ in $F_C$ counts Dyck paths below $C$ when $q=t=1$.

**Conjecture** ([BHMPS 20], [GL21]):

$C$ - convex $\Rightarrow \frac{1}{(1-t)^{k(C)-1}} F_C$ - Schur-positive

$C$ - straight $\to$ shuffle conjecture

- Ex. $m,n = 2,3$

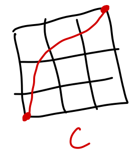



$F_{C_+} = s_{11} + (q+t)s_2$

$F_{C_-} = s_2$

$F_{C_0} = s_{11} + (q+t-qt)s_2$

- Ex. $m,n = 3,3$

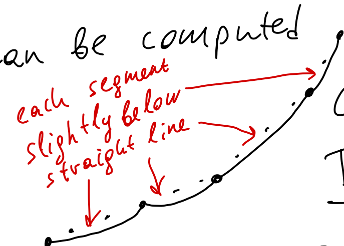



$F_C = (q+t-1)s_{111} + (\dots)s_{21} + \left(\begin{array}{c} q^4+q^3t+q^2t^2+qt^3+t^4 \\ +q^2t+qt^2 \\ -qt \end{array}\right) s_3$

not s-positive

$F_C$ can be computed recursively. Base case:

Convex piecewise almost linear curves.

$C =$ 


**Thm [GL].** The corresponding links are algebraic.

**Failed conjecture** ([GL21]):

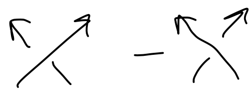
$C$ - convex $\Rightarrow$ $F_C$ computes KR homology of the link.

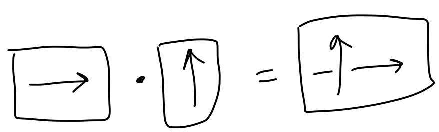
(Counterexample: Mazin/Mellit/Gorsky)

## ② Skein algebras.

Let $S$ be a surface.

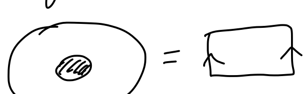
$Sk(S) = \left\{ \begin{array}{l} \text{linear combinations of} \\ \text{link diagrams drawn on } S \end{array} \right\} / \text{skein relation}$

skein relation:  $= (q^{1/2} - q^{-1/2})$ 

Multiplication: $L_1 \cdot L_2 =$ (draw $L_2$ on top of $L_1$) 

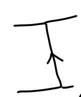
unit: empty link.

Ex. $S = D^2 \rightsquigarrow$ HOMFLY polynomial.

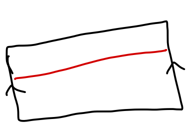
Ex. $S = A^2$ (Annulus) 

commutative!

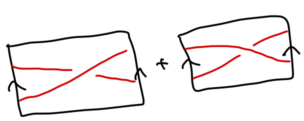
**Thm.** (Turaev '88) $Sk^+(A^2) \cong \Lambda_q$

$Sk^+$: only cross from left to right. 

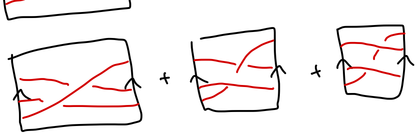
($p_k = x_1^k + x_2^k + \cdots$ generate $\Lambda_q$)

Ex: $p_1^A =$  $\mapsto p_1$

(cf. irreps of Hecke algebra)

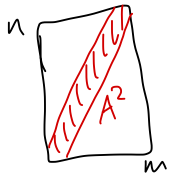
$p_2^A =$  $\mapsto p_2$

$p_k^A := \sum_{i=0}^{k-1} \sigma_1 \cdots \sigma_i \cdot \sigma_{i+1}^{-1} \cdots \sigma_{k-1}^{-1} \mapsto p_k$

$p_3^A =$  $\mapsto p_3$

# ③ Skein of $T^2$

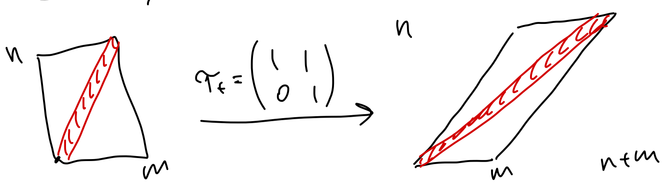
- $Sk(T^2)$ – non-commutative!
- For each slope $\frac{n}{m}$, $\gcd(m,n)=1$, $T^2$ contains a copy of $A^2$ in neighborhood of $y = \frac{n}{m}x$.



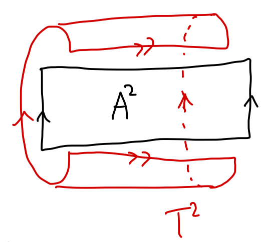

$\Rightarrow Sk(T^2)$ contains a copy of $\Lambda_q$ for each slope $\frac{n}{m}$.
Denote it by $\varphi_{m,n}(\Lambda_q) \subset Sk(T^2)$

- $SL_2(\mathbb{Z})$-action: $SL_2(\mathbb{Z})$ preserves $\mathbb{Z}^2 \Rightarrow$ acts on $T^2 = \mathbb{R}/\mathbb{Z}^2$

Ex.



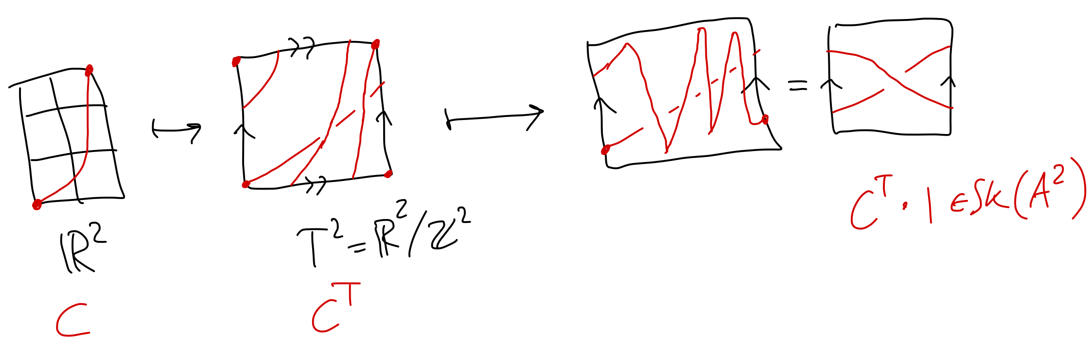

- $Sk(T^2)$ acts on $Sk(A^2)$ by wrapping $T^2$ around $A^2$:




$\varphi_{1,0}(P_k^A)$ acts by multiplication by $p_k$.

- Curve $C \longmapsto C^T \in Sk(T^2)$: project $C \subset \mathbb{R}^2$ to $T^2 = \mathbb{R}^2/\mathbb{Z}^2$
If $(x_1,y_1),(x_2,y_2) \in C$ project to $(x,y) \in T^2$
$x_1 < x_2 \Rightarrow$ draw $(x_2,y_2)$ above $(x_1,y_1)$.

Ex.



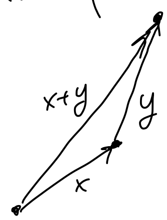

<u>Claim.</u> $F_C\big|_{t=1/q} = C^T \cdot 1$, where $C^T \in Sk(T^2)$
$C^T \cdot 1 \in Sk(A^2) \cong \Lambda_q$

## 4 EHA $\mathcal{E}_{q,t}$

Generators: $u_{m,n}$ $\quad m,n \in \mathbb{Z}^2$

Basic relation: $\cdot\ [u_x, u_y] = 0$ if $x = \lambda y$, $\lambda \in \mathbb{Q}$. (comm. subalgebra of fixed slope)

$[u_x, u_y] = u_{x+y}$ if triangle




contains no other lattice pts.

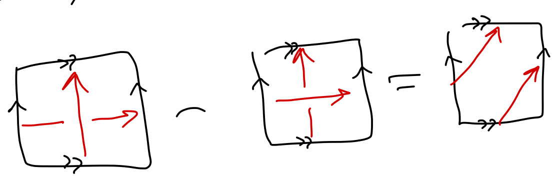
$\mathcal{E}_{q,t}$ acts on $\Lambda_{q,t}$: $\quad u_{m,0}$ – mult. by $p_m$

$u_{0,n}$ – Macdonald eigenoperator.

Thm. [MS 17] $\quad Sk(T^2) \xrightarrow{\sim} \mathcal{E}_{q,1/q}$

$$\varphi_{m,n}(P_k^A) \longmapsto u_{km,kn}$$

Ex. $[u_{1,0}, u_{0,1}] = u_{1,1}$



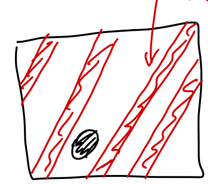
## 5 Punctured tores.

Thm. [MS21] [BCMN 23]

$$Sk(T^2 - D^2) \twoheadrightarrow \mathcal{E}_{q,t}$$

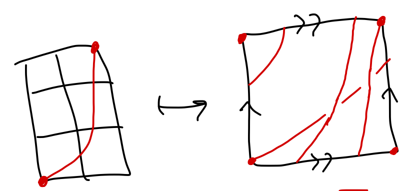
$$\varphi'_{m,n}(P_k^A) \longmapsto u_{km,kn}$$

$\varphi'_{m,n} =$

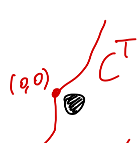



Q: How to define $C^{T-D}$ ???

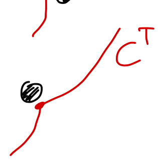
(Where to put the puncture?)




A: If $C$-primitive, let $C_+^{T-D} =$




$C_-^{T-D} :=$




$$\boxed{C^{T-D} := \frac{C_+^{T-D} - q\, C_-^{T-D}}{1-q}}$$

If $C = c_1 c_2 \cdots c_k \Rightarrow C^{T-D} = c_1^{T-D} \cdots c_k^{T-D}$